10th World Congress on Structural and Multidisciplinary Optimization
May 19 -24, 2013, Orlando, Florida, USA

Extension of Michell's theory to exact stress-based multi-load truss optimization

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1. Abstract
Michell's century old truss optimization theory is being extended – the first time in the history – to *exact* optimal *elastic* stress-constrained, multi-load truss design in a research project, the first part of which is reviewed briefly in this conference paper. A complete description of the results will be presented in a two-part journal paper.
Based on extensions of the Prager-Rozvany layout theory, optimal plastic design of single and multi-load trusses is briefly reexamined, with a view to exploring its connections to optimal elastic truss design. In doing so, superposition principles are revisited, extensions to prestressed elastic design discussed, and examples shown, in which the optimal plastic and optimal elastic designs are either the same or different. It is shown that certain approaches to multi-load optimization lead to significant errors. Finally, some fundamental properties of optimal multi-load truss topologies are shown and remaining topics of the full length paper listed.
2. Keywords: topology optimization, Michell-trusses, stress constraints, optimality criteria, plastic design, elastic design, superposition principles

3. Introduction
The historic implications of the present study can be explained as follows. Single-load truss topology optimization was first introduced over a century ago by Michell (1904) [1], but nobody has extended Michell's theory to *exact* optimal *elastic* stress-based optimal design of multiply loaded trusses ever since. The aim of this study is to fill this significant gap in our knowledge.
All past attempts in the literature (e.g. by Hemp, Prager and the first author) used 'plastic design' for multi-load truss optimization, which has a somewhat limited applicability. It is based on a constraint on the ultimate load (collapse load) of structures having a rigid-perfectly plastic or elastic-perfectly plastic material. It requires (by the lower bound theorem of plasticity, see Drucker, Greenberg and Prager 1951) [2]) fulfillment of the equilibrium conditions only.
On the other hand, in 'elastic design' we must also satisfy elastic compatibility. For a single load condition, optimal plastic and optimal elastic truss designs yield the same solution, due to statical determinacy of the optimal topology (e. g. Sved 1954 [3]). This is not so in general for trusses with multiple load conditions. Although exact elastic multi-load truss optimization for displacement constraints has been discussed in several early papers by the first author (starting with Rozvany 1992 [4]), this investigation has not been extended to stress constraints until the present study.

4. An outline of topics discussed in this paper
Optimality criteria for single-load trusses are briefly explained on the basis of the optimal layout theory (Prager and Rozvany 1977 [5]) in Section 5, where their extension to plastically designed multi-load trusses is also summarized. It is explained in Section 6, that optimal plastic design of trusses with multiple load conditions can be greatly facilitated by using superposition principles. In the same section, detailed examples are given, in which the optimal truss topology is either the same or different in optimal plastic and optimal elastic design. Some erroneous approaches to multi-load optimization are looked at in Section 7. Based on Sections 5 and 6 above, some fundamental properties of exact multi-load trusses are outlined in Section 8. It is shown in Section 9 that even some statically indeterminate optimal *plastic* designs can be utilized in optimal elastic design, if we apply a suitable prestress in some of the members.
We wish to stress that in this paper we are discussing *exact* analytical truss optimization, which can provide the most reliable benchmarks for numerical topology optimization.
Some additional research results in the full length paper (Rozvany, Sokol and Pomezanski 2013) [6], *for which there was not enough space in this brief conference paper, are summarized in a NOTE at the end of this text.*

5. Optimal plastic truss design via optimal layout theory (Prager and Rozvany 1977 [5], Rozvany 1976 [7])
The lower bound theorem (see Section 3) was used by Prager and Shield (1967) [8] for the optimal plastic design of structures with a single load condition. It was extended to topology optimization by Prager and Rozvany (1977)

[5], but applied already much earlier to flexural structures (beams, grillages frames, plates and shells) by the first author, for a review see the author's first book (Rozvany 1976) [7].

The difference between the Prager-Shield theory of optimal plastic design and the optimal layout theory is that the latter also gives optimality conditions for 'vanishing' members (of zero cross-sectional area). In other words, optimal layout theory starts off with a 'ground structure' of all potential members, and selects the optimal ones (of non-zero cross-sectional area) out of these.

Using either one of the above theories for a single load condition, it is necessary to find

(i) a statically admissible 'real' stress field for the given external loads (satisfying equilibrium and static support conditions),

(ii) a fictitious kinematically admissible 'adjoint' strain field (satisfying compatibility and kinematic support conditions), such that

(iii) the above fields fulfill certain 'static-kinematic' optimality criteria.

These criteria state that the adjoint strains must equal the subgradient of the specific cost function for the given stress values. The 'specific cost' is the cost, weight or volume of the structure per unit length, area or volume, the specific cost function is the functional relation between the stresses or stress resultants and the specific cost. The subgradient of a function is the usual gradient in smooth regions, but at discontinuities of the gradient it is any convex combination of the adjacent gradient values.

Returning to classical Michell structures with equal permissible stress in tension and compression, we have the specific cost function (see Fig. 1a)

$$A = k|F|, \ k = 1/\sigma_p \tag{1}$$

where A is the cross-sectional area of a truss member, F is the member force, k is a constant, and σ_p is the permissible stress in both tension and compression.

