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Reduction of the *Free Material Design* problem to a locking material formulation

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1. Abstract
The *Free Material Design* (FMD) with the trace constraint, originally put forward in [Bendsøe M.P., A.R. Diaz, R. Lipton, J.E. Taylor, Optimal design of material properties and material distribution for multiple loading conditions. Int. J. Numer. Meth. Eng. **38**, 1149-1170, 1995] is reduced to a locking material problem. A proof is given that at least 3 load conditions in 2D and 6 load conditions in 3D assure the optimal Hooke tensor being positive definite.
2. Keywords: Free Material Design, compliance minimization, anisotropy

3. Introduction
The *Free Material Design* (FMD) problem in its original setting concerns the optimum distribution of the elastic moduli within a given feasible domain Ω to make the body as stiff as possible under the isoperimetric condition of the form

$$\frac{1}{|\Omega|}\int_{\Omega} \operatorname{tr} \boldsymbol{C}\, dx = E_0 \tag{1}$$

Here $\boldsymbol{C}$ represents the Hooke tensor and E_0 is a referential elastic modulus, see Bendsøe et al. (1994). In the case of multiple loads the minimization concerns the weighted sum of the compliances corresponding to the subsequent loads, applied non-simultaneously, see Bendsøe et al. (1995). Both the problems reduce to saddle point formulations with the displacement fields assumed as the primal behavioral unknowns, see Kočvara et al. (2008), Haslinger et al. (2010).
The FMD problem for a single load condition can be rearranged to the stress-based setting. Then it assumes a remarkably simple form, equivalent to a locking material problem, and to some extent similar to the Michell problem formulation, see Czarnecki and Lewiński (2012). The aim of the present paper is to extend the stress-based approach to the multiple load case.
The following conventions are adopted. The stress and strain description refers to a Cartesian parameterization of the feasible domain Ω. Points $x \in \Omega$ are identified with the coordinates $(x_1, \ldots, x_d)$ d being the problem dimension. The orthonormal Cartesian basis is $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$. The set of symmetric tensors of second rank is denoted by $\mathbb{E}^2_s$. The stress and strain fields are referred to the $\boldsymbol{e}_i$ basis by:

$$\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j, \qquad \boldsymbol{\varepsilon} = \varepsilon_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

with the usual summation convention over repeated indices $i,\ j = 1, \ldots, d$. The scalar product in $\mathbb{E}^2_s$ is denoted by $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \cdot \tau_{ij}$ for $\boldsymbol{\sigma},\ \boldsymbol{\tau} \in \mathbb{E}^2_s$. The k-th eigenvalue of a square matrix $\boldsymbol{B}$ is denoted by $\mu_k(\boldsymbol{B})$.
The k-th singular value of an arbitrary matrix $\boldsymbol{A}$ is defined by

$$s_k(\boldsymbol{A}) = \sqrt{\mu_k(\boldsymbol{A}\boldsymbol{A}^T)}$$

The partial derivative $\dfrac{\partial}{\partial x_i}$ is denoted by $(\ )_{,i}$. Let $\boldsymbol{v} = (v_1, \ldots, v_d)$ be a displacement field in Ω. We define

$$\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i,j} + v_{j,i})$$

Thus $\boldsymbol{\varepsilon}(\boldsymbol{v})$ is the symmetric part of $\boldsymbol{\nabla v}$.
The indices: $i,\ j,\ k,\ l$ run over $1, \ldots, d$; the indices K, L run over $1, \ldots, m$; $m = d(d+1)/2$; the index

α of the load condition assumes the values: $1,\ \ldots,\ n$; n is the number of the load conditions.

4. The problem setting

Let $C_{ijkl}(x)$ be elastic moduli at $x \in \Omega$ referred to the basis: $\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$. The tensor $\boldsymbol{C}(x)$ satisfies the well known symmetry and positivity conditions. The spectral decomposition of the tensor $\boldsymbol{C}(x)$ has the form, cf Rychlewski (1984), Sutcliffe (1992)

$$\boldsymbol{C}(x) = \sum_{K=1}^{m} \lambda_K(x) \boldsymbol{\omega}_K(x) \otimes \boldsymbol{\omega}_K(x), \tag{2}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > \lambda_0 > 0$ are called Kelvin moduli and $\boldsymbol{\omega}_K(x) \in \mathbb{E}^2_s$, $\boldsymbol{\omega}_K(x) \cdot \boldsymbol{\omega}_L(x) = \delta_{KL}$, $K,\ L = 1,\ \ldots,\ m$ at $x \in \Omega$.

