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Sensitivity Analysis in the Level Set Method for Electromagnetic Problems

S.M. Townsend[1], S.W. Zhou[2] and Q. Li[1]

[1] School of Aerospace, Mechanical and Mechatronic Engineering, The University of Sydney, Sydney, NSW 2006, Australia, scott.townsend, qing.li@sydney.edu.au,
[2] Centre for Innovative Structures and Materials, School of Civil, Environmental and Chemical Engineering, RMIT University, GPO Box 2476, Melbourne 3001, Australia, shiwei.zhou@rmit.edu.au

1. Abstract
Interest in electromagnetic design problems has been recently refuelled as structural optimization rapidly propagates to other scientific and technological fields. Undertaking a sensitivity analysis of an objective function with respect to topological changes is a critical (and usually the most technical) step in any gradient-based topology optimization study. These types of analyses have been well documented for the density-based (SIMP) optimization methods, however clear instruction for the level set method is still scarce in the literature. As we will show, the level set method is a powerful alternative to the density-based methods, and has been shown to be superior in regard to some recent design problems, such as the design of electromagnetic metamaterials and plasmonic devices.

This study will explain the steps required in a level set sensitivity analysis. We will begin by describing the level set method in detail and show how one can use it to represent and transform a topology. We will continue by explaining the so-called shape gradient and derive some key formulas for its use. We will then show how to transform the governing PDE equations of a general engineering problem to a “level set method”-friendly form, and show how discrete methods can be used to evolve a topology toward optima.

We will conclude by applying the described steps to some electromagnetic design problems which have been solved previously by gradient methods in order to demonstrate the generality of the level set method and compare the results. We will also suggest some current problems which may require the level set method to be solved.
2. Keywords: Level set method, sensitivity analysis, electromagnetic, metamaterials.

3. Introduction
Topology and shape optimization techniques have been most noticeably applied to mechanical design problems, however recently the methodology has been extended to design problems in many other fields including photonics [1], phononics [2] and thermal energy transfer [3].

As design complexity increases, so too do the number of design variables; it becomes increasingly important to use sensitivity information to make design choices. Traditionally, the density-based (SIMP) methods have been employed. Though highly versatile and relatively easy to implement, the methods suffer drawbacks when discrete $\{0, 1\}$ solutions are required, namely in the form of intermediate density elements or checker-boarding [4].

The introduction of the level set method to topology optimization in the seminal works [5, 6] provided an alternate avenue of solution. This new method naturally produced discrete $\{0, 1\}$ designs and has since proven its versatility in several fields.

Although the method itself is easy enough to understand, calculation of the so-called “shape gradient”, which drives the design evolution, involves significant mathematical derivation, and clear instruction regarding its computation in the general case is scarce in the literature.

This paper aims to explain in detail the steps needed to undertake a sensitivity analysis for use with the level set method. Section 4 will explain the details of the level set method and its use in representing and transforming a topology. Section 5 will discuss sensitivity analysis in the level set framework and derive several key formulas. Finally, section 6 will apply the method to some electromagnetic design problems which have been solved previously in order to demonstrate the generality of the method.

4. Level Set Method
4.1 Topology Representation
In this paper, a topology Ω will refer to the distribution of material in space. Ω is bounded by both

external and internal boundaries, and the salient effect is that some material property m is different inside Ω compared to outside of it, i.e. $m = \{m_1 \mid \vec{x} \in \Omega,\ m_0 \mid \vec{x} \notin \Omega\}$.

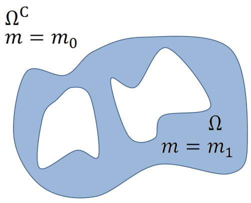


Figure 1: Topology and material properties

A problem becomes immediately obvious: How do we represent a complex topology like the one shown in figure 1? The level set method addresses this by first embedding Ω into a larger "hold-all domain" $U \mid \Omega \subset U$. We then define a scalar field $\psi\ \forall\ \vec{x} \in U$. The relation between ψ and Ω is such that

$$\begin{aligned} &\psi < 0 \rightarrow \vec{x} \in \Omega \\ &\psi > 0 \rightarrow \vec{x} \in U \setminus \overline{\Omega} \\ &\psi = 0 \rightarrow \vec{x} \in \partial\Omega \end{aligned} \tag{1}$$

as shown in figure 2. We are then able to define material properties in terms of ψ. For example, the following is common in the level set framework:

$$m = m_1 + (m_0 - m_1)H(\psi) \tag{2}$$

where $H(.)$ denotes the Heaviside step function. As (2) shows, the level set method restricts Ω to be strictly two-phase i.e. $m \in \{m_0, m_1\}$; there are no "grey" areas where $m \in (m_0, m_1)$. This property is a great advantage of the level set method when grey material properties are not physically achievable.

