Rayleigh-Ritz continued

• Choice of mode is an issue of number vs complexity
  – Using modes that satisfy only kinematic BC allows simple functions
  – Using modes that satisfy all BC allows accurate solution with small number of modes
  – What other condition does the expansion needs to satisfy?

• Can use R-R also for transient response problem
Effect of choice of trial functions

• Trial functions that satisfy also the natural boundary conditions are called comparison functions
• An easy choice for comparison functions is the natural modes of a similar structures
• Thus we can use the vibration modes of a uniform beam to find the frequencies of a tapered beam
Problem 9.20 revisited

- A tapered cantilever beam has stiffness and mass distribution of

\[ m(x) = \rho h \left(1 - \frac{x}{2L}\right) \quad EI(x) = \frac{1}{12} Eh^3 \left(1 - \frac{x}{2L}\right)^3 \]

- Estimate lowest frequency by using a two term Ritz solution \( Y = a_1 x^2 + a_2 x^3 \). **Compare to using first two modes of uniform beam**
Vibration modes of uniform beam

• The general form of a beam vibration mode is (Example 8.4)

\[
Y(x) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x
\]

• From BC:

\[
Y(0) = B + D = 0 \quad Y'(0) = \beta(A + C) = 0
\]

\[
Y''(L) = -\beta^2 \left[ A(\sin \beta L + \sinh \beta L) + B(\cos \beta L + \cosh \beta L) \right] = 0
\]

\[
Y'''(L) = -\beta^3 \left[ A(\cos \beta L + \cosh \beta L) - B(\sin \beta L - \sinh \beta L) \right] = 0
\]

• Mode shapes

\[
Y_r(x) = \sin \beta_r x - \sinh \beta_r x - \frac{\sin \beta_r L + \sinh \beta_r L}{\cos \beta_r L + \cosh \beta_r L} (\cos \beta_r x - \cosh \beta_r x)
\]

\[
\cos \beta_r L \cosh \beta_r L = -1
\]
Calculation of stiffness matrix

- For a beam

\[ V_{\text{max}} = \frac{1}{2} \int_0^L EI \left( \frac{\partial^2 Y}{\partial x^2} \right)^2 dx \]

- With the expansion

\[ Y(x) = a_1 \phi_1 + a_2 \phi_2 \]

\[ V_{\text{max}} = \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 k_{ij} a_i a_j \quad k_{ij} = \int_0^L EI(x) \frac{d^2 \phi_i}{dx^2} \frac{d^2 \phi_j}{dx^2} dx \]

\[ K_{\text{polynomial}} = ELh^3 \begin{bmatrix} 0.1563 & 0.1625L \\ 0.1625L & 0.2625L^2 \end{bmatrix} \quad K_{\text{modes}} = \frac{Eh^3}{L} \begin{bmatrix} 1.439 & 1.472 \\ 1.472 & 21.20 \end{bmatrix} \]
Calculation of mass matrix and frequencies

- **Kinetic energy**

\[
T_{ref} = \int_0^L m \ Y^2 \, dx = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} m_{ij} a_i a_j
\]

\[
m_{ij} = \int_0^L m(x) \ \phi_i(x) \phi_j(x) \, dx = \int_0^L m(x) \ \phi_i(x) \phi_j(x) \, dx
\]

\[
M_{\text{polynomial}} = \rho h L^5 \begin{bmatrix}
0.1167 & 0.0952L \\
0.0952L & 0.0804L^2
\end{bmatrix}
\]

\[
M_{\text{modes}} = \rho h L \begin{bmatrix}
1.107 & 0.1027 \\
0.1027 & 0.6777
\end{bmatrix}
\]

Checks?
Solution

- Convergence with increasing number of terms:
- Polynomial
  - First frequency: 1.1573, 1.1102, 1.1040, 1.1040, 1.1039
  - Second: 6.2369, 5.4518, 5.2993, 5.2945
- Modes
  - First frequency: 1.1400, 1.1042, 1.1038, 1.1038, 1.1038
  - Second: 5.6052, 5.2960, 5.2909, 5.2880
9.8: System response

- Same principle of series expansion, this time for transient response
- Known as the Rayleigh-Ritz method, even though textbook qualifies that some
- We still need to satisfy kinematic boundary conditions
- The most commonly used series is actually vibration modes of the structure or a similar structure
Reading assignment

Section 9.9

Source: www.library.veryhelpful.co.uk/Page11.htm