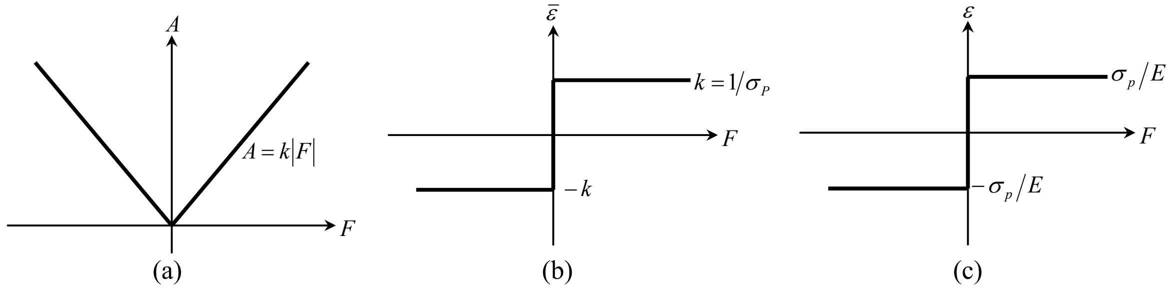


Fig. 1: (a) Specific cost function, (b) optimality criteria for Michell trusses with equal permissible stresses in tension and compression, (c) the relation between member forces and elastic ('real') strains.

The adjoint strains are given by the subgradient of the specific cost function in (1),

$$\bar{\varepsilon} = k \,\mathrm{sgn}\, F \quad (\text{for } F \neq 0), \quad |\bar{\varepsilon}| \leq k \quad (\text{for } F = 0), \tag{2}$$

where $\bar{\varepsilon}$ is the adjoint strain (see Fig. 1b). Note that, for a zero member force, the adjoint strain is subject only to an inequality. For a comparison, the 'real' or elastic strains are shown in Fig. 1c. It will be seen that this is a 'self-adjoint' problem, because the adjoint strains are linearly proportional to the elastic strains.

For the above truss problem with a single load conditions the optimality conditions arising from the layout theory reduce to those of Michell (1904 [1].

Considering now several alternate load conditions (j = 1, 2, …, m) equilibrating the internal forces F_j $(j = 1, 2, \ldots m)$, the specific cost is subject to the inequalities

$$A \geq k|F_j|, (j = 1, 2, \ldots m), \tag{3}$$

yielding the optimality condition

$$\bar{\varepsilon}_j = \lambda_j k \,\mathrm{sgn}\, F_j \quad (\text{for } F_j \neq 0), \quad |\bar{\varepsilon}_j| \leq \lambda_j k \quad (\text{for } F_j = 0), \tag{4}$$

where $\bar{\varepsilon}_j$ are adjoint strains, and λ_j the Lagrange multiplier for the j-th load condition, with

$$\lambda_j \geq 0, \ \sum_{j=1}^{m} \lambda_j = 1, \ \lambda_j > 0 \ \text{ only if } \ k|F_j| = A \ . \tag{5}$$

The multiplier λ_j may vary with the spatial coordinates and directions.

Proof of the optimality conditions (4) and (5) will be given in the full paper (Rozvany, Sokol and Pomezanski 2013) [6] . These conditions also follow from more general relations in the author's books (Rozvany 1976, [7] p. 89, or Rozvany 1989, [9] p. 47).

Considering now trusses with ***two*** load conditions, if a member of non-zero cross-sectional area is fully stressed under one of the loads only, then we have by (4) and (5)

$$(\text{for } k|F_l| = A,\ k|F_2| < A,\ F_1 \neq 0)\ \bar{\varepsilon}_1 = k \text{ sgn } F_1,\ \bar{\varepsilon}_2 = 0,$$
$$(\text{for } k|F_2| = A,\ k|F_1| < A,\ F_2 \neq 0)\ \bar{\varepsilon}_2 = k \text{ sgn } F_2,\ \bar{\varepsilon}_1 = 0 \quad . \tag{6}$$

If a member is fully stressed under both loads, then (5) implies

$$(\text{for } k|F_1| = k|F_2| = A,\ F_1 \neq 0,\ F_2 \neq 0)\ \bar{\varepsilon}_1 = \lambda k \,\text{sgn} F_1,\ \bar{\varepsilon}_2 = (1-\lambda)\, k \,\text{sgn} F_2 \tag{7}$$

Finally, if a potential member has zero force in it under both loading conditions (and therefore takes on a zero cross-sectional area), then the optimality condition becomes

$$(\text{for } F_1 = F_2 = A = 0)\ |\bar{\varepsilon}_1| + |\bar{\varepsilon}_2| \leq k\,. \tag{8}$$

6. Review of superposition principles for optimal plastic design considering multiple load conditions

As mentioned, optimal plastic design for multiple load conditions on the basis of (4) to (8) is rather difficult, but it can be facilitated greatly by using superposition principles.

Such principles for two load conditions have been proposed by Nagtegaal and Prager (1973), Spillers and Lev (1971) and Hemp (1973), and for any number of load conditions by Rozvany and Hill (1978). However, the latter is subject to certain restrictions. Here we illustrate the superposition principles with the two-load case, but use the more general expressions by Rozvany and Hill (1978).

Let the two load conditions on a truss consist of

$$\mathbf{P}_1 = (\mathbf{P}_{1,1}, \mathbf{P}_{1,2}, \ldots\ \mathbf{P}_{1,n})\ ,\quad \mathbf{P}_2 = (\mathbf{P}_{2,1}, \mathbf{P}_{2,2}, \ldots\ \mathbf{P}_{2,n})\ , \tag{9}$$

where $\mathbf{P}_{j,i}$ $(j = 1,2,\ \ i = 1, \ldots, n,)$ are point loads (vectors) for the two load conditions $\mathbf{P}_j = 1,2$ .We define the 'component loads' as

$$\mathbf{P}_1^* = (\mathbf{P}_1 + \mathbf{P}_2)/\sqrt{2},\qquad \mathbf{P}_2^* = (\mathbf{P}_1 - \mathbf{P}_2)/\sqrt{2}\quad . \tag{10}$$

Then the optimal plastic design for the original alternate loads can be obtained by determining the optimal designs for the component loads $\mathbf{P}_1^*$, $\mathbf{P}_2^*$ separately, and then superimposing those two optimal designs. This means adding the cross-sectional areas A_1^*, A_2^* for the two component loads with the following scaling

$$A = (A_1^* + A_2^*)/\sqrt{2} \tag{11}$$

In (11) either of A_1^*, A_2^*, or both may take on zero value.