4.2 Evolving the Topology

This paper deals with sensitivity, meaning quantitative information about changes in Ω. The "changes" in Ω are those brought about via a velocity field $\vec{v}$ defined $\forall\ \vec{x} \in U$ which transforms the original topology to Ω_t (see figure 3). Thus Ω (and ψ) can be thought of as being time-dependent. The velocity field causes $\psi(t, \vec{x})$ to behave such that

$$\psi(\tau, \vec{x} + \vec{v}\tau) = \psi(0, \vec{x})$$

where $\vec{v} = \vec{v}(0, \vec{x})$ is the imposed velocity field defined at $t = 0$ and τ is some sufficiently small time step. Using a first order Taylor approximation, we have

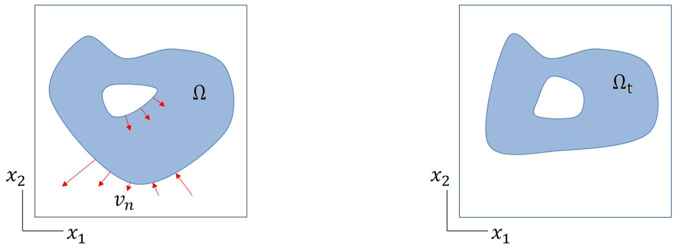
$$\begin{aligned} \psi(\tau, \vec{x}) &= \psi(\tau, \vec{x} + \vec{v}\tau) - \nabla\psi(\tau, \vec{x} + \vec{v}\tau) \cdot \vec{v}\tau \\ &= \psi(0, \vec{x}) - \nabla\psi(0, \vec{x}) \cdot \vec{v}\tau \end{aligned}$$

So then

$$\begin{aligned} \lim_{\tau \to 0} \frac{\psi(\tau, \vec{x}) - \psi(0, \vec{x})}{\tau} &= \lim_{\tau \to 0} \frac{\psi(0, \vec{x}) - \nabla\psi(0, \vec{x}) \cdot \vec{v}\tau - \psi(0, \vec{x})}{\tau} \\ \frac{d\psi}{dt}(0, \vec{x}) = \dot{\psi}(0, \vec{x}) &= -\nabla\psi(0, \vec{x}) \cdot \vec{v} \end{aligned}$$

(a) Level set function $\psi(\vec{x})$ (b) Resulting topology Ω

Figure 2: Topology representation in the level set method



(a) Velocity field on Ω (b) Evolved topology Ω_t

Figure 3: Evolving the topology in the level set method

The above can be re-written in a more familiar form by noting that every point $\vec{x}$ can be considered to lie on a level set of the ψ field; that is, every $\vec{x}$ lies on some contour $\psi(0, \vec{x}) = c$. As such, $\nabla\psi$ points in the direction normal to this level set, and only the normal component of $\vec{v}$, which we call v_n, will have any effect. We can thus write

$$\dot{\psi}(0, \vec{x}) = -\left|\nabla\psi(0, \vec{x})\right| v_n \tag{3}$$

5. Sensitivity Analysis

5.1 Objective Functionals and Level Set Embedding

In design situations, our aim will be to extremize some objective functional $J : \Omega \to \mathbb{R}$. We assume that J will take the form of a boundary or volume integral on Ω, i.e.

$$J = \int\limits_{\partial\Omega} \phi \; .dS \quad \text{or} \quad J = \int\limits_{\Omega} \phi \; .dV \tag{4}$$

where $\phi = \phi(\vec{x})$ represents local (i.e. spatially varying) properties of the topology. The value of ϕ at a given $\vec{x}$ may or may not change as Ω changes.

Our aim for a sensitivity analysis is to find an expression for the so-called "shape gradient" $\frac{dJ}{dt} = \dot{J}(v_n)$, then choose v_n and evolve Ω such that J changes favourably. The issue we have in finding $\dot{J}$ is that the domain of integration of J depends on Ω; we cannot differentiate (4) directly.

The way around this issue is to recall our definition of U from before, which is the "container" of Ω, and is the domain on which ψ is defined. The salient point is that the shape of U does not change with time, so any objectives on this domain can be readily differentiated akin to Leibnitz' rule. The strategy, then, is to convert (4) into an integral on U and then calculate $\dot{J}$. We will then usually want to convert the solution back onto Ω or $\partial\Omega$, since ϕ may contain terms that don't exist outside of Ω. We can use the results of this process to select v_n.

