5.1-2: Consider a uniform, simple supported beam of length L. For each of the following loadings, use the Galerkin method to determine generalized d.o.f. a in the approximating lateral deflection field \( \ddot{v} = \alpha x (L - x) \) at midspan, determine the percentage errors of deflection \( \ddot{v} \) and bending moment \( \ddot{M} = EI \dddot{v} xx \).

\[
\int_0^L W[EI \dddot{v}_{xx} - \dddot{q}] \, dx = 0, \quad \dddot{v} = ax(L - x)
\]

Integrate by parts twice: \( W \)

\[
\int_0^L W_x[EI \dddot{v}_{xx}] \, dx - \int_0^L Wq \, dx + EI \int_0^L \dddot{v}_{xx} \, dx = 0
\]

\[W = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \quad \Rightarrow \quad \int_0^L W_{xx}[EI \dddot{v}_{xx}] \, dx - \int_0^L Wq \, dx - EI \int_0^L \dddot{v}_{xx} \, dx = 0
\]

But, as nonessential boundary condition, ends are simply supported; \( \dddot{v}_{xx} = 0 \) at ends.

Also \( \dddot{v}_{xx} = -2 \alpha \) and \( W_{xx} = -2 \), so

\[
\int_0^L -2EI(-2\alpha) \, dx - \int_0^L x(L-x)q \, dx = 0 \quad \text{and}
\]

\[a = \frac{1}{4EI} \int_0^L x(L-x)q \, dx; \quad \text{center} \quad \dddot{v} = a \frac{L^2}{4}
\]

(a) \( q = q_o \sin \frac{\pi x}{L} \). Let \( \Theta = \frac{\pi x}{L} \); then

\[a = \frac{1}{4EI} \int_0^L x(L-x)q_o \sin \frac{\pi x}{L} \, dx \quad \text{becomes}
\]

\[a = \frac{L^2 q_o}{4EI \pi^2} \int_0^\pi \Theta(1-\Theta) \sin \Theta \, d\Theta = 0.03225 \frac{q_o L^2}{EI}
\]

At center, \( \ddot{v} = a L^2 = 0.00806 \frac{q_o L^2}{EI} \)

Exact \( \ddot{v} = v_c \sin \frac{\pi x}{L} \) where \( v_c = \text{center} \ddot{v} \).

\[EI \dddot{v}_{xxx} = q, \quad EI \dddot{v}_{xx} = \frac{q_o L^4}{L^4} v_c \sin \frac{\pi x}{L} = q_o \sin \frac{\pi x}{L}
\]

\[v_c = \frac{q_o L^4}{L^4 EI} = 0.01027 \frac{q_o L^2}{EI}
\]

Approx. center deflection is 21.5% low.

\[
M = EI \dddot{v}_{xx} \quad \text{at center},
\]

exact \( M_c = EI \frac{q_o L^4}{L} \frac{(-\pi^2)}{n^2 + EI L^2} \)

\[
= 0.1013 \frac{q_o L^2}{36\% \text{ low}}
\]

Approx. \( M = EI(-2\alpha) = 0.0645 \frac{q_o L^2}{36\% \text{ low}} \)
(b) \( q = q_0 \)

\[
a = \frac{q_0}{4EI} \int_0^L x (L-x) \, dx = \frac{q_0}{4EI} \frac{L^2}{4} \frac{2L}{3} = \frac{q_0 L^2}{24EI}
\]

At center, \( \bar{V} = a \frac{L^4}{4} = \frac{q_0 L^4}{4EI} = 0.01042 \frac{q_0 L^4}{EI} \)

Exact is \( \frac{5q_0 L^4}{384EI} = 0.01302 \frac{q_0 L^4}{EI} \)

Approx. center deflection is 20.0 \% low.

At center, exact \( M_c = \frac{q_0 L^2}{8} \)

Approx. \( M = EI (-2a) = \frac{q_0 L^2}{12} \)

(33 \% low)
5.3-2: A cable of negligible flexural stiffness carries axial tension and contacts an elastic foundation of modulus $B$ (force per unit length per unit lateral deflection $w$). The sketch shows the cable in its deflected position. Left and right ends of the cable are loaded by the respective lateral forces $F_L$ and $F_R$.
Cable tension can be considered constant over the span shown, and equal to horizontal forces $T$, because $|w_x| \ll 1$.

