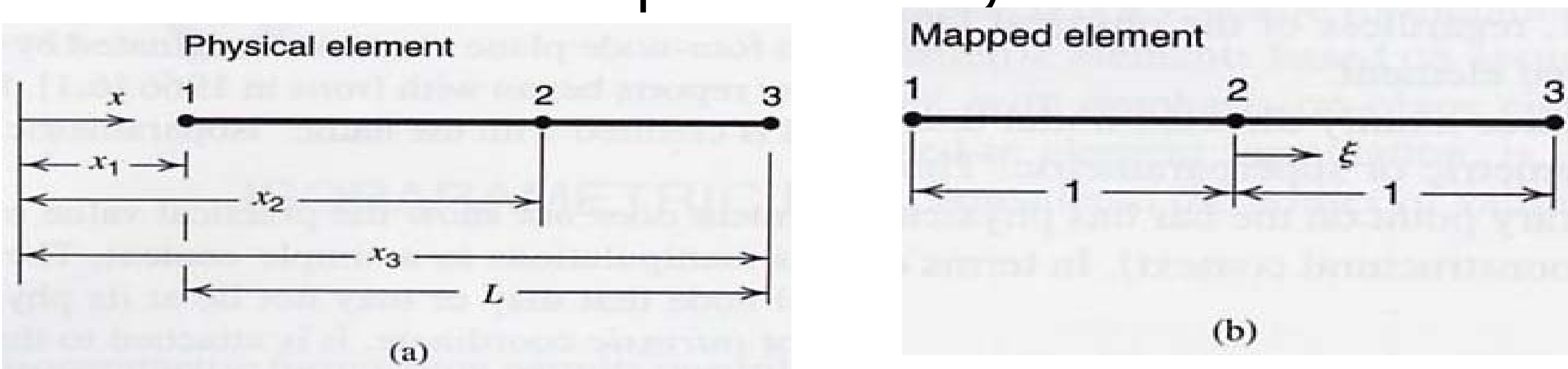
Chapter 6: Isoparametric elements

- Same shape functions are used to interpolate nodal coordinates and displacements
- Shape functions are defined for an idealized mapped element (e.g. square for any quadrilateral element)
- Advantages include more flexible shapes and compatibility
- We pay the price in complexity and require numerical integration to calculate stiffness matrices and equivalent loads

Bar element example

- Three-node bar example used only for illustration



- Quadratic variation of both coordinate and displacement in terms of ideal element coordinate

$$x = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \textit{and} \quad u = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

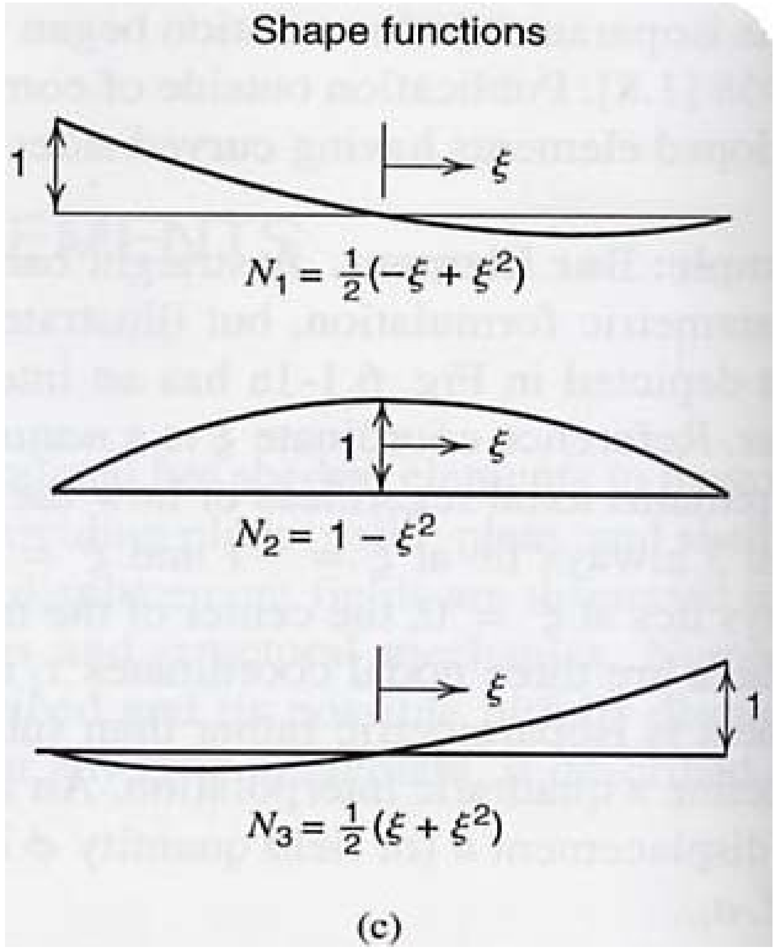
$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \textit{hence} \quad x = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Interpolation functions