Three results that will be very useful to us in this regard are:

$$\int\limits_{\partial\Omega} \phi \; .dS = \int\limits_{U} \phi \; \delta(\psi) \; |\nabla\psi| \; .dV \tag{5}$$

$$\frac{d}{dt}\int\limits_{\partial\Omega} \phi \; .dS = \int\limits_{\partial\Omega} (\nabla\phi \cdot \hat{\nu} + \phi \; \nabla \cdot \hat{\nu}) \, v_n \; .dS \tag{6}$$

$$\frac{d}{dt}\int\limits_{\Omega} \phi \; .dV = \int\limits_{\partial\Omega} \phi \; v_n \; .dS \tag{7}$$

Note that equations (6) and (7) are valid only when ϕ is a static field (independent of Ω). (5) is valid for any ϕ. In order to prove (5) - (7), we first consider the boundary functional

$$J = \int\limits_{\partial\Omega} \phi \; .dS$$

First note that the above can be re-written using Gauss' theorem as

$$\begin{aligned} J &= \int\limits_{\partial\Omega} \phi \; \hat{\nu} \cdot \hat{\nu} \; .dS \\ &= \int\limits_{\Omega} \nabla \cdot (\phi \; \hat{\nu}) \; .dV \end{aligned}$$

where $\hat{\nu}$ is the unit normal to the shape boundary $\partial\Omega$. Now embedding Ω into a larger hold-all domain U and recalling (1), we can write

$$J = \int\limits_{U} (1 - H(\psi)) \; \nabla \cdot (\phi \; \hat{\nu}) \; .dV$$

A question arises as to what $\hat{\nu}$ should mean when used in the context of a volume integral. Recalling the level set formulation, we know that on the boundary $\partial\Omega$, we have $\hat{\nu} = \frac{\nabla\psi}{|\nabla\psi|}$. The scalar field ψ is also defined everywhere in U, so we can write

$$J = \int\limits_{U} (1 - H(\psi)) \; \nabla \cdot \left(\phi \; \frac{\nabla\psi}{|\nabla\psi|} \right) \; .dV$$

Applying the first scalar Green's theorem, we have

$$J = \int\limits_{\partial U} (1 - H(\psi)) \; \phi \; \left(\frac{\nabla\psi}{|\nabla\psi|} \cdot \hat{n} \right) \; .dS - \int\limits_{U} \phi \; \nabla(1 - H(\psi)) \; \cdot \frac{\nabla\psi}{|\nabla\psi|} \; .dV$$

where $\hat{n}$ is the unit normal to the hold-all boundary ∂U. We stated previously that U is large enough to completely contain $\overline{\Omega}$, i.e. $\overline{\Omega} \cap \partial U = \emptyset$. It follows that $\psi > 0 \; \forall \; \vec{x} \in \partial U$, and accordingly the boundary integral above drops out. Then

$$\begin{aligned} J &= -\int\limits_{U} \phi \; \nabla(1 - H(\psi)) \; \cdot \frac{\nabla\psi}{|\nabla\psi|} \; .dV \\ &= \int\limits_{U} \phi \; \delta(\psi) \; |\nabla\psi| \; .dV = \int\limits_{U} G(\phi, \psi, \nabla\psi) \; .dV \end{aligned}$$

which completes the proof of (5). Now since U is a static domain and assuming ϕ is a static field, it follows that we can write

$$\frac{dJ}{dt} = \dot{J} = \int\limits_{U} G_{,\psi}\dot{\psi} \; .dV + \int\limits_{U} G_{,\nabla\psi} \cdot \dot{\nabla\psi} \; .dV$$

where

$$G_{,\nabla\psi} \cdot \dot{\nabla\psi} = \frac{\partial G}{\partial \nabla\psi} \cdot \frac{d\nabla\psi}{dt} = \sum_i \frac{\partial G}{\partial \nabla\psi_i} \frac{d\nabla\psi_i}{dt}$$

Note that in what follows, it will been assumed that time and space differential operators are commutative, such that $\frac{d\nabla\psi}{dt} = \nabla\frac{d\psi}{dt}$. Then

$$\dot{J} = \int\limits_{U} \phi \; \delta(\psi)_{,\psi} \; |\nabla\psi| \; \dot{\psi} \; .dV + \int\limits_{U} \phi \; \delta(\psi) \; \frac{\nabla\psi}{|\nabla\psi|} \cdot \nabla\dot{\psi} \; .dV$$

Now using

$$\nabla\delta = \delta_{,\psi}\nabla\psi \quad \text{and} \quad \delta_{,\psi}|\nabla\psi| = \nabla\delta \cdot \frac{\nabla\psi}{|\nabla\psi|} \; ,$$