We write $\lambda_K \in L(\Omega,\ \mathbb{R}^+)$ which stands for more accurate information on the regularity of the moduli, not discussed in the present paper. The tensors $\boldsymbol{\omega}_K$, called eigenstates, will be further viewed as vectors in $\mathbb{R}^m$ according to the rules:

for $d = 2$;

$$\boldsymbol{\omega}_K(x) = \left[(\boldsymbol{\omega}_K)_{11},\ (\boldsymbol{\omega}_K)_{22},\ \sqrt{2}(\boldsymbol{\omega}_K)_{12} \right]^T \in \mathbb{R}^3$$

for $d = 3$

$$\boldsymbol{\omega}_K(x) = \left[(\boldsymbol{\omega}_K)_{11},\ (\boldsymbol{\omega}_K)_{22},\ (\boldsymbol{\omega}_K)_{33},\ \sqrt{2}(\boldsymbol{\omega}_K)_{23},\ \sqrt{2}(\boldsymbol{\omega}_K)_{13},\ \sqrt{2}(\boldsymbol{\omega}_K)_{12} \right]^T \in \mathbb{R}^6$$

Assume that $V(\Omega)$ is the set of kinematically admissible displacement fields $\boldsymbol{v} = (v_1,\ v_2,\ \ldots,\ v_d)$.
They vanish on Γ_2, a part of the boundary of Ω.
The remaining part Γ_1 of the boundary is subject to given tractions $\boldsymbol{T}^{(\alpha)}$, $\alpha \in \{1,\ \ldots,\ n\}$. Let $\Sigma_\alpha(\Omega)$ be the set of the stress fields capable of transmitting the load $\boldsymbol{T}^{(\alpha)}$ to the support Γ_1. The body forces are omitted. The linear form

$$f^{(\alpha)}(\boldsymbol{v}) = \int_{\Gamma_1} \boldsymbol{T}^{(\alpha)} \cdot \boldsymbol{v}\, ds \tag{3}$$

represents the virtual work of the $\boldsymbol{T}^{\alpha)}$ loading on the trial displacement $\boldsymbol{v} \in V(\Omega)$. The set $\Sigma_\alpha(\Omega)$ is composed of such fields $\boldsymbol{\tau}$ which locally are of $\mathbb{E}^2_s$ class and satisfy the variational equation of equilibrium

$$\int_\Omega \boldsymbol{\tau} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) dx = f^{(\alpha)}(\boldsymbol{v}) \qquad \forall\ \boldsymbol{v} \in V(\Omega) \tag{4}$$

called the equation of virtual work. Let

$$\begin{aligned} &\boldsymbol{u}^{(\alpha)} \in V(\Omega), \qquad \boldsymbol{\sigma}^{(\alpha)}(x) = \boldsymbol{C}(x) \boldsymbol{\varepsilon}\left(\boldsymbol{u}^{(\alpha)}(x)\right). \\ &x \in \Omega, \quad \boldsymbol{\sigma}^{(\alpha)} \in \Sigma_\alpha(\Omega). \end{aligned} \tag{5}$$

Then

$$\boldsymbol{u}^{(\alpha)},\ \boldsymbol{\varepsilon}^{(\alpha)} = \boldsymbol{\varepsilon}\left(\boldsymbol{u}^{(\alpha)}\right),\ \boldsymbol{\sigma}^{(\alpha)}$$

is the solution to the elasticity problem corresponding to the α-th loading. The quantity

$$\Upsilon^{(\alpha)} = f^{(\alpha)}\left(\boldsymbol{u}^{(\alpha)}\right) \tag{6}$$

or, equivalently

$$\Upsilon^{(\alpha)} = \min \left\{ \int_\Omega \boldsymbol{\tau} \cdot \left(\boldsymbol{C}^{-1} \boldsymbol{\tau}\right) dx \;\middle|\; \boldsymbol{\tau} \in \Sigma_\alpha(\Omega) \right\} \tag{7}$$

is called the compliance referred to the load case α. The equality (7) reflects the celebrated Castigliano theorem.

Define the weighted sum of the compliances

$$\eta_1 \Upsilon^{(1)} + \eta_2 \Upsilon^{(2)} + \ldots + \eta_n \Upsilon^{(n)} \tag{8}$$

corresponding to the subsequent loading conditions;

$$\eta_\alpha \in [0,\ 1], \quad \text{and} \quad \eta_1 + \ldots + \eta_n = 1.$$

The FMD problem reads:
find the Hooke tensor fields in Ω satisfying the isoperimetric condition (1) minimizing the functional (8), cf.Bendsøe et al. (1995).
The elastic moduli $C_{ijkl}(x)$ are at each point $x \in \Omega$ parameterized by the Kelvin moduli

$$\boldsymbol{\lambda}(x) = (\lambda_1(x),\ \ldots,\ \lambda_m(x))$$

and the eigenstates $(\boldsymbol{\omega}(x),\ \ldots,\ \boldsymbol{\omega}_m(x))$ satisfying the mentioned conditions.
Note that the isoperimetric condition (1) is directly expressed by $\boldsymbol{\lambda}$, since

$$\begin{aligned}
&\operatorname{tr} \boldsymbol{C} = C_{ijij} \quad \text{or} \quad \operatorname{tr} \boldsymbol{C} = ||\boldsymbol{\lambda}||_1 \\
&\text{or} \\
&\operatorname{tr} \boldsymbol{C} = \lambda_1 + \ldots + \lambda_m \\
&\text{because} \qquad \lambda_K \geq 0
\end{aligned}$$