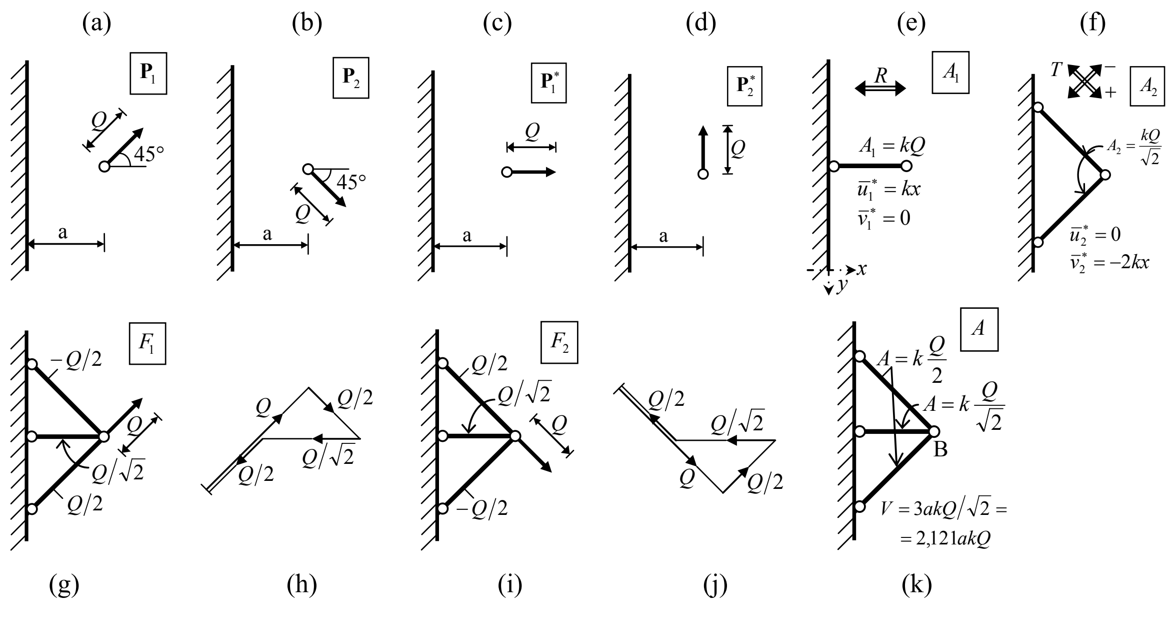


Fig. 2. Example with different optimal plastic design and optimal elastic design.

6.1. Example in which the optimal plastic design is different from the optimal elastic design

The above procedure will be illustrated with a simple example, in which each of the two load conditions consists of a single point load (Figs. 2a and b). These are at $\pm$ 45 degrees to the horizontal, and we have a vertical line support. The corresponding component loads given by (10) are shown in Figs. 2c and d. For these two component loads, the well known optimal topologies (e. g. Rozvany and Gollub 1990) [14] are shown in Figs. 2e and f. These figures also show the adjoint displacements in x and y directions $\bar{u}_j^*, \bar{v}_j^*$ $(j = 1,2)$ for the component loads $\mathbf{P}_1^*, \mathbf{P}_2^*$. These give (see Fig 2e) for the first component load the principal strain of $\bar{\varepsilon}_I = k$ in the horizontal direction and zero principal strain $\bar{\varepsilon}_{II} = 0$ in the vertical direction. The adjoint displacements for the second component load (see Fig. 2f) give the principal strains $\bar{\varepsilon}_I = -\bar{\varepsilon}_{II} = k$ at $\pm$45 degrees to the horizontal (for the derivation of these, see again Rozvany and Gollub (1990) or Rozvany, Bendsoe and Kirsch (1995), p.48, Eqs. (11)). Fig. 2e and f also show the types of optimal regions (R and T, for an explanation see the above papers), and the cross-sectional areas of the optimal members for the component loads, which are to be superimposed to get the final solution for the two alternate loads.

The superimposed final topology for the original problem is shown, together with the first loading condition and the corresponding internal forces in Fig. 2g . The vector diagram for these forces is indicated in Fig. 2h. The internal forces and vector diagram for the second load condition are given in Figs. 2i and j. It can be seen that for both load conditions all members are fully stressed. The final cross-sectional areas based on (11) and the optimum volume for this *optimal plastic design* is shown in Fig. 2k.

It can be seen easily that the solution in Fig. 2k is not feasible in optimal elastic design. This is because for either loading case in Figs. 2g and i the two sloping members would give a zero horizontal displacement at the member intersection (Point B in Fig. 2k), but the horizontal bar would be subject to a horizontal displacement of $a\sigma_p / E$ at point B. This means that elastic compatibility is violated, and therefore *optimal plastic design and optimal elastic design are different for this problem*.

6.2 Proof that the optimality conditions in (4) to (8) are fulfilled by the above optimal plastic design

Although the superposition principles presented in Section 6 guarantee an optimal plastic design, we also check the optimality of the above example by using the original optimality criteria in (4) to (8).

This will be done in two parts. First we consider adjoint strains along members of nonzero cross-sectional area, and then along vanishing members (with zero cross-sectional area).