we can write

$$\begin{aligned} \dot{J} &= \int\limits_{U} \phi \; \nabla\delta(\psi) \cdot \frac{\nabla\psi}{|\nabla\psi|} \; \dot{\psi} \; .dV + \int\limits_{U} \psi \; \delta(\psi) \; \frac{\nabla\psi}{|\nabla\psi|} \cdot \nabla\dot{\psi} \; .dV = \int\limits_{U} \phi \; \nabla \left(\delta(\psi) \; \dot{\psi} \right) \cdot \frac{\nabla\psi}{|\nabla\psi|} \; .dV \\ &= \int\limits_{\partial U} \phi \; \delta(\psi) \; \dot{\psi} \; \frac{\nabla\psi}{|\nabla\psi|} \cdot \hat{n} \; .dS - \int\limits_{U} \delta(\psi) \; \dot{\psi} \; \nabla \cdot \left(\phi \; \frac{\nabla\psi}{|\nabla\psi|} \right) \; .dV \end{aligned}$$

where Green's first scalar theorem was reapplied from the first to the second line. It was previously argued that U can be chosen such that $\psi > 0 \; \forall \; \vec{x} \in \partial U$. Thus, the $\delta(\psi)$ term will be 0 on the whole of ∂U, and we conclude that the boundary integral term above drops out. So now we have

$$\begin{aligned}
\dot{J} &= -\int_U \delta(\psi)\; \dot{\psi}\; \nabla \cdot \left(\phi \; \frac{\nabla\psi}{|\nabla\psi|}\right) \; .dV \\
&= -\int_U \delta(\psi)\; \dot{\psi}\; \nabla\phi \; \cdot \frac{\nabla\psi}{|\nabla\psi|} \; .dV - \int_U \delta(\psi)\; \dot{\psi}\; \phi\; \nabla \cdot \left(\frac{\nabla\psi}{|\nabla\psi|}\right) \; .dV \\
&= \int_U \delta(\psi)\; v_n \; |\nabla\psi| \; \nabla\phi \; \cdot \frac{\nabla\psi}{|\nabla\psi|} \; .dV + \int_U \delta(\psi)\; v_n \; |\nabla\psi| \; \phi\; \nabla \cdot \left(\frac{\nabla\psi}{|\nabla\psi|}\right) \; .dV
\end{aligned}$$

where the substitution from (3) was made from the second to third line. Now we compare the above integrals to the relation (5), and recalling that $\frac{\nabla\psi}{|\nabla\psi|} = \hat{\nu}$ on $\partial\Omega$, we can complete the proof of (6):

$$\frac{d}{dt}\int_{\partial\Omega} \phi \; .dS = \int_{\partial\Omega} \left(\nabla\phi \cdot \hat{\nu} + \phi \; \nabla \cdot \hat{\nu}\right) v_n \; .dS$$

Now consider the volume functional

$$J = \int_\Omega \phi \; .dV$$

Embedding the above in the hold-all domain U, we have

$$J = \int_U \left(1 - H(\psi)\right) \phi \; .dV = \int_U G\left(\phi, \psi\right) \; .dV$$

As before, since ϕ is static,

$$\dot{J} = \int_U G_{,\psi} \; \dot{\psi} \; .dV = -\int_U \delta(\psi)\; \phi \; \dot{\psi} \; .dV = \int_U \delta(\psi)\; \phi \; v_n \; |\nabla\psi| \; .dV$$

where the substitution (3) has been made. Now comparing the above to (5), we conclude

$$\frac{d}{dt}\int_\Omega \phi \; .dV = \int_{\partial\Omega} \phi \; v_n \; .dS \tag{8}$$

thus completing the proof of (7).

5.2 Finding $\dot{J}$ for non-static ϕ

Above, we assumed that the integrand ϕ was static ($\dot{\phi} = 0$). However in engineering we will rarely be blessed with a simple problem like this, and sensitivity analysis will turn up terms involving $\dot{\phi}$, which is non-calculable. When this happens, we use an "adjoint method" to cancel the terms. Such a method is difficult to demonstrate in a general sense, and so we will conduct it by example in the following section.

As for actually evolving ψ with v_n, the details are outside the scope of this paper. We simply state that it is best practice to make ψ a signed distance function, in which case $|\nabla\psi| = 1 \; \forall \; \vec{x} \in U$. Reference [7] is an excellent resource for developing algorithms to evolve ψ with v_n.

6. Electromagnetic Design Problems

6.1 Electrostatic Permittivity

It has been shown in [8] that the effective electrostatic permittivity $\bar{\epsilon}$ of a two-phase composite can be determined via

$$\bar{\epsilon} = \frac{1}{a}\int_{\Omega_1 \cup \Omega_2} \epsilon \; \nabla u \cdot \nabla u \; .dV \tag{9}$$

where a is a constant, u is the electric potential field (voltage), ϵ is the local permittivity and the boundary conditions are as shown in figure 4. The problem we consider here is to maximize the effective permittivity for given constituent materials ϵ_1, ϵ_2 and volume fraction V^*. In terms of physics, we need to solve a Poisson field, and our problem is thus formulated as:

$$\max_{\Omega} \quad J = \int\limits_{\Omega_1 \cup \Omega_2} \epsilon \ \nabla u \cdot \nabla u \ .dV \tag{10a}$$