We are ready now to write down the FMD problem with (8) as the merit function and (1) as the isoperimetric condition. By combining (7), (8) and using (2) we arrive at the problem

$$J_\eta = \min \left\{ I_\eta(\boldsymbol{\lambda}) \,\middle|\, \lambda_K \in L(\Omega,\ \mathbb{R}^+),\ \lambda_K \geq \lambda_3^0 > 0,\ \int_\Omega (\lambda_1 + \ldots + \lambda_m) dx = \Lambda \right\} \tag{9}$$

where $\Lambda = E_0 |\Omega|$ and

$$I_\eta(\boldsymbol{\lambda}) = \min \left\{ \int_\Omega W_\eta \left(\boldsymbol{\lambda}(x),\ \boldsymbol{\tau}^1(x),\ \ldots,\ \boldsymbol{\tau}^n(x) \right) dx \,\middle|\, \quad \boldsymbol{\tau}^\alpha \in \Sigma_\alpha(\Omega), \quad \alpha = 1,\ \ldots,\ n \right\} \tag{10}$$

with the integrand:

$$\begin{aligned}
W_\eta(\boldsymbol{\lambda};\ \boldsymbol{\tau}^1,\ \ldots,\ \boldsymbol{\tau}^n) = \min \Bigg\{ \sum_{K=1}^{m} \frac{1}{\lambda_K} \sum_{\alpha=1}^{m} \left(\sqrt{\eta_\alpha} \boldsymbol{\tau}^\alpha \cdot \boldsymbol{\omega}_K \right)^2 \,\Bigg| \\
\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL},\ \boldsymbol{\omega}_K \in \mathbb{R}^m,\ K,\ L = 1,\ \ldots,\ m \Bigg\}
\end{aligned} \tag{11}$$

The algebraic problem in (11) can be solved analytically. The details of the derivation are given in the next section. Prior to displaying this result let us define the matrix

$$\boldsymbol{S} = \left[\boldsymbol{\sigma}^1,\ \boldsymbol{\sigma}^2,\ \ldots,\ \boldsymbol{\sigma}^n \right]_{m \times n}, \qquad \hat{\boldsymbol{S}} = \boldsymbol{S}\boldsymbol{S}^T \tag{12}$$

where $\boldsymbol{\sigma}^\alpha$ are column vectors in $\mathbb{R}^m$.
Note that $\hat{\boldsymbol{S}}$ is a square matrix of dimensions $m \times m$. Recall that

$$\mu_1 \left(\hat{\boldsymbol{S}} \right) \geq \mu_2 \left(\hat{\boldsymbol{S}} \right) \geq \ldots \geq \mu_m \left(\hat{\boldsymbol{S}} \right)$$

are eigenvalues of $\hat{\boldsymbol{S}}$ and $s_K(\boldsymbol{S})$ are singular values of $\boldsymbol{S}$. They are interrelated by

$$\mu_K \left(\hat{\boldsymbol{S}} \right) = \left[s_K(\boldsymbol{S}) \right]^2 \tag{13}$$

For given $\boldsymbol{\sigma}^1,\ \ldots,\ \boldsymbol{\sigma}^n$ we determine the matrix $\hat{\boldsymbol{S}}$ and the potential

$$W_\lambda(\boldsymbol{\sigma}^1,\ \ldots,\ \boldsymbol{\sigma}^n) = \sum_{K=1}^{m} \frac{1}{\lambda_K} \mu_K \left(\hat{\boldsymbol{S}} \right) \tag{14}$$

We shall prove in the sequel that

$$W_\eta(\lambda;\ \boldsymbol{\tau}^1,\ \ldots,\ \boldsymbol{\tau}^n) = W_\lambda \left(\sqrt{\eta_1} \boldsymbol{\tau}^1,\ \ldots,\ \sqrt{\eta_n} \boldsymbol{\tau}^n \right) \tag{15}$$

with W_λ given by (14).
Having the results (15), (14) one can rearrange (9), (10) to the form

$$J_\eta = \min \left\{ Y_\eta(\boldsymbol{\tau}^1,\ \ldots,\ \boldsymbol{\tau}^n) \;\middle|\; \boldsymbol{\tau}^\alpha \in \Sigma_\alpha(\Omega) \right\} \tag{16}$$

where

$$Y_\eta\left(\boldsymbol{\tau}^1,\ \dots,\ \boldsymbol{\tau}^n\right)=\min\left\{\int_\Omega \sum_{K=1}^m \frac{1}{\lambda_K(x)}\left[s_K\left(\boldsymbol{S}_\eta(x)\right)\right]^2 dx \;\middle|\; \lambda_K\in L(\Omega,\ \mathbb{E}^+),\ \int_\Omega \sum_{K=1}^m \lambda_K(x)dx=\Lambda\right\} \quad (17)$$

Here

$$\boldsymbol{S}_\eta(x)=\left[\sqrt{\eta_1}\boldsymbol{\tau}^1(x),\ \dots,\ \sqrt{\eta_n}\boldsymbol{\tau}^n(x)\right].$$

Minimization over λ_K can be performed analytically. The result is

$$J_\eta=\frac{1}{\Lambda}(Z_\eta)^2 \quad (18)$$

with

$$Z_\eta=\min\left\{\int_\Omega \sum_{K=1}^m s_K\left(\boldsymbol{S}_\eta(x)\right)dx \;\middle|\; \boldsymbol{\tau}^\alpha\in\Sigma_\alpha(\Omega),\qquad \alpha=1,\ \dots,\ n\right\} \quad (19)$$

