2.1 Undamped single-degree-of freedom

- Example: Problem 2.1: A cylindrical buoy of cross-sectional area $A$ and total mass $m$ is first depressed from equilibrium and then allowed to oscillate. Determine its natural frequency in terms of mass density of the liquid $\gamma$.

\[
\text{Buoyancy force} \quad F = -g\gamma Ax
\]

\[
m\ddot{x} + g\gamma Ax = 0 \quad \omega_n = \sqrt{\frac{g\gamma A}{m}}
\]
Basic equations

• Equations of motion

\[ m\ddot{x} + kx = 0 \quad \text{or} \quad \ddot{x} + \omega_n x = 0 \]

\[ \omega_n = \sqrt{k/m} \]

\[ x(0) = x_0, \quad \dot{x}(0) = v_0 \]

• Solution

\[ x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t} \]

\[ = C \cos(\omega_n t - \phi) \]

\[ C = \sqrt{x_0 + \left( \frac{v_0}{\omega_n} \right)^2}, \quad \phi = \tan^{-1} \left( \frac{v_0}{x_0 \omega_n} \right) \]
2.2: Viscously damped single-dof systems

• Basic equations

\[ m\ddot{x} + c\dot{x} + kx = 0 \quad \ddot{x} + 2\zeta\omega_n\dot{x} + k\omega_n^2 x = 0 \]

\[ \zeta = \frac{c}{2m\omega_n} \]

\[ x(0) = x_0 \quad \dot{x}(0) = v_0 \]

• Solution

\[ x = Ae^{st} \quad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]

\[ s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \]

\[ x = \frac{-s_2x_0 + v_0}{s_1 - s_2} e^{s_1t} + \frac{s_1x_0 - v_0}{s_1 - s_2} e^{s_2t} \]
Under damped systems

- What is the damping ratio for your car's suspension?

\[ \zeta < 1 \quad x = Ce^{-\zeta \omega_n t} \cos(\omega_n t - \phi) \]
Over-damped systems

\[ x = e^{-\zeta \omega_n t} \left( \frac{\zeta \omega_n x_0 + v_0}{\omega_n \sqrt{\zeta^2 - 1}} \sinh \sqrt{\zeta^2 - 1} \omega_n t + x_0 \cosh \sqrt{\zeta^2 - 1} \omega_n t \right) \]

Where would you use a critically damped system?

FIGURE 2.10
Response of a critically damped ($\zeta = 1$) and overdamped ($\zeta > 1$) system
2.3 Measurement of damping

- Amplitude drop in one cycle
- Fig 2.11

\[
\frac{x_1}{x_2} = e^{\zeta \omega_n T} = e^{2\pi \zeta / \sqrt{1-\zeta^2}} \quad \delta = \ln \frac{x_1}{x_2} = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}}
\]

\[
\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}
\]

\[
\ln x_j = \ln x_1 - \delta(j-1)
\]
Example 2.5

- Excel spreadsheet

\[ z_j = 2.4211 - 0.3131 y_j \]

**FIGURE 2.13**
Determination of viscous damping factor by the least squares method
2.4 Coulomb Friction

- Basic equations

\[ m\ddot{x} + F_d \operatorname{sgn}(\dot{x}) + kx = 0 \]

\[ \operatorname{sgn}(\dot{x}) = \frac{\dot{x}}{|\dot{x}|} \]

\[ \ddot{x} + f_d \operatorname{sgn}(\dot{x}) + \omega_n^2 x = 0 \]

Analytical solution in textbook by breaking motion into intervals

**FIGURE 2.14**
Mass-spring system subjected to Coulomb damping
Numerical solution (Matlab)

- Reduce to first order equations
- In Example 2.6

function [f]=exam2_6(t,y)
% RHS for example 2.6 as 2 first order ODE
x=y(1);
xprime=y(2);
g(1)=xprime;
g(2)=-101.605*sign(xprime)-350*x;
f=g';
end

\[
\begin{align*}
\dot{x} &= x' \\
\dot{x}' &= -f_d \cdot \text{sgn}(x') - \omega_n^2 x
\end{align*}
\]

\[ f_d = 0.2803 \text{cm} \quad \omega_n^2 = 350 \left(1/\text{sec}^2\right) \]
Response

\[
[t,y] = \text{ode15s}(@\text{exam2_6},[0 \ 1], [3 \ 0]);
\]

Warning: Failure
at \( t=8.397061\times10^{-1} \). Unable
to meet integration
tolerances without reducing
the step size below the
smallest value allowed
\( (2.983235\times10^{-15}) \) at time \( t \).

\>
In ode15s at 686

\>
\>
plot(t,y(:,1));
Reading assignment

Sections 3.1-3.4

Source: www.library.veryhelpful.co.uk/ Page11.htm