$$\text{s.t.}$$

$$\nabla \cdot (\epsilon \ \nabla u) = 0 \quad \forall \ \vec{x} \in \Omega_1 \cup \Omega_2 \tag{10b}$$

$$u = 1 \quad \forall \ \vec{x} \in S_1 \tag{10c}$$

$$u = 0 \quad \forall \ \vec{x} \in S_2 \tag{10d}$$

$$\nabla u \cdot \hat{n} = 0 \quad \forall \ \vec{x} \in S_3 \cup S_4 \tag{10e}$$

$$\int\limits_{\Omega_2} .dV = V^* \int\limits_{\Omega_1 \cup \Omega_2} .dV \tag{10f}$$

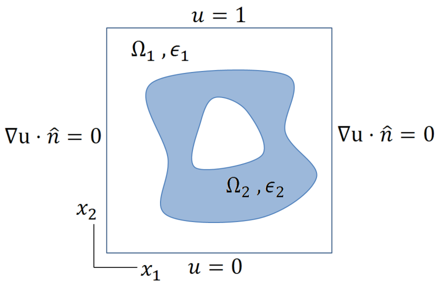


Figure 4: Determination of electrostatic permittivity

We embed J into the level set framework as

$$J = \int\limits_U H(\psi) \ \epsilon_1 \ \nabla u \cdot \nabla u \ .dV + \int\limits_U (1 - H(\psi)) \ \epsilon_2 \ \nabla u \cdot \nabla u \ .dV$$

$$\rightarrow \dot{J} = \int\limits_U \delta(\psi) \ \dot{\psi} \ (\epsilon_1 - \epsilon_2) \ \nabla u \cdot \nabla u \ .dV + 2 \int\limits_{\Omega_1 \cup \Omega_2} \epsilon \ \nabla u \cdot \nabla(\dot{u}) \ .dV$$

$$= \int\limits_U \delta(\psi) \ (\epsilon_2 - \epsilon_1) \ |\nabla \psi| \ v_n \ \nabla u \cdot \nabla u \ .dV + 2 \int\limits_{\Omega_1 \cup \Omega_2} \epsilon \ \nabla u \cdot \nabla(\dot{u}) \ .dV$$

$$= \int\limits_{\partial \Omega} (\epsilon_2 - \epsilon_1) \ v_n \ \nabla u \cdot \nabla u \ .dV + 2 \int\limits_{\Omega_1 \cup \Omega_2} \epsilon \ \nabla u \cdot \nabla(\dot{u}) \ .dV$$

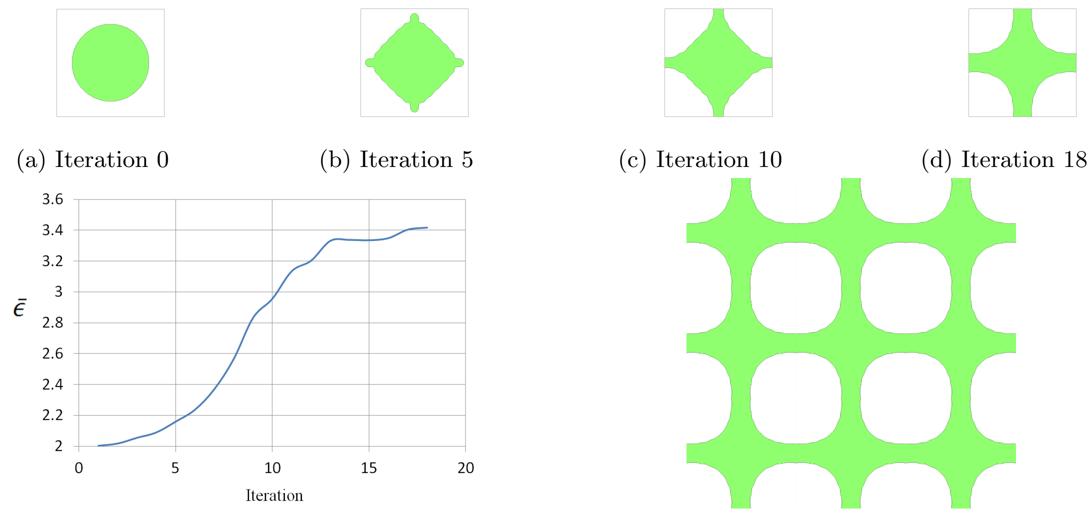
where (3) and (5) have been used in the above. We note that the second integrand in the above contains $\dot{u}$, which cannot be evaluated. Generally, the adjoint method is employed here to cancel this term (as will be done in the next example problem). However in this instance, applying the first scalar Green's theorem to the second integral,

$$\int\limits_{\Omega_1 \cup \Omega_2} \epsilon \ \nabla u \cdot \nabla(\dot{u}) \ .dV = \int\limits_{\partial(\Omega_1 \cup \Omega_2)} \epsilon \ \dot{u} \ \nabla u \cdot \hat{n} \ .dS - \int\limits_{\Omega_1 \cup \Omega_2} \dot{u} \ \nabla \cdot (\epsilon \ \nabla u) \ .dV \tag{11}$$