![Sketch of cable and forces](image)

**Problem 5.3-2**

a) Show that governing differential equation is $T w_{xx} - B w = 0$

b) Use the Galerkin method to establish formulas for element matrices in terms of shape functions and constants. Clearly identify the stiffness matrix and the load vector (which is expressed in terms of $F_L$ and $F_R$).

$$
\begin{align*}
5.3-2 & \\
(a) & \quad \text{Sum vertical forces:} \\
& \quad -T w_x + T [w_x + d(w_{xx})] - B w \, dx = 0 \\
& \quad T d \left( w_x \right) = B w \, dx, \quad T w_{xx} - B w = 0 \\
(b) & \quad \tilde{w} = \left[ N \right] \{ d \} \quad \text{and} \quad \sum \int_0^L \left\{ \left[ N_1 \right]^T \tilde{w}_x - B \tilde{w} \right\} \, dx = 0 \\
& \quad \text{Integrate by parts} \\
& \quad \sum \int_0^L \left( -\left[ N \right]_{1x} \right)^T \tilde{w}_x \left| w_x \right| \, dx + \sum \left[ \left[ N \right]^T \tilde{w} \right]_0^L = 0 \\
& \quad \text{Also substitute} \quad \tilde{w} = \left[ N \right] \{ d \} \quad \text{and} \quad \tilde{w}_x = \left[ N \right]_{1x} \{ d \} \\
& \quad \sum \left( \int_0^L \left[ N \right]_{1x}^T \left[ N \right]_{1x} \, dx \right) \{ d \} = \left( \begin{array}{c} F_L \\ F_R \end{array} \right) \\
& \quad \left[ k \right] \quad \text{with} \quad \left[ k \right]_i = \int_0^L \left[ N \right]_i^T B \left[ N \right]_i \, dx \\
& \quad \text{the last term enters} \\
& \quad F_L & F_R \\
& \quad \text{at right end} & \text{at x=0} \\
& \quad F_L \& F_R \quad \text{enter} \\
& \quad \text{the last term} \\
& \quad \text{Also substitute} \\
& \quad \tilde{w} = \left[ N \right] \{ d \} \quad \text{and} \quad \tilde{w}_x = \left[ N \right]_{1x} \{ d \} \\
& \quad \sum \left( \int_0^L \left[ N \right]_{1x}^T \left[ N \right]_{1x} \, dx \right) \{ d \} = \left( \begin{array}{c} F_L \\ F_R \end{array} \right) \\
& \quad \left[ k \right] \quad \text{with} \quad \left[ k \right]_i = \int_0^L \left[ N \right]_i^T B \left[ N \right]_i \, dx \\
& \quad \text{the last term enters} \\
& \quad F_L & F_R \\
& \quad \text{at right end} & \text{at x=0} \\
& \quad F_L \& F_R \quad \text{enter} \\
& \quad \text{the last term}
5.5-2: The Helmholtz equation, \( p_{xx} + p_{yy} + p_{zz} + \left(\frac{\omega}{c}\right)^2 p = 0 \), governs acoustic modes of vibration in a cavity with rigid walls. Here \( p = p(x,y,z) \) is the amplitude of sinusoidally varying pressure, \( \omega \) is the circular frequency, and \( c \) is the speed of sound in the medium. The boundary condition is \( p_n = 0 \), where \( n \) is an axis normal to the wall. Let the assumed pressure amplitude field be \( p = \int N \{ p_e \} \). By using the Galerkin method, derive formulas for element matrices in terms of shape functions and constants.

\[
\begin{aligned}
\text{Residual equation is:} \\
\int_{V} N^T \left( \tilde{p}_{xx} + \tilde{p}_{yy} + \tilde{p}_{zz} + \frac{\omega^2}{c^2} \tilde{p} \right) dV = 0 \quad (A)
\end{aligned}
\]

Integrate by parts (see Eq. 5.5-5, with \( k_x = k_y = k_z = 1 \)).

\[
\int_{V} N^T \nabla^2 \tilde{p} dV = \int_{S} N^T (\tilde{p}_{xx} \ell + \tilde{p}_{yy} \ell + \tilde{p}_{zz} n) dS
\]

\[-\int \left( N_{xx}^T \tilde{p}_{xx} + N_{yy}^T \tilde{p}_{yy} + N_{zz}^T \tilde{p}_{zz} \right) dV
\]