Let $\boldsymbol{\tau}^K=\boldsymbol{\tau}^{*K}$ are minimizers of (19). The minimizers $\lambda_K=\lambda_K^*$ of (17) read

$$\lambda_K^*(x)=\Lambda\frac{s_K\left(\boldsymbol{S}_\eta^*(x)\right)}{\displaystyle\int_\Omega \sum_{K=1}^m s_K\left(\boldsymbol{S}_\eta^*(x)\right)dx} \quad (20)$$

for $K=1,\ \dots,\ m$. The optimal $\boldsymbol{\omega}_K(x)=\boldsymbol{\omega}_K^*(x)$ is the K-th eigenvector of the m by m matrix $\hat{\boldsymbol{S}}_\eta^*(x)=\boldsymbol{S}_\eta^*\left(\boldsymbol{S}_\eta^*\right)^T$ and $||\boldsymbol{\omega}_K^*(x)||=1$.
The algorithm of constructing the optimal Hooke tensor at each point x is as follows. For given $\eta_1,\ \dots,\ \eta_n$:

1. Solve (19) or find the tensors $\boldsymbol{\tau}^{*1}(x),\ \dots,\ \boldsymbol{\tau}^{*n}(x)$ for each $x\in\Omega$. Construct the matrices $\boldsymbol{S}_\eta^*$ and $\hat{\boldsymbol{S}}_\eta^*$.
2. Solve the eigenvalue problem for the matrix $\hat{\boldsymbol{S}}_\eta^*$. Find $(\mu_K(x),\ \boldsymbol{\omega}_K^*)$ for each $x\in\Omega$. Compute $s_K(x)$ by (13).
3. Compute $\lambda_K^*(x)$ by (20).
4. Construct $\boldsymbol{C}$ by (2).

The crucial step of the algorithm is point 1. For $n=1$ a numerical method for solving (19) has been put forward in Czarnecki and Lewiński (2012). The extension to the case of $n>1$ is the subject of the current research.

5. Derivation of the potential W_λ

The aim of this problem is to solve the minimization problem (11). Let us introduce the $m\times n$ matrix

$$\boldsymbol{S}_\eta=\left[\sqrt{\eta_1}\boldsymbol{\tau}^1,\ \dots,\ \sqrt{\eta_n}\boldsymbol{\tau}^n\right] \quad (21)$$

in which $\boldsymbol{\tau}^i$ are columns representing the tensors $\boldsymbol{\tau}^i\in\mathbb{E}_s^2$ by the rules given in Sec. 2. Let us define

$$\hat{\boldsymbol{S}}_\eta=\boldsymbol{S}_\eta\boldsymbol{S}_\eta^T \quad (22)$$

The derivation of (14), (15) is based on the formula

$$\sum_{K=1}^m \frac{1}{\lambda_K}\sum_{\alpha=1}^n\left(\sqrt{\eta_\alpha}\boldsymbol{\tau}^\alpha\cdot\boldsymbol{\omega}_K\right)^2=\sum_{K=1}^m\frac{1}{\lambda_K}\boldsymbol{\omega}_K\cdot\left(\hat{\boldsymbol{S}}_\eta\boldsymbol{\omega}_k\right) \quad (23)$$

which can be easy checked. Let us define

$$a_K=\boldsymbol{\omega}_K\cdot\left(\hat{\boldsymbol{S}}_\eta\boldsymbol{\omega}_K\right),\qquad K=1,\ \dots,\ m \quad (24)$$

and re-write the potential (11) in the following form

$$W_\eta\left(\boldsymbol{\lambda};\ \boldsymbol{\tau}^1,\ \ldots,\ \boldsymbol{\tau}^n\right) = \min\left\{\sum_{K=1}^{m}\frac{1}{\lambda_K}a_K\ \middle|\ \boldsymbol{\omega}_K\cdot\boldsymbol{\omega}_L = \delta_{KL},\quad K,\ L=1,\ \ldots,\ m\right\} \tag{25}$$

Assume that

$$a_1 \geq a_2 \geq \ldots \geq a_m \tag{26}$$

and recall the ordering

$$\frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq \ldots \leq \frac{1}{\lambda_m} \tag{27}$$

Let us invoke the "rearrangement inequality", cf. Hardy et al. (1999)

$$\sum_{K=1}^{m}\frac{a_K}{\lambda_K} \leq \sum_{K=1}^{m}\frac{a_{\sigma(K)}}{\lambda_K} \tag{28}$$

where $\sigma(K)$ is a permutation of the indices $\{1,\ 2,\ \ldots,\ m\}$. We see that minimum in (25) refers to the case of a_K ordered as in (26). To make a_1 the biggest we define it by

$$a_1 = \max\left\{\boldsymbol{y}\cdot\left(\hat{\boldsymbol{S}}_\eta\boldsymbol{y}\right)\ \middle|\ ||\boldsymbol{y}|| = 1\right\} \tag{29}$$

hence

$$a_1 = \mu_1\left(\hat{\boldsymbol{S}}_\eta\right) \tag{30}$$

where $\mu_1\left(\hat{\boldsymbol{S}}_\eta\right)$ is the biggest eigenvalue of $\hat{\boldsymbol{S}}_\eta$. Moreover,

$$a_1 = \boldsymbol{\omega}_1^*\cdot\left(\hat{\boldsymbol{S}}_\eta\boldsymbol{\omega}_1^*\right) \tag{31}$$

where $(\mu_1,\ \boldsymbol{\omega}_1^*)$ solves the eigenvalue problem

$$\hat{\boldsymbol{S}}_\eta\boldsymbol{\omega}_1^* = \mu_1\left(\hat{\boldsymbol{S}}_\eta\right)\boldsymbol{\omega}_1^* \tag{32}$$