Clearly, the volume integral on the right side of (11) is exactly zero, as it contains the state equation. The boundary integral is also zero as $\dot{u} = 0 \ \forall \ \vec{x} \in S_1 \cup S_2$, $\nabla u \cdot \hat{n} = 0 \ \forall \ \vec{x} \in S_3 \cup S_4$ and $\partial(\Omega_1 \cup \Omega_2) = \bigcup_i S_i$. Hence, there is no adjoint system to solve for this problem, and thus,

$$\dot{J} = \int\limits_{\partial \Omega} \alpha \ v_n \ .dS \tag{12}$$

where $\alpha = (\epsilon_2 - \epsilon_1) \ \nabla u \cdot \nabla u$ and $\partial \Omega$ should be understood as the intersection between regions of ϵ_1 and ϵ_2. We can use (12) to select v_n in order to increase our objective. The constant volume constraint can



(a) Iteration 0 (b) Iteration 5 (c) Iteration 10 (d) Iteration 18

(e) Optimization Process (f) Visualized material

Figure 5: Optimization with $\epsilon_1 = 1$, $\epsilon_2 = 10$, $V^* = 0.4$

be enforced by setting $\int_{\partial\Omega} v_n \, .dS = 0$, or by incorporating a penalty term into the objective as is done in [6].

The results of such an analysis for $\epsilon_1 = 1$, $\epsilon_2 = 10$ and $V^* = 0.4$ are shown in figure 5. The cross shape obtained is very similar to that reported in [9] which used a density (BESO) method to address the same problem.

6.2 Metamaterial Permeability

An exciting new area in electromagnetics is that of metamaterials: composites with effective properties much different from those of the constituent materials. It has been shown [10] that the effective permeability $\bar{\mu}$ of a metamaterial composite can be determined via analysis of a unit cell as shown in figure 6, where it is expressed as

$$\bar{\mu} = \frac{1}{a} \cos^{-1}\left(\frac{1}{2T}(1 - R^2 + T^2)\right) \sqrt{\frac{(1+R)^2 - T^2}{(1-R)^2 - T^2}} \tag{13}$$

where

$$T = \frac{1}{b} \int_{S_2} \vec{E} \cdot \hat{y} \, .dS, \quad R = \frac{1}{b} \int_{S_1} \vec{E} \cdot \hat{y} \, .dS - 1$$

In the above, a and b are constants and $\vec{E}$ is the time-dependent electric field vector for the system as shown in figure 6. In this problem, we aim to design a composite which has an effective magnetic response ($\bar{\mu} \neq 1$) using constituent materials which do not. The constituent materials have dielectric constants ϵ_1 and ϵ_2 and permeabilities $\mu_1 = \mu_2 = 1$. The problem is formulated as

$$\max_{\Omega} \quad J = \left(\Re\{\bar{\mu}\} - 1\right)^2 \tag{14a}$$

$$\text{s.t.}$$

$$\nabla \times \mu_r^{-1}(\nabla \times \vec{E}) - k_0^2 \, \epsilon_r \, \vec{E} = 0 \qquad \forall \, \vec{x} \in \Omega_1 \cup \Omega_2 \tag{14b}$$

$$\hat{n} \times (\nabla \times \vec{E}) + jk_0 \hat{n} \times (\hat{n} \times \vec{E}) = -2jk_0 \vec{E}_i \qquad \forall \, \vec{x} \in S_1 \tag{14c}$$

$$\hat{n} \times (\nabla \times \vec{E}) + jk_0 \hat{n} \times (\hat{n} \times \vec{E}) = 0 \qquad \forall \, \vec{x} \in S_2 \tag{14d}$$

$$\hat{n} \times \vec{E} = 0 \qquad \forall \, \vec{x} \in S_3 \cup S_4 \tag{14e}$$

where $\vec{E}_i = E_0 \, \vec{y} \, e^{-jk_0 x}$ is the field incident on S_1. Now since $\bar{\mu} = \bar{\mu}(R, T)$, the sensitivity of J should be

$$\dot{J} = 2\left(\Re\{\bar{\mu}\} - 1\right) \, \Re\left\{\bar{\mu}_{,T} \, \dot{T} + \bar{\mu}_{,R} \, \dot{R}\right\}$$

