Vibrations of beams

- Obtaining the equations of motion of beams via Newton’s method
- Demonstrating the advantage of Hamilton’s principle for boundary conditions
- Understanding why a vibration problem is an eigenproblem even when no matrices are involved
8.3: Bending vibration of beams

• Beam is more complicated because we have both force and moment
Equations of motion

- Vertical motion

\[
\left[ Q(x, t) + \frac{\partial Q(x, t)}{\partial x} \right] - Q(x, t) + f(x, t)dx = m(x)dx \frac{\partial^2 y(x, t)}{\partial t^2}
\]

\[
\frac{\partial Q(x, t)}{\partial x} + f(x, t) = m(x) \frac{\partial^2 y(x, t)}{\partial t^2}
\]

- For moment equilibrium we neglect inertia. Why?

\[
\left[ M(x, t) + \frac{\partial M(x, t)}{\partial x} \right] - M(x, t) + \left[ Q(x, t) + \frac{\partial Q(x, t)}{\partial x} \right] dx
\]

\[
+ f(x, t) dx \frac{dx}{2} = 0
\]

\[
\frac{\partial M(x, t)}{\partial x} + Q(x, t) = 0
\]
Putting it all together

• Equation of motion in terms of moments

\[- \frac{\partial^2 M(x, t)}{\partial x^2} + f(x, t) = m(x) \frac{\partial^2 y(x, t)}{\partial t^2}\]

• We can now use stiffness relation

\[M = EI \frac{\partial^2 y}{\partial x^2}\]

• See various boundary conditions in text
  – Clamped end: \(y = \frac{\partial y}{\partial x} = 0\)
  – Pinned end: \(y = M = 0, \ EI \frac{\partial^2 y}{\partial x^2} = 0\)
  – Free end \(M = Q = 0, \ EIy'' = 0, (EIy'')' = 0\)
Using Hamilton’s principle

• Problem 8.24, using integration by parts

\[ V = \frac{1}{2} \int_0^L EI \left[ \frac{\partial^2 y}{\partial x^2} \right]^2 dx \]

\[ T = \frac{1}{2} \int_0^L m \left[ \frac{\partial y}{\partial t} \right]^2 dx + \frac{1}{2} M \left[ \frac{\partial y(0,t)}{\partial t} \right]^2 \]

\[ \int_{t_1}^{t_2} \delta T dt = -\int_{t_1}^{t_2} \left( \int_0^L m \frac{\partial^2 y}{\partial t^2} \delta y dx + M \frac{\partial^2 y(0,t)}{\partial t^2} \delta y(0,t) \right) dt \]

\[ \delta V = \int_0^L EI \frac{\partial^2 y}{\partial x^2} \delta \left( \frac{\partial^2 y}{\partial x^2} \right) dx = \int_0^L EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 (\delta y)}{\partial x^2} dx \]

\[ = EI \frac{\partial^2 y}{\partial x^2} \frac{\partial (\delta y)}{\partial x} \bigg|_0^L - \frac{\partial}{\partial x} \left( EI \frac{\partial^2 y}{\partial x^2} \right) \delta y \bigg|_0^L + \int_0^L \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) \delta y dx \]

Why is the crossed terms zero?
Solution completed

• Collecting all terms

\[
\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) dt = \int_{t_1}^{t_2} \int_0^L \left( m \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[ EI \frac{\partial^2 y}{\partial x^2} \right] - f \right) \delta y dx dt
\]

\[
+ \int_{t_1}^{t_2} \left[ \frac{\partial}{\partial x} \left( EI \frac{\partial^2 y}{\partial x^2} \right) + M \frac{\partial^2 y}{\partial t^2} \right]_{x=0} \delta y(0, t) dt - \int_{t_1}^{t_2} \left[ \frac{\partial}{\partial x} \left( EI \frac{\partial^2 y}{\partial x^2} \right) \right]_{x=L} \delta y(L, t) dt
\]

\[
\int_{t_1}^{t_2} EI \frac{\partial^2 y}{\partial x^2} \frac{\partial (\delta y)}{\partial x} \bigg|_{x=L} dt - \int_{t_1}^{t_2} EI \frac{\partial^2 y}{\partial x^2} \frac{\partial (\delta y)}{\partial x} \bigg|_{x=0} dt
\]

• To get PDE and BC

\[
m \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[ EI \frac{\partial^2 y}{\partial x^2} \right] - f = 0
\]

\[
\frac{\partial}{\partial x} \left( EI \frac{\partial^2 y}{\partial x^2} \right) + M \frac{\partial^2 y}{\partial t^2} \bigg|_{x=0} = 0, \quad u(t, L) = 0, \quad EI \frac{\partial^2 y}{\partial x^2} \bigg|_{x=0} = 0, \quad \frac{\partial y(L, t)}{\partial x} = 0
\]
8.4: Free vibration: The differential eigenproblem

- Free vibration of a string

\[ \frac{\partial}{\partial x} \left[ T(x) \frac{\partial y(x, t)}{\partial x} \right] = \rho \frac{\partial^2 y(x, t)}{\partial t^2}, \quad y(0, t) = y(L, t) = 0 \]

- Separation of variables: \( y(x,t)=Y(x)F(t) \)

- Substitute back and rearrange

\[ \frac{1}{\rho(x)Y(x)} \frac{d}{dx} \left[ T(x) \frac{dY(x)}{dx} \right] = \frac{1}{F(t)} \frac{d^2 F}{dt^2} = \lambda \]

\[ \frac{d^2 F}{dt^2} - \lambda F = 0 \]

\[ \frac{d}{dx} \left[ T(x) \frac{dY(x)}{dx} \right] = \lambda \rho(x)Y(x) \]
Solution of time and space problems

• Time: \[ \frac{d^2 F}{dt^2} - \lambda F = 0, \quad F(t) = C \cos(\omega t - \phi), \quad \omega^2 = \lambda \]

• Space \[ \frac{d}{dx} \left[ T(x) \frac{dY(x)}{dx} \right] = \omega^2 \rho(x) Y(x) \quad Y(0) = Y(L) = 0 \]

• This is called a differential eigenvalue problem.

• What is “eigenvalue” about it? That is what is the common denominator of the algebraic and differential eigenvalue problems?

• For beams obtain \[ \frac{d^2}{dx^2} \left[ EI \frac{d^2 Y}{dx^2} \right] = \omega^2 m Y \]
Reading assignment

Sections 8.4-5

Source: www.library.veryhelpful.co.uk/Page11.htm