Having determined a_1 we choose a_2 as

$$a_2 = \boldsymbol{\omega}_2^*\cdot\left(\hat{\boldsymbol{S}}_\eta\boldsymbol{\omega}_2^*\right) \tag{33}$$

where $(\mu_2,\ \boldsymbol{\omega}_2^*)$ solves the eigenvalue problem

$$\hat{\boldsymbol{S}}_\eta\boldsymbol{\omega}_2^* = \mu_2\left(\hat{\boldsymbol{S}}_\eta\right)\boldsymbol{\omega}_2^* \tag{34}$$

and $\mu_2 \geq \mu_K$ $K = 3,\ \ldots,\ m$. Proceeding the same way we find eventually

$$a_K = \mu_K\left(\hat{\boldsymbol{S}}_\eta\right),\qquad K = 1,\ 2,\ \ldots,\ m \tag{35}$$

where μ_K are ordered as follows

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$$

The result ends the derivation of (13).

6. Derivation of (18), (19)

Consider the problem

$$f(s_1,\ \ldots,\ s_m) = \min\left\{\int_\Omega\sum_{K=1}^{m}\frac{(s_K(x))^2}{\lambda_K(x)}dx\ \middle|\ \sum_{K=1}^{m}\int_\Omega\lambda_K(x)dx = \Lambda\right\} \tag{36}$$

where $s_K \geq 0$.

Introduce the lagrangian

$$L = \int_\Omega\left(\sum_{K=1}^{m}\frac{(s_K(x))^2}{\lambda_K(x)} + \beta\left[\sum_{K=1}^{m}\lambda_K(x) - \Lambda\right]\right)dx \tag{37}$$

where β is a constant multiplier. The variation of L with respect to λ_K should be zero, hence

$$\lambda_K(x) = \frac{1}{\sqrt{\beta}} s_K(x) \tag{38}$$

The isoperimetric condition yields

$$\frac{1}{\sqrt{\beta}} = \frac{1}{\Lambda} \int_\Omega \sum_{K=1}^{m} s_K(x) dx \tag{39}$$

Substitution of this result into (38) gives (19). Substitution of (20) into (36) confirms (18).

7. Case studies

7.1. A single load case. Assume that $n = 1$

Denote $\Sigma_1(\Omega) = \Sigma(\Omega)$. Then $\eta_1 = 1$, $\boldsymbol{S}_\eta = \boldsymbol{S}$, $\boldsymbol{\tau}^1 = \boldsymbol{\tau}$. Then $\boldsymbol{S} = \boldsymbol{\tau}_{m\times 1}$ and $\hat{\boldsymbol{S}} = \boldsymbol{\tau}\boldsymbol{\tau}^T$. We know that $\hat{\boldsymbol{S}}$ has one positive eigenvalue μ_1 and $\mu_K\left(\hat{\boldsymbol{S}}\right) = 0$ for $K = 2, \ldots, m$. Note that $\mu_1 = ||\boldsymbol{\tau}||^2$ and $s_1 = ||\boldsymbol{\tau}||$. The problem (19) reduces to

$$Z_\eta = \min\left\{ \int_\Omega ||\boldsymbol{\tau}||dx \;\middle|\; \boldsymbol{\tau} \in \Sigma(\Omega) \right\} \tag{40}$$

Let $\boldsymbol{\tau} = \boldsymbol{\tau}^*$ be the minimizer of (40). Then $\mu_1 = ||\boldsymbol{\tau}^*||^2$ and $s_1 = ||\boldsymbol{\tau}^*||$.
By (20) we have

$$\lambda_1^*(x) = \Lambda \cdot \frac{||\boldsymbol{\tau}^*(x)||}{\displaystyle\int_\Omega ||\boldsymbol{\tau}^*(x)||dx} \tag{41}$$

and

$$\begin{aligned} &\lambda_2 = 0,\ \lambda_3 = 0 \quad \text{if} \quad d = 2,\ m = 3 \\ &\lambda_2 = 0,\ \ldots,\ \lambda_6 = 0 \quad \text{if} \quad d = 3,\ m = 6. \end{aligned} \tag{42}$$

The first eigenvector reads:

$$\boldsymbol{\omega}_1^* = \frac{\boldsymbol{\tau}^*}{||\boldsymbol{\tau}^*||}$$

Other eigenvectors $\boldsymbol{\omega}_1^*$ are chosen such that the orthogonality conditions hold. The optimal tensor $\boldsymbol{C}$ has components given by (2). This formula reduces to the first term

$$C_{ijkl} = \lambda_1^* \left(\boldsymbol{\omega}_1^*\right)_{ij} \left(\boldsymbol{\omega}_1^*\right)_{kl} \tag{43}$$

or

$$C_{ijkl}(x) = \frac{\Lambda}{\displaystyle\int_\Omega ||\boldsymbol{\tau}^*(x)||dx} \frac{\tau_{ij}^*(x)\tau_{kl}^*(x)}{||\boldsymbol{\tau}^*(x)||} \tag{44}$$

The results (40)–(44) have been for the first time reported in Czarnecki and Lewiński (2012).