Hence, with \( \tilde{p}_{xx} = \int N_{xx}^T p_e \) etc., Eq. (A) becomes

\[
\int \left( N_{xx}^T N_{xx} + N_{yy}^T N_{yy} + N_{zz}^T N_{zz} \right) dV p_e
\]

\[-\omega^2 \int \frac{L}{c^2} N N dV p_e = 0
\]

5.6-2: For the “Mixed” beam formulation described in problem 4.7-6, use the Galerkin method to obtain expressions for element matrices in terms of shape functions and constants. Governing equations are \( v_{xx} - M/EI = 0 \) and \( M_{xx} - q = 0 \). Consider a four–d.o.f. beam element of length \( L \), and assume that \( M \) and \( v \) fields use linear interpolation from nodal d.o.f. at element ends. Apply the formulation to a one-element cantilever beam that carries uniformly distributed lateral load \( q \) and transverse tip force \( P \).
5.6-2  Governing eqs. are \( \dddot{y}_{xx} - \frac{M}{EI} = 0 \) and \( M_{xx} - q = 0 \).

Assume, for element fields, \( \dddot{y} = \dddot{\bar{y}}_e \) and \( \dddot{M} = \dddot{\bar{M}}_e \).

First eq.: \( \int_0^L \dot{N}^T (\dddot{\bar{y}}_{xx} - \frac{\dddot{\bar{M}}}{EI}) \, dx = 0 \). Integrate 1st term by parts:

\[
\int_0^L \dddot{N}^T \dddot{\bar{y}}_{xx} \, dx = \left[ \dddot{N}^T \dddot{\bar{y}}_x \right]_0^L - \int_0^L \dot{N}^T \dddot{\bar{y}}_x \, dx \quad \text{Hence 1st eq. becomes}
\]

\[
- \int_0^L \dot{N}^T \dot{N} \, dx \, \dot{y}_e - \int_0^L \frac{1}{EI} \dddot{N}^T \dddot{N} \, dx \, \dddot{M}_e = - \left[ \dddot{N}^T \dddot{\bar{y}}_x \right]_0^L
\]

\( \quad \quad \text{\( L \) C cancels upon assembly of elements.} \)

Second eq.: \( \int_0^L \dddot{M} (\dddot{\bar{M}}_{xx} - \dddot{q}) \, dx = 0 \). Integrate 1st term by parts:

\[
\int_0^L \dddot{N}^T \dddot{M}_{xx} \, dx = \left[ \dddot{N}^T \dddot{M}_x \right]_0^L - \int_0^L \dot{N}^T \dddot{M}_x \, dx \quad \text{Hence 2nd eq. becomes}
\]

\[
- \int_0^L \dot{N}^T \dot{N} \, dx \, \dot{M}_e = \int_0^L \dddot{q} \, dx - \left[ \dddot{N}^T \dddot{M}_x \right]_0^L
\]

\( \quad \quad \text{\( L \) C cancels upon assembly of els.} \)

Put together: \[
\begin{bmatrix}
\dddot{N}_0 \\
\dddot{N}_1 \\
\dddot{N}_2
\end{bmatrix} \begin{bmatrix}
M_e \\
Q_e \\
\ddot{q}_f
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

in which, if

\[
\int \frac{L-x}{L} \, dx = \frac{L^2}{8}
\]

\[
[H]_0 = \begin{bmatrix}
0 & L & 0 \\
0 & 0 & L
\end{bmatrix}, \quad [H]_1 = \frac{1}{L} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

hence

\[
\begin{bmatrix}
L/3EI & L/6EI & 0 \\
L/6EI & L/3EI & 0 \\
1/L & -1/L & 0
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
\ddot{v}_1
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
-\ddot{q}_f
\end{bmatrix}
\]

Example: \( M_1 = 0, \, \ddot{v}_2 = 0 \), so

\[
\begin{bmatrix}
L/3EI & -1/L \\
-1/L & 0
\end{bmatrix}
\begin{bmatrix}
M_2 \\
\ddot{v}_1
\end{bmatrix} = \begin{bmatrix}
0 \\
-\ddot{q}_f
\end{bmatrix}
\]

Gives \( M_2 = PL + \frac{qL^2}{2} \), \( \ddot{v}_1 = \frac{PL^3}{3EI} + \frac{qL^4}{8EI} \). Should be \( qL^4/8EI \); other terms in \( M_2 \) and \( \ddot{v}_1 \) are exact.