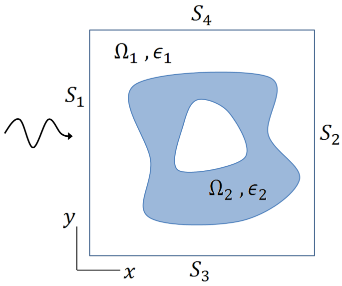


Figure 6: Determination of metamaterial permeability

First we will derive $\dot{T}$. Now

$$T = \frac{1}{b} \int\limits_{S_2} \vec{E} \cdot \hat{y} \ .dS \rightarrow \dot{T} = \frac{1}{b} \int\limits_{S_2} \dot{\vec{E}} \cdot \hat{y} \ .dS$$

We are not able to calculate $\dot{\vec{E}}$, so we apply the adjoint method by augmenting the original expression. Let

$$T_A = T - \int\limits_{\Omega_1 \cup \Omega_2} \vec{\Lambda} \cdot (\nabla \times \mu_r^{-1} (\nabla \times \vec{E}) - k_0^2 \ \epsilon_r \ \vec{E}) \ .dV$$

which is equal to the original expression as the integral term is exactly zero. Thus

$$\dot{T} = \frac{d}{dt} \left\{ \int\limits_{S_2} \frac{1}{b} \ \vec{E} \cdot \hat{y} \ .dS - \int\limits_{\Omega_1 \cup \Omega_2} \vec{\Lambda} \cdot (\nabla \times \mu_r^{-1} (\nabla \times \vec{E}) - k_0^2 \ \epsilon_r \ \vec{E}) \ .dV \right\} \tag{15}$$

Now the task is to choose the $\vec{\Lambda}$ field such that all terms involving $\dot{\vec{E}}$ will be cancelled. Applying the second vector Green's theorem to the above,

$$\begin{aligned} \dot{T} =& \frac{d}{dt} \left\{ \int\limits_{S_2} \frac{1}{b} \ \vec{E} \cdot \hat{y} \ .dS + \int\limits_{\partial(\Omega_1 \cup \Omega_2)} \mu_r^{-1} \left[\vec{\Lambda} \times (\nabla \times \vec{E}) - \vec{E} \times (\nabla \times \vec{\Lambda}) \right] \cdot \hat{n} \ .dS \right\} \\ & - \frac{d}{dt} \left\{ \int\limits_{\Omega_1 \cup \Omega_2} \vec{E} \cdot \left(\nabla \times \mu_r^{-1} (\nabla \times \vec{\Lambda}) - k_0^2 \ \epsilon_r \ \vec{\Lambda} \right) \ .dV \right\} \end{aligned}$$

We insert the boundary conditions (14c), (14d) and (14e) into the above. Then using the scalar triple product identity $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$, the above can be written as

$$\begin{aligned} \dot{T} =& - \frac{d}{dt} \left\{ \int\limits_{S_1} \mu_r^{-1} \left[(\hat{n} \times \vec{E}) \cdot \left((\nabla \times \vec{\Lambda}) + (jk_0 \ \hat{n} \times \vec{\Lambda}) \right) - 2jk_0 \ \vec{\Lambda} \cdot \vec{E}_i \right] \ .dS \right\} \\ & - \frac{d}{dt} \left\{ \int\limits_{S_2} \mu_r^{-1} \left[(\hat{n} \times \vec{E}) \cdot \left((\nabla \times \vec{\Lambda}) + (jk_0 \ \hat{n} \times \vec{\Lambda}) - \frac{\mu_r}{b} \ \hat{z} \right) \right] \ .dS \right\} \\ & + \frac{d}{dt} \left\{ \int\limits_{S_3 \cup S_4} \mu_r^{-1} \ (\nabla \times \vec{E}) \cdot (\hat{n} \times \vec{\Lambda}) \ .dS \right\} - \frac{d}{dt} \left\{ \int\limits_{\Omega_1 \cup \Omega_2} \vec{E} \cdot \left(\nabla \times \mu_r^{-1} (\nabla \times \vec{\Lambda}) - k_0^2 \ \epsilon_r \ \vec{\Lambda} \right) \ .dV \right\} \end{aligned}$$

At this point we recall that μ_r is constant everywhere and $\vec{E}_i$ does not change with Ω. Now looking at the above equation, in order to cancel all the terms which would involve $\dot{\vec{E}}$, we can choose

$$\nabla \times \mu_r^{-1}(\nabla \times \vec{\Lambda}) - k_0^2\ \epsilon_r\ \vec{\Lambda} = 0 \qquad \forall\ \vec{x} \in \Omega_1 \cup \Omega_2 \tag{16a}$$

$$\hat{n} \times (\nabla \times \vec{\Lambda}) + jk_0\ \hat{n} \times (\hat{n} \times \vec{\Lambda}) = 0 \qquad \forall\ \vec{x} \in S_1 \tag{16b}$$

$$\hat{n} \times (\nabla \times \vec{\Lambda}) + jk_0\ \hat{n} \times (\hat{n} \times \vec{\Lambda}) = -\frac{\mu_r}{b}\ \hat{y} \qquad \forall\ \vec{x} \in S_2 \tag{16c}$$

$$\hat{n} \times \vec{\Lambda} = 0 \qquad \forall\ \vec{x} \in S_3 \cup S_4 \tag{16d}$$