7.2. Case of: $1 < n < m$

The matrix $\boldsymbol{S}_\eta$ has dimensions $m \times n$, while $\hat{\boldsymbol{S}} = \boldsymbol{S}_\eta \boldsymbol{S}_\eta^T$ has dimensions $m \times m$. The number of non-zero eigenvalues of $\hat{\boldsymbol{S}}_\eta$ equals n:

$$\begin{aligned} &\mu_K\left(\hat{\boldsymbol{S}}_\eta\right) > 0, \qquad K = 1,\ \ldots,\ n \\ &\mu_K\left(\hat{\boldsymbol{S}}_\eta\right) = 0, \qquad K = n+1,\ \ldots,\ m \end{aligned} \tag{45}$$

Consequently: $s_K(\boldsymbol{S}_\eta) = 0$ for $K = n+1, \ldots, m$.
Thus the problem (19) assumes the form

$$Z_\eta = \min\left\{ \int_\Omega \sum_{K=1}^{n} s_K\left(\boldsymbol{S}_\eta(x)\right) dx \;\middle|\; \boldsymbol{\tau}^\alpha \in \Sigma_\alpha(\Omega), \quad \alpha = 1,\ \ldots,\ n \right\} \tag{46}$$

The rule (20) yields the optimal Kelvin moduli. The first n moduli are positive (in general)

$$\lambda_K^* = \Lambda \frac{s_K\left(\boldsymbol{S}_\eta^*(x)\right)}{\displaystyle\int_\Omega \sum_{K=1}^{n} s_K\left(\boldsymbol{S}_\eta^*(x)\right) dx} \tag{47}$$

while the other Kelvin moduli vanish: $\lambda_K^* = 0$ for $K = n+1, \ \ldots, \ m$.
The unit eigenvectors $\boldsymbol{\omega}_K^*$, $K = 1, \ \ldots, \ n$ are uniquely determined. They appear in the representation:

$$C_{ijkl}(x) = \sum_{K=1}^{n} \lambda_K^*(x) \left(\boldsymbol{\omega}_K^*(x)\right)_{ij} \left(\boldsymbol{\omega}_K^*(x)\right)_{kl} \tag{48}$$

The optimal material is degenerated, since m-n Kelvin moduli vanish.

7.3. Case of $n = m$

The smallest number n of the load conditions assuring that all the optimal Kelvin moduli λ_K^*, $K = 1, \ \ldots, \ m$ are positive is equal m. In case of $n = m$ the matrix $\boldsymbol{S}_\eta$ is a square $m \times m$ matrix whose singular values $s_K(\boldsymbol{S}_\eta)$ are positive. The problem (19) holds for $n = m$ and λ_K^* given by (20) are all positive for $K = 1, \ \ldots, \ m$. The eigenvectors $\boldsymbol{\omega}_K$ are uniquely determined provided that all $\mu_K \left(\hat{\boldsymbol{S}}_\eta\right)$ are different. The formula for the optimal moduli is kept in the form of (2).

7.4. Case of $n > m$

In general, all singular values $s_K(\boldsymbol{S}_\eta)$ are positive. The problem (19) holds. The optimal Kelvin moduli are given by (20). The eigenvectors $\boldsymbol{\omega}_K^*$ of $\hat{\boldsymbol{S}}_\eta^*$ along with λ_K^* determine the optimal elastic modul by (2).

7.5. Case of $n = 2$

This case is a special case of Sec. 5.2 since $1 < n < 3$. Here $\boldsymbol{S}_\eta$ can be written by

$$\boldsymbol{S}_\eta = [\boldsymbol{\sigma}, \ \boldsymbol{\tau}]_{m\times 2} \tag{49}$$

with

$$\boldsymbol{\sigma} = \sqrt{\eta_1}\boldsymbol{\tau}^1, \qquad \boldsymbol{\tau} = \sqrt{\eta_2}\boldsymbol{\tau}^2.$$

The two positive eigenvalues of $\hat{\boldsymbol{S}} = \boldsymbol{S}_\eta \boldsymbol{S}_\eta^T$ are the same as the eigenvalues of the 2×2 matrix:

$$\boldsymbol{g}(\boldsymbol{\sigma}, \ \boldsymbol{\tau}) = \begin{bmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} & \boldsymbol{\sigma}\cdot\boldsymbol{\tau} \\ \boldsymbol{\tau}\cdot\boldsymbol{\sigma} & \boldsymbol{\tau}\cdot\boldsymbol{\tau} \end{bmatrix} \tag{50}$$