The adjoint strain fields $\bar{\varepsilon}_1, \bar{\varepsilon}_2$ for the original alternate loads are given by the adjoint strain fields $\bar{\varepsilon}_1^*, \bar{\varepsilon}_2^*$ for the component loads $\mathbf{P}_1^*, \mathbf{P}_2^*$ in Fig. 2c and d, using the relations

$$\bar{\varepsilon}_1 = \frac{\bar{\varepsilon}_1^* + \bar{\varepsilon}_2^*}{2}, \quad \bar{\varepsilon}_2 = \frac{\bar{\varepsilon}_1^* - \bar{\varepsilon}_2^*}{2} \ , \tag{12}$$

in which $\bar{\varepsilon}_1^*, \bar{\varepsilon}_2^*$ can be calculated from the adjoint displacement fields $\bar{u}_j^*, \bar{v}_j^*$ $(j = 1,2)$ in Figs. 2e and f. The adjoint strains along members of non-zero cross sectional area ($\bar{\varepsilon}_1^*, \bar{\varepsilon}_2^*$) for the component loads are shown in Figs 3a and b, and the adjoint strains ($\bar{\varepsilon}_1, \bar{\varepsilon}_2$) for the original alternate loads (given by (12)) can be seen in Figs. 3c and d. Since in this problem all members are fully stressed for both loading conditions, we have to use the optimality condition (7), which is completely satisfied with the multiplier values λ_1, λ_2 in Figs. 3e and f.

It is more difficult to check on the optimality condition (8) for vanishing members (of zero cross sectional area), i. e. for any point and in any direction in the design domain for which there are no optimal members of non-zero cross-sectional area. The optimal principal strains for $\bar{\varepsilon}_1^*, \bar{\varepsilon}_2^*$ are shown in Figs. 3g and i, and the corresponding Mohr circles in Figs. 3h and j. It can be seen from the latter that the adjoint component strains as functions of the angle α are

$$\bar{\varepsilon}_1^*(\alpha) = \ k(1+\cos 2\alpha)/2, \ \ \bar{\varepsilon}_2^*(\alpha) = k\cos(90° - 2\alpha) = k\sin 2\alpha \ . \tag{13}$$

Then by (12) we have

$$\begin{aligned} \bar{\varepsilon}_1(\alpha) &= k[(1+\cos 2\alpha)/2 + \sin 2\alpha]/2, \\ \bar{\varepsilon}_2(\alpha) &= k[(1+\cos 2\alpha)/2 - \sin 2\alpha]/2 \ . \end{aligned} \tag{14}$$

The curves for $|\bar{\varepsilon}_1| + |\bar{\varepsilon}_2|$ as a function of α can be seen in Fig. 4, which shows that the considered solution clearly satisfies the optimality condition (8). The cusps in Fig. 4 in the $|\bar{\varepsilon}_1| + |\bar{\varepsilon}_2|$ curves occur when $\bar{\varepsilon}_1$ or $\bar{\varepsilon}_2$ takes on a zero value. It can be shown easily that this slope discontinuity is at $\alpha = \pm\arctan(1/2) = \pm 26.565051...$

6.3. Calculation of the truss volume from dual formulation

The optimal volume for trusses with multiple load conditions is given by

$$V = \sum_i \ \sum_j \mathbf{P}_{ij} \overline{\mathbf{\Delta}}_{ij} \ , \tag{15}$$

where i is the number identifying a particular point load $\mathbf{P}_{ij}$ within a load condition j, and $\overline{\mathbf{\Delta}}_{ij}$ is the adjoint displacement at the point of application of the point load $\mathbf{P}_{ij}$. Within the summation we have scalar products of the point loads and the corresponding adjoint displacements.

In the example in Sections 6.1 and 6.2, we have only one point load for both loading condition. For the adjoint displacements $\overline{\mathbf{\Delta}}_1, \overline{\mathbf{\Delta}}_2$ derived from Figs. 3k and ℓ, and the corresponding external loads (shown in Figs. 2a and b), the dual formula(15) implies the optimum volume

$$V = 3akQ/\sqrt{2}\,, \tag{16}$$

which agrees with the primal result in in Fig. 2k.

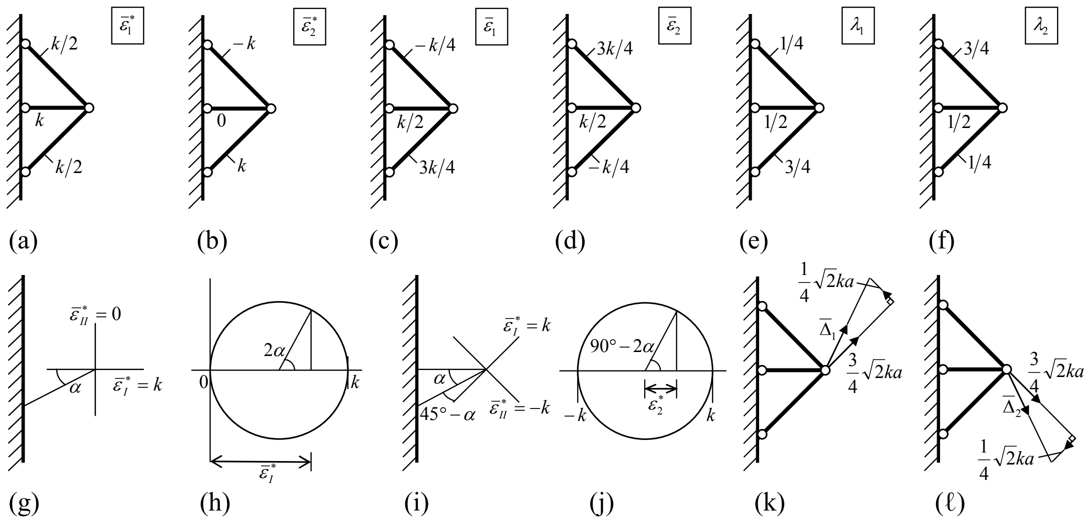


Fig. 3. (a to j) Check on the optimality of the solution in Fig. 2k, using the optimality criteria in (4) to (8), (k,ℓ) check on the optimal volume by dual formulation