The above equation set is called the "adjoint system". If $\vec{\Lambda}$ is chosen to satisfy (16), then we can write (now treating $\vec{E}$ as constant)

$$\dot{T} = \frac{d}{dt}\left\{ \int_{\Omega_1 \cup \Omega_2} k_0^2\ \epsilon_r\ \vec{\Lambda} \cdot \vec{E}\ .dV \right\} = \frac{d}{dt}\left\{ \int_{\Omega_1} k_0^2\ \epsilon_1\ \vec{\Lambda} \cdot \vec{E}\ .dV + \int_{\Omega_2} k_0^2\ \epsilon_2\ \vec{\Lambda} \cdot \vec{E}\ .dV \right\}$$

Now embedding the level set field into ϵ_r similarly to the last example, and recalling (3) and (5), we get

$$\dot{T} = \int_{\partial\Omega} v_n\ k_0^2\ (\epsilon_2 - \epsilon_1)\ \vec{\Lambda} \cdot \vec{E}\ .dS \tag{17}$$

The calculation of $\dot{R}$ is very similar to the above and will not be repeated here. The form of $\dot{J}$ then becomes

$$\dot{J} = 2\,(\Re\{\bar{\mu}\} - 1)\ \Re\left\{ \int_{\partial\Omega} v_n\ k_0^2\ (\epsilon_2 - \epsilon_1)\ (\bar{\mu}_{,T}\ \vec{\Lambda} \cdot \vec{E} + \bar{\mu}_{,R}\ \vec{\Gamma} \cdot \vec{E}).dS \right\}$$

where Γ is the adjoint field associated with solving for $\dot{R}$. Since $v_n \in \Re$, we can write the above as

$$\dot{J} = \int_{\partial\Omega} \alpha\ v_n\ .dS \tag{18}$$

where $\alpha = 2\,(\Re\{\bar{\mu}\} - 1)\ \Re\left\{k_0^2\ (\epsilon_2 - \epsilon_1)\ (\bar{\mu}_{,T}\ \vec{\Lambda} \cdot \vec{E} + \bar{\mu}_{,R}\ \vec{\Gamma} \cdot \vec{E})\right\}$. We can then use (18) to select a desirable v_n.

The results of such an analysis for $\epsilon_1 = 1$, $\epsilon_2 = 100$, and input frequency 0.3 THz is shown in figure 7. The cross section shape obtained is very similar to that reported as optimum in [11], which addressed the same problem. The frequency response of this composite is the classic resonant shape, and thus the composite clearly has a magnetic response at the desired frequency.

7. Conclusion

This paper has described the salient features of the level set method in detail: how it is used to represent a topology and how it can be evolved using information from a shape gradient sensitivity analysis. Instruction was given on a method to calculate shape gradients for a general case engineering problem.

The described method was applied to two electromagnetic design problems which had been addressed previously in the literature, in an aim to demonstrate the generality of the method.

The primary advantage that the level set method has over its density-based counterparts is that it is a pure two-phase method. Any problem where "grey" regions are non-manufacturable can directly benefit from the level set method. In addition, many problems, especially in electromagnetics, greatly depend on edge conditions (such as interfaces between conductors and insulators), and the two-phase level set method offers a clear advantage in this respect also.

In light of the above, it is considered that the level set method will continue to find applications in the future, for example in the areas of waveguide, antennae, metamaterial and solar panel design.

8. References

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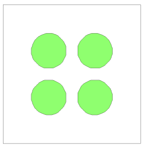
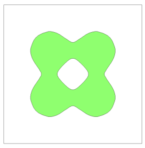
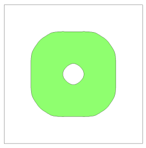


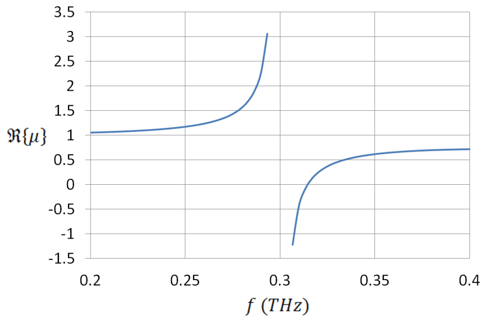
(a) Iteration 0



(b) Iteration 5



(c) Iteration 15



(d) Frequency response

Figure 7: Initial 2×2 configuration

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