Hence

$$\begin{aligned} s_1(\boldsymbol{S}_\eta) &= \sqrt{\mu_1(\boldsymbol{g}(\boldsymbol{\sigma}, \ \boldsymbol{\tau}))} \\ s_2(\boldsymbol{S}_\eta) &= \sqrt{\mu_2(\boldsymbol{g}(\boldsymbol{\sigma}, \ \boldsymbol{\tau}))} \\ s_3(\boldsymbol{S}_\eta) &= 0 \end{aligned} \tag{51}$$

and $\mu_1 \geq \mu_2$ are ordered eigenvalues of $\boldsymbol{g}$. The problem (19) reduces to

$$Z_\eta = \min\left\{ \int_\Omega \left[\sqrt{\mu_1(\boldsymbol{g}(\boldsymbol{\sigma}, \ \boldsymbol{\tau}))} + \sqrt{\mu_2(\boldsymbol{g}(\boldsymbol{\sigma}, \ \boldsymbol{\tau}))}\right] dx \ \middle| \ \boldsymbol{\sigma} = \sqrt{\eta_1}\boldsymbol{\tau}^1, \ \boldsymbol{\tau} = \sqrt{\eta_2}\boldsymbol{\tau}^2, \ \boldsymbol{\tau}^\alpha \in \Sigma_\alpha(\Omega), \ \alpha = 1, \ 2 \right\} \tag{52}$$

The eigenvalues μ_1, μ_2 are the roots of the algebraic equation

$$\mu^2 - \left(||\boldsymbol{\sigma}||^2 + ||\boldsymbol{\tau}||^2\right)\mu + \left[||\boldsymbol{\sigma}||^2||\boldsymbol{\tau}||^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{\tau})^2\right] = 0 \tag{53}$$

hence

$$\begin{aligned} \mu_1 + \mu_2 &= ||\boldsymbol{\sigma}||^2 + ||\boldsymbol{\tau}||^2 \\ \mu_1\mu_2 &= ||\boldsymbol{\sigma}||^2||\boldsymbol{\tau}||^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{\tau})^2 \end{aligned} \tag{54}$$

Therefore the problem (52) can be put in the form

$$Z_\eta = \min\left\{ \int_\Omega U\left(\sqrt{\eta_1}\boldsymbol{\tau}^1(x), \ \sqrt{\eta_2}\boldsymbol{\tau}^2(x)\right) dx \ \middle| \ \boldsymbol{\tau}^\alpha \in \Sigma_\alpha(\Omega), \ \alpha = 1, \ 2 \right\} \tag{55}$$

where

$$U(\boldsymbol{\sigma}, \ \boldsymbol{\tau}) = \sqrt{||\boldsymbol{\sigma}||^2 + ||\boldsymbol{\tau}||^2 + 2\sqrt{||\boldsymbol{\sigma}||^2||\boldsymbol{\tau}||^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{\tau})^2}} \tag{56}$$

Upon finding the solution to this problem one can find the optimal Kelvin moduli by (47) or

$$\lambda_K^*(x) = \Lambda \frac{\sqrt{\mu_K(x)}}{\displaystyle\int_\Omega \left(\sqrt{\mu_1(x)} + \sqrt{\mu_2(x)}\right) dx} \tag{57}$$

$K = 1,\ 2$. Other Kelvin moduli vanish.
Thus the optimal Hooke tensor is degenerated. The eigenstates $\boldsymbol{\omega}_K^*$ are eigenvectors of the eigenvalue problem for the matrix $\hat{\boldsymbol{S}}_\eta$.
The result (55), (56) has been for the first time announced in Czarnecki and Lewiński (2011), see Czarnecki et al. (2011), for $d = 2$.

8. Kinematic formulation of the FMD problem

The characteristic feature of the problem (19) is the integrand being of linear growth with respect to $\boldsymbol{\tau}^1,\ \ldots,\ \boldsymbol{\tau}^n$. This property is well seen in the formula (40) for $n = 1$ and in the formula (55) for $n = 2$ and $d = 2$. Such variational problems appear in the theory of minimal surfaces and in the theory of mechanics of the locking materials, see Demengel and Suquet (1986). The latter theory teaches us that the problem dual to (19) assumes the form

$$\max \left\{ \sum_{\alpha=1}^{n} \sqrt{\eta_\alpha} f^{(\alpha)}(\boldsymbol{v}^{(\alpha)}) \;\middle|\; \left(\boldsymbol{\varepsilon}(\boldsymbol{v}^{(1)}(x)),\ \ldots,\ \boldsymbol{\varepsilon}(\boldsymbol{v}^{(n)}(x))\right) \in B \quad \text{for a.e. } x \in \Omega, \boldsymbol{v}^{(\alpha)} \in V(\Omega), \quad \alpha = 1,\ \ldots,\ n \right\} \tag{58}$$

where B is a ball in $\mathbb{R}^{m \cdot n}$ with respect to the norm dual to the norm

$$|||(\boldsymbol{\sigma}^1,\ \ldots,\ \boldsymbol{\sigma}^n)||| = \sum_{K=1}^{m} s_K \left([\boldsymbol{\sigma}^1,\ \ldots,\ \boldsymbol{\sigma}^n]\right) \tag{59}$$

in the sense of Yang (1998). This ball has been explicitly characterized in the case of $n = 2$, $d = 2$ in Czarnecki et al. (2011).