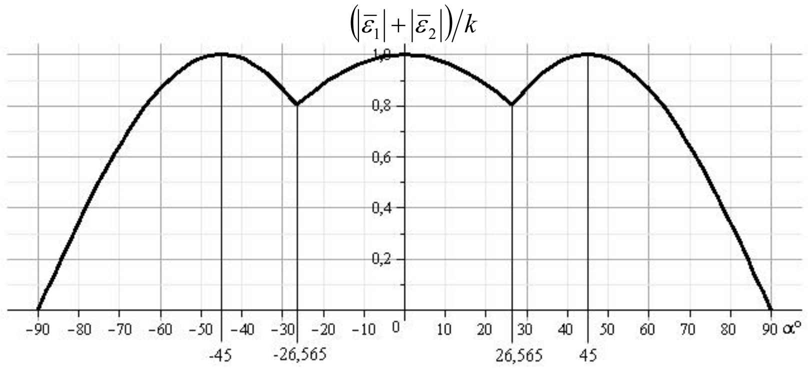


Fig. 4. Check on the optimal topology in Fig. 2k at points without members

6.4 The optimal two-bar solution for the above problem (possible elastic optimal design, but at least upper bound on it)

In this section, we determine the optimal plastic design within a two-bar topology. Since two-bar topologies are statically determinate, they are also valid for elastic design. The solution that follows is possibly the optimal elastic design for any topology, but at present this has not been proved. It's volume is certainly an upper bound on that of the optimal elastic design.

Considering the problem in Fig 5a, with the force diagram in Fig. 5b, the sine rule implies

$$\frac{F_2}{Q} = \frac{\sin(135° - \alpha)}{\sin(2\alpha)}\,. \tag{17}$$

Since F_2 is the bigger member force for both load conditions, the total truss volume becomes

$$V = 2kaQ\frac{\sin(135° - \alpha)}{\cos\alpha \sin(2\alpha)} = \frac{\cos\alpha + \sin\alpha}{\sqrt{2}\sin\alpha\cos^2\alpha} = \frac{\sqrt{2}}{\sin 2\alpha} + \frac{1}{\sqrt{2}\cos^2\alpha}\,. \tag{18}$$

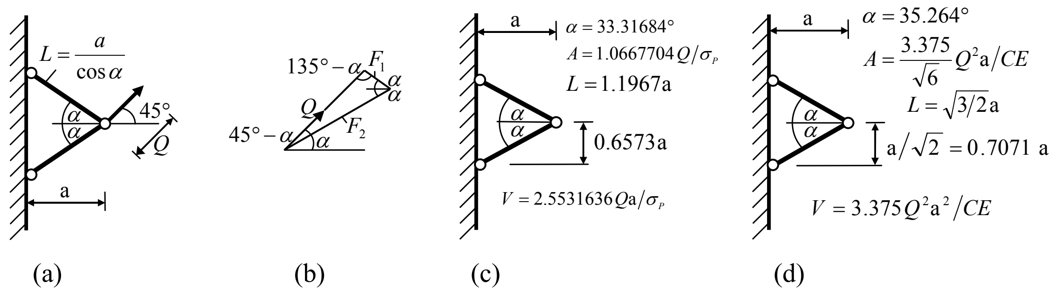


Fig. 5. Optimal two-bar solution for the problem in Sections 6.1 to 6.3

Then the stationarity condition $dV/d\alpha = 0$ leads to

$$4\cos 2\alpha \cos^4 \alpha = (\sin 2\alpha)^3 , \qquad (19)$$

giving

$$\alpha = 33.31684° . \qquad (20)$$

The corresponding truss volume is by (18)

$$V = 2.5531636\, kaQ \qquad (21)$$

The resulting optimal topology is shown in Fig. 5c. If we optimize the truss for the same two loading conditions but for a compliance constraint, then we get the optimal design in Fig. 5d (see Rozvany (1992) [4], Rozvany and Zhou 1992) [16] or Rozvany et al. (1993) [17].

6.5. Example having a statically determinate optimal plastic design, and therefore the same optimal elastic design

We shall call this example the 'conjugate' of the example in Section 6.1, meaning that the component loads of the previous example (Fig. 2c and d) are now the alternate loads (Fig. 6a and b), and the earlier alternate loads (Figs 2a and b) are now the component loads (Fig, 6c and d). However, the optimal topologies are far from being the same as in the previous example.