$$x = \lfloor N \rfloor \lfloor x_1 \quad x_2 \quad x_3 \rfloor^T$$

$$u = \lfloor N \rfloor \lfloor u_1 \quad u_2 \quad u_3 \rfloor^T$$

$$\lfloor N \rfloor = \left\lfloor \frac{1}{2}\left(-\xi + \xi^2\right) \quad 1-\xi^2 \quad \frac{1}{2}\left(\xi + \xi^2\right) \right\rfloor$$



Strains and stiffness matrix

- Because of coordinate transformation taking derivatives is more complicated

$$\varepsilon_x = \frac{du}{dx} = \left(\frac{d}{dx}\lfloor N \rfloor\right)\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad where \quad \frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi}$$

- Jacobian

$$J = \frac{dx}{d\xi} = \frac{d}{d\xi}\lfloor N \rfloor \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \left\lfloor \frac{1}{2}(-1+2\xi) \quad -2\xi \quad \frac{1}{2}(1+2\xi) \right\rfloor \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

- Hence strain matrix

$$\lfloor B \rfloor = \frac{1}{J}\frac{d}{d\xi}\lfloor N \rfloor = \frac{1}{J}\left\lfloor \frac{1}{2}(-1+2\xi) \quad -2\xi \quad \frac{1}{2}(1+2\xi) \right\rfloor$$

- And stiffness matrix

$$\lfloor k \rfloor = \int_0^L \lfloor B \rfloor^T E \lfloor B \rfloor A dx = \int_{-1}^{1} \lfloor B \rfloor^T AE \lfloor B \rfloor J d\xi$$

6.2: Bilinear quadrilateral

- We were limited to rectangles because of compatibility

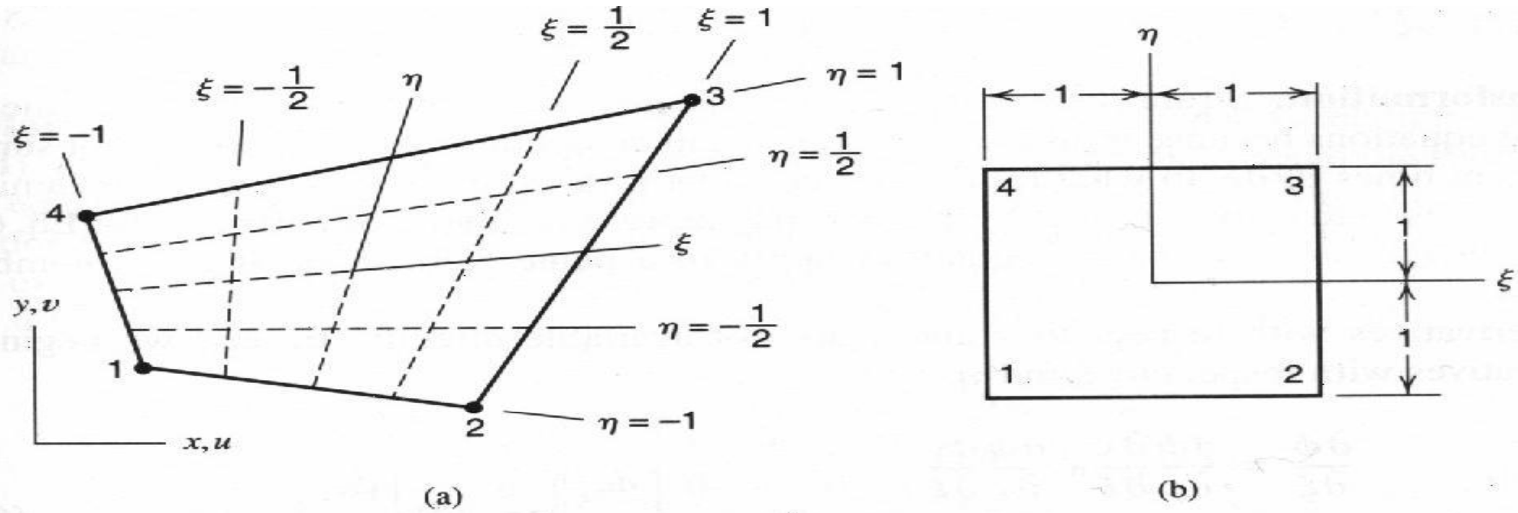


Figure 6.2-1. (a) Four-node plane element in physical space. (b) The same element, mapped into $\xi\eta$ space.