9. On the numerical analysis of the stress-based problem (19)

To attack the stress-based problem (19) numerically one should find an explicit numerical characterization of statically admissible stress fields. To this end we introduce a division of the domain Ω into polygons. In 3D case we use eight node hexahedrons-the isoparametric elements with trilinear shape functions interpolating six stress fields within each element. The equilibrium will be satisfied by fulfilling the virtual work equation (4) with the field $\boldsymbol{v}$ interpolated within the finite elements similarly as the stress fields. Eq. (4) leads to a system of underdeterminate algebraic equations linking the nodal values of the stress components. This system is solved by using the singular value decomposition (SVD) method. The solution involves free parameters. They are determined by the minimization condition in (19), which reduces to a finite dimensional unconstrained minimization problem.
Both single and multiload cases can be numerically solved by the method sketched above. The details of the algorithm can be found in Czarnecki and Lewiński (2012). To illustrate the method, consider the single load problems: a) cantilever, b) clamped thick plate under a transverse load and c) deep beam clamped at both the ends, see Fig. 1. For the subsequent problems the dimensions of the domains and the finite element meshes are
a) $L_x = 1$ m, $L_y = L_z = 0.5$ m; the mesh: $16 \times 8 \times 8 = 1024$
b) $L_x = L_y = 1.6$ m, $L_z = 0.4$ m; the mesh: $16 \times 16 \times 4 = 1024$
c) $L_x = 2.4$ m, $L_y = 0.4$ m, $L_z = 0.8$ m.; the mesh: $24 \times 4 \times 8 = 768$
The load T is modeled by the weight function emulating the concentrated force $T = T(\zeta) = T_{max} e^{-\left(\frac{\zeta - \zeta_0}{w}\right)}$, where $T_{max} \approx -31.8$ [N/m^2], $w = 0.1$ (surface integral $\int T(\zeta) da \approx 1.0$ [N]). Upon solving problem (40) one can find the distribution of the Kelvin modulus λ_1 by the rule (41). The solutions found, illustrated in Fig. 1 by the isovalue plots, compare favorably with the results of similar problems of topology optimization by using different approaches.

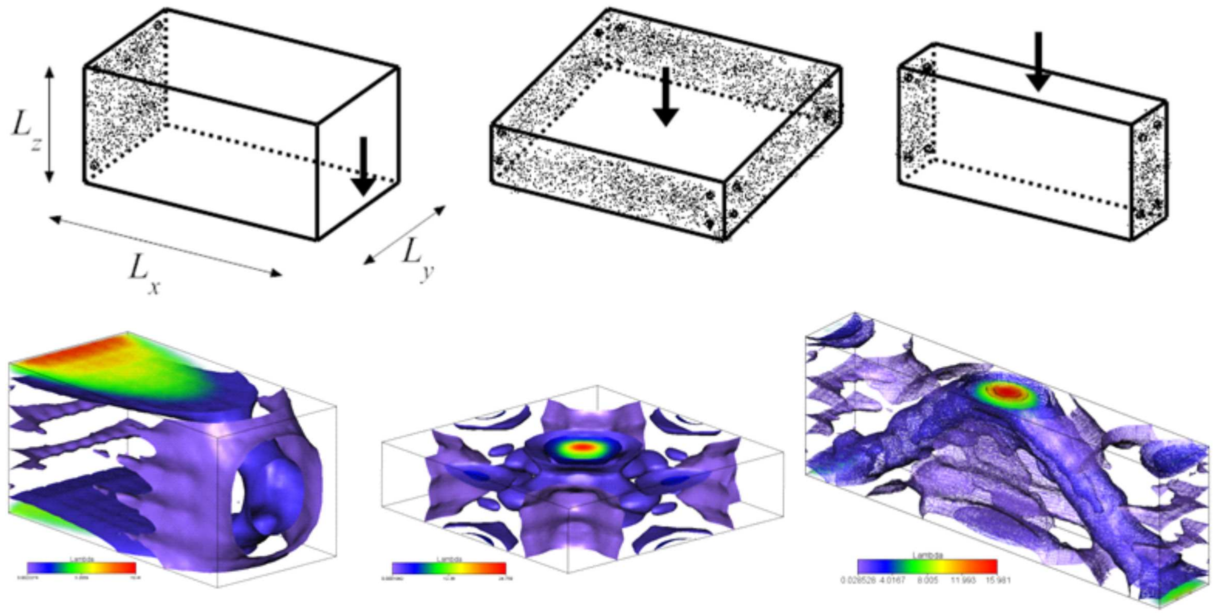

Figure 1: Selected isovalues of the optimal Kelvin modulus λ_1/E_0

10. Final remarks

A characteristic feature of both the Michell-like and locking material solutions is that there could appear sub-domains of Ω of positive measure where all the stresses $\boldsymbol{\tau}^{*i}$ (the minimizers of (19)) vanish identically. Note that this cannot happen in the linear elasticity problem in which the functional is of the quadratic growth in $\boldsymbol{\tau}$. In these domains where all $\boldsymbol{\tau}^{*i} = \mathbf{0}$ the material is not necessary, which means that there a hole is created. Thus the formulation (19) delivers the algorithm of creating all the necessary holes in one step, thus circumventing many evolutionary techniques as well as the topology derivative method by Sokołowski and Żochowski (1999) developed for the same aim. It should be stressed, however, that the latter method applies to a broad class of shape functionals, while the method proposed here applies only to the minimization of the weighted sum of the compliances.

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