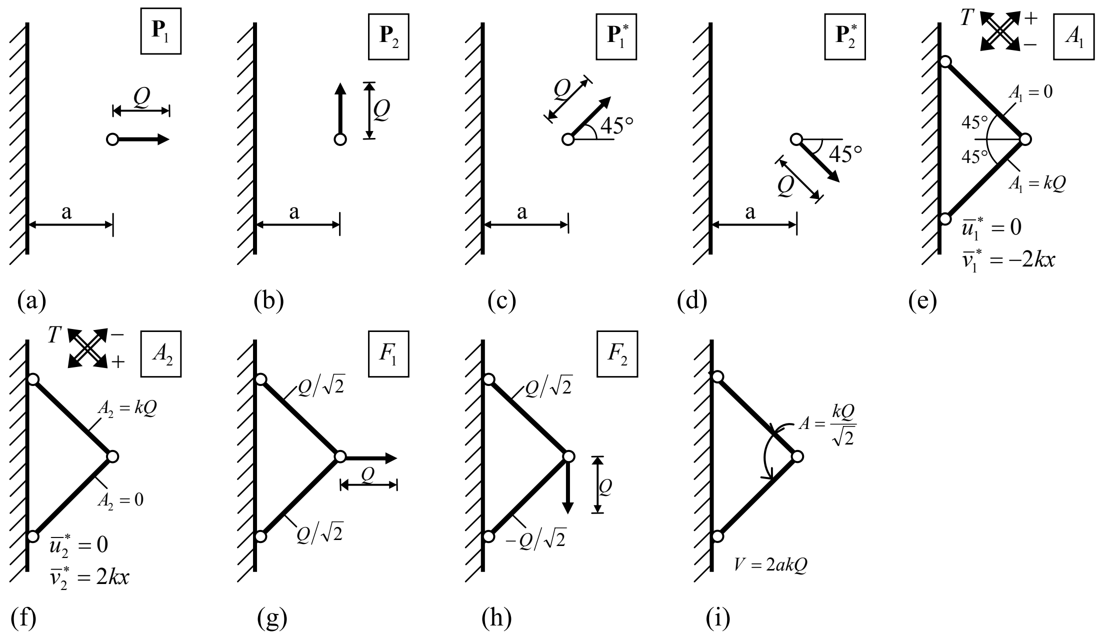


Fig. 6. Example of a statically determinate optimal topology for two alternative load conditions

The adjoint displacement fields for the two component loads are shown in Figs. 6e and f, together with the corresponding optimal truss members.. They are both T-regions. The internal forces for the alternative loads are indicated in Figs. 6g and h, and the final cross-sectional areas given by (11) and the volume in Fig. 6i.

Since the optimal plastic design is statically determinate in this example, it is also the optimal elastic design.

7. Incorrect methods for designing multi-load trusses (and other structures) by separate topology optimization for each load condition and superposition.

It has been suggested that for optimal multi-load trusses one of the following procedure could be used.

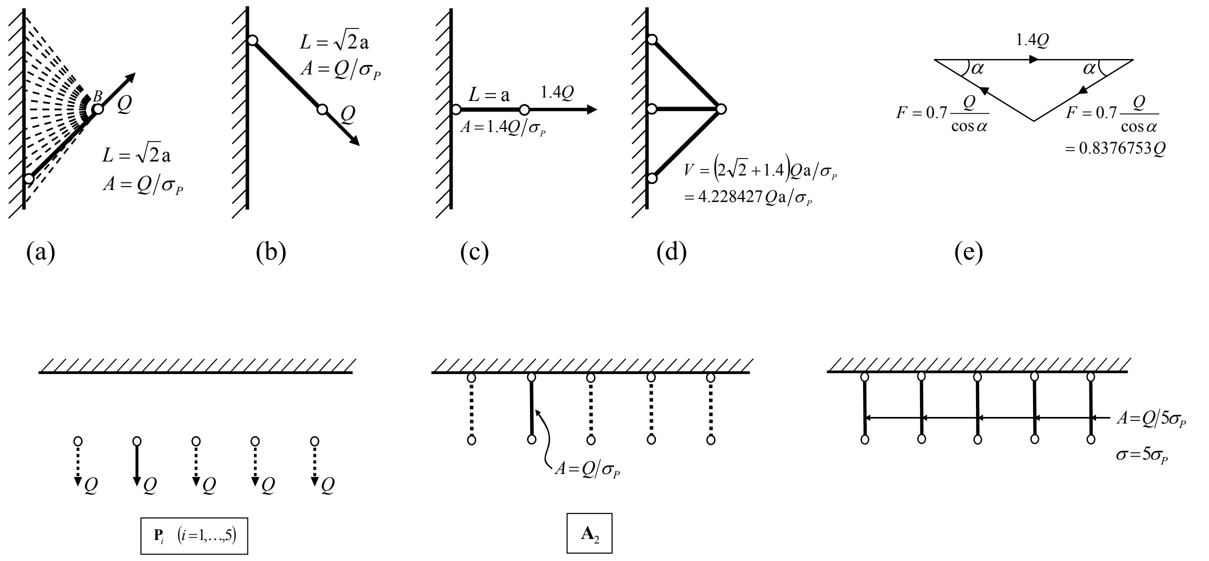


Fig. 7. Incorrect methods for multi-load topology optimization. (a)-(e) Worst case strategy, (f)-(h) mean value strategy

7.1. Worst case strategy

This method consists of two steps as follows.

(i) Considering a given ground structure, optimize the truss separately for each loading condition. For trusses with a large number of members in the ground structure, parallel computers have been recommended.

(ii) For any given truss member, adopt a cross sectional area, which equals the highest value for that member, out of all loading cases considered in step (i).

It will be demonstrated on an extended version of the example in Figs. 2a and b that the above procedure can result in highly non-optimal solutions.

In Figs. 7a to c we show three loading conditions, together with the corresponding optimal topologies considering those loads separately. The ground structure may consist of an infinite number of truss members (at all points and in all directions of the half-plane), or a finite number of members that include those three shown in Figs. 7a to c (one such ground structure is shown in broken lines in Fig.7a).

If we now superimpose the above three solutions, we get the topology in Fig 7d, which also indicates the volume of this design.

Clearly our plastic design for two loading conditions in Fig. 2k does not change if we add the alternative (horizontal) load of the magnitude 1.4 Q (shown in Fig. 7c). It can be seen from Fig. 2k that that *plastic* design can safely transmit a horizontal load of $2\left[(Q/2)(1/\sqrt{2})\right]+Q/\sqrt{2}=\sqrt{2}Q$, which is greater than 1.4Q.