- Now both displacement and coordinates are bi-linear functions of ξ and η
- How does that preserve compatibility?

Interpolation

- Mapping

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \sum N_i x_i \\ \sum N_i y_i \end{Bmatrix} = [N]\{c\} \qquad \& \qquad \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \sum N_i u_i \\ \sum N_i v_i \end{Bmatrix} = [N]\{d\}$$

$$\{c\} = \lfloor x_1 \quad y_1 \quad x_2 \quad y_2 \quad x_3 \quad y_3 \quad x_4 \quad y_4 \rfloor^T$$

$$\{d\} = \lfloor u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \rfloor^T$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

- Interpolation functions

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Derivatives

- Chain rule of differentiation

$$\left.\begin{aligned}\frac{\partial \phi}{\partial \xi} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial \phi}{\partial \eta} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \eta}\end{aligned}\right\rangle \quad or \quad \begin{Bmatrix}\phi,_{\xi} \\ \phi,_{\eta}\end{Bmatrix} = [J]\begin{Bmatrix}\phi,_{x} \\ \phi,_{y}\end{Bmatrix}$$

- Jacobian

$$[J] = \begin{bmatrix} x,_{\xi} & y,_{\xi} \\ x,_{\eta} & y,_{\eta} \end{bmatrix} = \begin{bmatrix} \sum N_{i,\xi} x_i & \sum N_{i,\xi} y_i \\ \sum N_{i,\eta} x_i & \sum N_{i,\eta} y_i \end{bmatrix}$$

Derivatives-completed

- Using shape functins

$$[J]=\frac{1}{4}\begin{bmatrix}-(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta)\\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi)\end{bmatrix}\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\\ x_4 & y_4\end{bmatrix}=\begin{bmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix}$$

- Then

$$\begin{Bmatrix}\phi,_x\\ \phi,_y\end{Bmatrix}=\underbrace{\begin{bmatrix}\Gamma_{11} & \Gamma_{12}\\ \Gamma_{21} & \Gamma_{22}\end{bmatrix}}_{[\Gamma]}\begin{Bmatrix}\phi,_\xi\\ \phi,_\eta\end{Bmatrix}\quad where\quad [\Gamma]=[J]^{-1}=\frac{1}{J}\begin{bmatrix}J_{22} & -J_{12}\\ -J_{21} & J_{11}\end{bmatrix}$$

Strain and stiffness matrices

- Strains

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u,_x \\ u,_y \\ v,_x \\ v,_y \end{Bmatrix}$$

- Where

$$\begin{Bmatrix} u,_x \\ u,_y \\ v,_x \\ v,_y \end{Bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{Bmatrix} u,_\xi \\ u,_\eta \\ v,_\xi \\ v,_\eta \end{Bmatrix}$$

Laboring to obtain B matrix

- Finally expressed in terms of nodal displacements

$$\begin{Bmatrix} u,_{\xi} \\ u,_{\eta} \\ v,_{\xi} \\ v,_{\eta} \end{Bmatrix} = \begin{bmatrix} N_{1,\xi} & 0 & N_{2,\xi} & 0 & N_{3,\xi} & 0 & N_{4,\xi} & 0 \\ N_{1,\eta} & 0 & N_{2,\eta} & 0 & N_{3,\eta} & 0 & N_{4,\eta} & 0 \\ 0 & N_{1,\xi} & 0 & N_{2,\xi} & 0 & N_{3,\xi} & 0 & N_{4,\xi} \\ 0 & N_{1,\eta} & 0 & N_{2,\eta} & 0 & N_{3,\eta} & 0 & N_{4,\eta} \end{bmatrix} \underset{1*8}{\{d\}}$$

- So B matrix obtained by multiplying these 3x4, 4x4 and 4x8 matrices

- Stiffness matrix

$$\underset{8*8}{[k]} = \iint \underset{8*3}{[B]^T} \underset{3*3}{[E]} \underset{3*8}{[E]} t\, dx\, dy = \int_{-1}^{1}\int_{-1}^{1} \underset{8*3}{[B]^T} \underset{3*3}{[E]} \underset{3*8}{[B]} t\, J\, d\xi\, d\eta$$