The (upper bound) elastic design in Fig. 5c does not change by the additional alternative load either. The bar forces in the truss in Fig. 5c can be determined by means of the vector diagram in Fig. 7e. The stress caused by the above load in both truss members in the design in Fig. 5c is $\sigma = F/A = 0.8376753Q/(1.0667704Q/\sigma_p) = 0.785244\sigma_p$, which indicates that this load condition is not active for the optimal solution in Fig. 5c (cf. also Property 3 in Rozvany et al. 2013).

This means that *the solution in Fig. 7d, given by the above worst case strategy is 99% heavier than the correct optimal plastic design in Fig. 2k, and at least 66 % heavier than the correct optimal elastic design* (an upper bound on which is given by the solution in Fig. 5c). A much higher degree of non-optimality could be found by adding further load conditions.

It is also noted that if we used the above incorrect superposition principle for the compliance-based problem problem in Fig. 5d, then we would get again a much higher volume than that the correct optimal volume given there. We may add that the same error occurs if we use a perforated plate model of high resolution and a SIMP-like algorithm (for SIMP see Bendsoe 1989 [18], Zhou and Rozvany 1991)[19], which result in truss-like optimal topologies for low volume fractions.

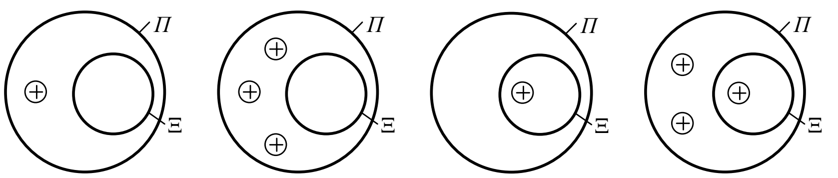
Summarizing: designs by the worst case strategy are certainly on the safe side, but they can be highly *uneconomical.*

7.2. Mean value strategy

Another erroneous suggestion is that instead of Step (ii) under Section (a) above, for any given truss member one should take the *mean value* of the cross-sectional areas in single-load optimal solutions (e. g. of those in Figs. 7a to c). For the considered problem, this would mean that we would have to multiply the areas in Figs. 7a to c by (1/3) for each member of the combined structure in Fig. 7d, with cross-sectional areas of $A = Q/3\sigma_p$, $A = Q/2\sigma_p$, $A = Q/3\sigma_p$. It can be seen easily that such a solution would be highly non-feasible (grossly violating stress constraints).
The absurdity of 'averaging' the cross-sectional areas of single load optima for multi-load optimization can be shown convincingly on a very simple example. Consider a horizontal line support, with five alternative load conditions $\mathbf{P}_1, \ldots \mathbf{P}_5$, each of which is a vertical point load of magnitude Q in a different location (see Fig. 6f, in which the second load condition is shown in continuous line, and the others in broken line). If we optimize the topology for any load condition separately, we get a vertical bar with a cross sectional area of Q/σ_p (see Fig. 7g). If we now calculate the mean value of the optimal cross-sectional areas for various load conditions, we get e. g.. for the second vertical bar the mean area of $(0 + Q/\sigma_p + 0 + 0 + 0)/5 = Q/5\sigma_p$, which value is the same for all other bars (see Fig 7h). This may appear to be an economical solution, but it is totally infeasible, because the corresponding stress in each bar for any one of the loading conditions is five times the permissible stress, $\sigma = 5\sigma_p$ (*500 per cent constraint violation*). *Summarizing: the mean value strategy may look economical, but it can be catastrophically unsafe.*

8. Some fundamental properties illustrated by the examples in Section 6

The properties listed below may be self-evident for topology optimization experts, but they may not be so obvious for the non-specialist reader. The considered class of problems is 'classical' Michell trusses, i. e. truss volume minimization for multiple loads, equal permissible stresses in tension and compression, and continuously varying, unconstrained cross-sectional areas.
PROPERTY 1. The volume of an optimal elastic design for trusses of the considered class is greater than or equal to that of the optimal plastic design for the same problem.
PROOF OUTLINE. The set Ξ of all elastic designs for the considered class of problems is a subset of the set Π of all plastic (limit) designs, because elastic designs have to satisfy all constraints for plastic design, and additional constraints representing elastic compatibility.
Then we have the following possibilities.
(a) The optimal plastic design is unique, but it is not contained in the set Ξ of elastic designs (as in the example in Section 6.1, see also Fig. 8a), or
(b) The optimal plastic design is non-unique, but none of the plastic optimal designs of equal minimum weight are contained in the set Ξ of elastic designs (Fig. 8b).
In both cases (a) and (b) all elastic designs in the set Ξ, including the optimal elastic design(s), have a greater volume than the optimal plastic designs, which by definition have a lower volume than any other (i. e. non-optimal) design in the set Π (which also contains set Ξ).



(a) (b) (c) (d)

⊕ optimal plastic design

Fig. 8. Proof of Property 1

(c)The optimal plastic design is unique, but it is contained in the set Ξ of elastic designs (as in the example in Section 6.5, see Fig. 8c), or
(d) The optimal plastic design is non-unique but some optimal plastic designs are contained in the set Ξ of elastic designs (Fig. 8d).

In both cases (c) and (d) some of the optimal plastic designs are also optimal elastic designs, and therefore both optimal plastic and optimal elastic designs have the same minimum volume.

PROPERTY 2. If an optimal plastic design is statically determinate, then it is also the optimal elastic design.

PROOF OUTLINE. In an optimal elastic design, the compatibility of the deformations must also be fulfilled. However, in a statically determinate optimal plastic design the deformations are always compatible, and therefore they are also optimal elastic designs.

9. Conversion of optimal plastic design into optimal prestressed elastic design

9.1 Trusses with a single load condition

The so-called 'reversed deformation method' (Rozvany 1964) [20], was illustrated with examples of arch-bridges, pressure vessels and beam systems (grillages), but here we discuss it in the context of trusses. We can use basically the following procedure for any linearly or nonlinearly elastic truss with a single load condition.

(1) Divide the structure into statically determinate sub-systems by applying 'cuts' at suitable locations. For trusses these cuts are usually at joints (i. e. points of member intersections).

(2) For each sub-system assign forces (termed 'connection forces') at the cuts such that they are in equilibrium (i) with similar forces for the other sub-systems at a particular cut, and (ii) also within each subsystem (this is equivalent to plastic design as described previously).

(3) Assign member sizes to each subsystem, such that under the combined effect of external loads and connection forces all members are fully stressed (reach the permissible stress) in any of the sub-systems.

(4) Calculate the relative displacements at the cuts of each statically determinate sub-system for the external load and the above described connection forces.

(5) The manufactured initial shape of each sub-system is obtained by using the negative value of the above displacements (i.e. 'reversed deformations').

At the fabrication stage, there will be a lack of fit between the sub-systems of the structure, and these are eliminated by prestressing ('pulling the cut parts together'). This will cause certain deformations in the assembled structure. However, at the given external loading, the elastic deformations and the manufactured reversed deformations cancel out, and the cuts will be subject to zero displacements.

In an extended version of this method, we can synthesize a prestressed structure for some prescribed non-zero displacements.

In ***optimal*** prestressed elastic design, in step (2) above we select the statically admissible connection forces optimally.

It is to be noted that classical Michell frames are statically determinate (or convex combinations of statically determinate solutions) for a single load conditions. However, other structures (e.g. generalized Michell structures with prescribed minimal cross-sections) are often statically indeterminate even for a single load condition. This is particularly so for non-self-adjoint problems (in which the 'real' and 'adjoint' strains are not proportional to each other). An example of such a non-self-adjoint problem with optimal plastic, optimal prestressed elastic and optimal (non-prestressed) elastic solutions is given in the full-length paper.

NOTE: *The reader is referred to the full length paper of the authors (Rozvany, Sokol and Pomezanski 2013) for (a) examples of optimizing prestressed elastic single and multi-load trusses, (b) further properties of optimal multi-load trusses, which help in deriving the least-volume solution, (c) classes of problems for which the optimal elastic and optimal plastic design is the same, and (d) highly accurate numerical checks by the second author on all analytical solutions in this paper.* For the numerical method of the second author see Sokol et al (2013) [21].

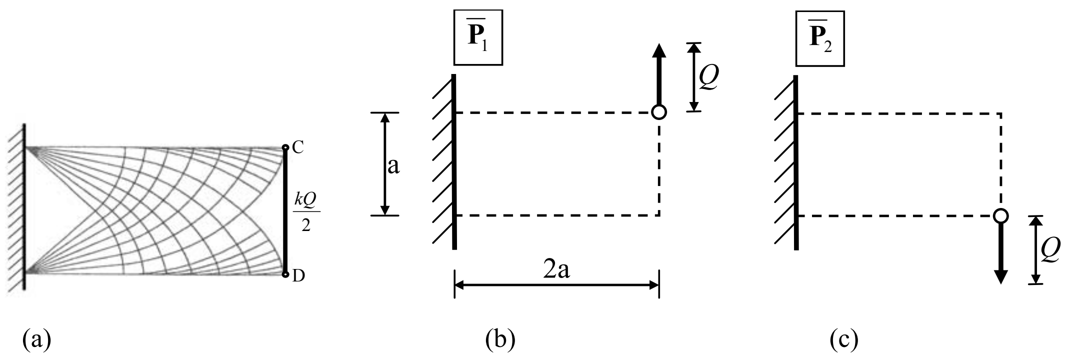


Fig. 9. Example from the full length paper. (a) Optimal topology for plastic design or prestressed elastic design for

the two alternative loads in (b) and (c)

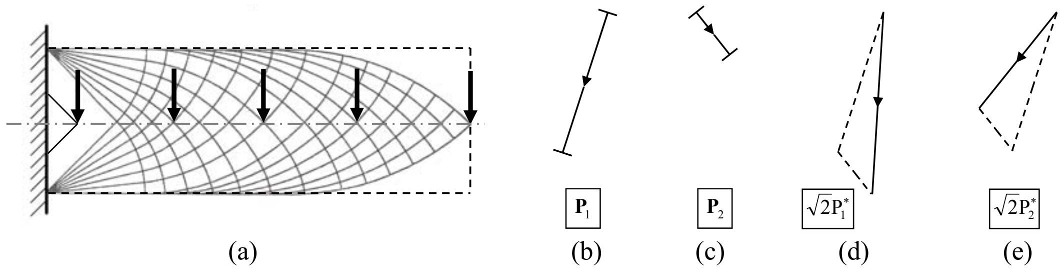


Fig. 10. Example from the full length paper. Problem class for which the optimal plastic and optimal elastic designs are the same. Consider the structure in Fig. 10a, but replace each of the vertical point loads with the two alternative loads in Figs 10b and c. The component loads (see Eq. (10)) are shown in Figs. 10d and f.

Conclusion
More than a century after the publication of Michell's classical (1904) paper, the authors are making significant inroads into exact stress-based multi-load truss optimization.

Acknowledgement The present study was supported by the Hungarian Research Foundation (OTKA, Research Grant number K81185)

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