Reliability-based Design Optimization Using A Maximum Confidence Enhancement based Sequential Sampling Approach

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1. Abstract
This paper presents a maximum confidence enhancement based sequential sampling approach for simulation-based design under uncertainty. In the proposed approach, the ordinary Kriging method is adopted to construct surrogate models for all constraints and thus Monte Carlo simulation (MCS) is able to be used to estimate reliability and its sensitivity with respect to design variables. A cumulative confidence level is defined to quantify the accuracy of reliability estimation using MCS based on the Kriging models. To improve the efficiency of proposed approach, a maximum confidence enhancement based sequential sampling scheme is developed to update the Kriging models based on the maximum improvement of the defined cumulative confidence level, in which a sample that produces the largest improvement of the cumulative confidence level is selected to update the surrogate models. Moreover, a new design sensitivity estimation approach based upon constructed Kriging models is developed to estimate the reliability sensitivity information with respect to design variables without incurring any extra function evaluations. This enables to compute smooth sensitivity values and thus greatly enhances the efficiency and robustness of the design optimization process. The case study are used to demonstrate the proposed methodology.
2. Keywords: RBDO, Kriging, Sequential sampling, Reliability analysis, Sensitivity

3. Introduction
In practical engineering design, various methods [1-5] have been developed to systematically treat uncertainties in engineering analysis and more recently to carry out Reliability-Based Design Optimization (RBDO). In RBDO, reliability [4] is defined as the probability that a system performance (e.g., fatigue, corrosion, and fracture) meets its marginal value while taking into account various uncertainties involved in system design, manufacturing and operations. By successfully solving RBDO problems, optimal system design that leads to minimum design cost while maintaining desired system reliability levels can be obtained. RBDO technique has been applied to many engineering applications to acquire optimal designs, such as structures [5-8], vehicle designs [9, 10] and manufacturing processes [11]. RBDO could be extremely expensive since it requires repeated reliability analysis during the iterative design process, thus high computational cost becomes a major challenge and bottleneck for its wide applications. In RBDO, there are two major tasks that need to be addressed: reliability analysis and design sensitivity analysis. In the literature, great efforts and improvements have been made to improve RBDO in term of accuracy and efficiency from various aspects. Some new approaches [12-14] for reliability estimation have been developed to improve numerical efficiency and stability. Different optimization strategies [15-17] including both double loop and decoupled approaches have been used to solve RBDO problems. To deal with RBDO problems with lack of data, the Bayesian approach has been investigated for the reliability modeling [18] and Bayesian Reliability-Based Design Optimization (Bayesian RBDO) methodology [19-21] has been proposed to deal with engineering design problems, which involve both aleatory and epistemic uncertainties. Also some studies [22-24] have developed dynamic reliability analysis methods and eventually resulted in optimal designs considering time-dependent uncertainties, which can minimize the life cycle cost and maintain desired dynamic reliability within life-cycle.

Employing an efficient yet accurate reliability analysis method is one of the most critical tasks in applying the RBDO technique. To improve the performance of reliability analysis, various methods , including both analytical [25] and simulation based approaches [26], have been developed. Some representatives include the most probable point (MPP) based methods [27-29], dimension reduction method (DRM)[30-32], polynomial chaos expansion (PCE) [33] and Kriging based methods [34]. MPP based reliability analysis approaches such as the first- or second-order reliability method (FORM/SORM) are to locate the MPP in the U-space and approximate reliability by calculating the reliability index which is the distance between MPP and the origin in the U-space. However, the MPP based methods may encounter convergence problems in some cases as reported in the literature and the accuracy might also be sacrificed due to the high nonlinearity of the limit states. Due to the iterative MPP search involved in each reliability analysis process, MPP based methods require sensitivity information of limit states with respect to design variables, however the accurate gradient information of the limit state functions is usually not readily available in practical design applications.

Other than analytical approaches, surrogate models such as Kriging have also been used widely in reliability analysis and design applications [35, 36]. In some recent studies, surrogate models have been integrated with the performance measure approach (PMA) in a sequential optimization and reliability analysis (SORA) framework to reduce the computational cost, where reliability is calculated by the PMA whereas limit state functions are estimated by Kriging models. Moreover, provided that the PMA is still adopted as a reliability analysis tool, a sequential sampling strategy has been proposed to build Kriging models sequentially to improve the computational efficiency, where an expected improvement criteria, similar with the efficient global optimization (EGO) technique, has been used to identify new samples for the updating of Kriging models. Another relatively more blunt way to utilize surrogate models in RBDO is to construct Kriging models for limit state functions and then estimate the reliability by directly sampling approach such as MCS based on the validated Kriging models, referred to as sampling-based RBDO. Lee et al developed a dynamic Kriging model for sampling-based RBDO, in which the order of the polynomial term in Kriging models is optimized at design iterations. Although this approach takes the advantage of direct MCS for reliability analysis, validation of the Kriging model is still a quite challenging task.

Despite accurate reliability analysis, another important task of RBDO is to compute the sensitivity information of reliability with respect to design variables. In the literature, the score function has been used to derive the stochastic sensitivity of reliability with respect to design variables without inducing any extra computational cost. However, the obtained sensitivity information using this approach is usually not smooth within the design space, as the sensitivity will become zero whenever the reliability of a particular system design approaches zero or one.

In summary, two grand challenges remain for sequential sampling-based RBDO using Kriging models: (1) effective sequential sampling scheme for updating of Kriging models, and (2) accurate estimation of smooth design sensitivity information. To address the challenges, this paper presents a maximum confidence enhancement (MCE) based sequential sampling approach for simulation-based design under uncertainty.

4. Maximum Confidence Enhancement Based Sequential Sampling

This section presents a sequential sampling approach for reliability analysis using Kriging model. Subsection 2.1 overviews reliability analysis using Kriging models. Subsection 2.2 introduces the proposed cumulative confidence level measure of reliability analysis employing a Kriging model. Subsection 2.3 details the developed maximum confidence enhancement based sequential sampling approach for sequential Kriging model development.

4.1. Kriging Surrogate Model

The title of the paper should be typed in Title Case with 12 point bold typeface. The title of the paper should be typed in Title Case with 12 point bold typeface. For time invariant reliability analysis, the limit state function $G(\mathbf{X})$ is used where the vector $\mathbf{X}$ represents random input variables with a joint probability density function $f_{\mathbf{x}}(\mathbf{X})$. The probability of failure is then defined as

$$P_f = P(G(\mathbf{X}) < 0) = \int \cdots \int_{G(\mathbf{X})<0} f_X(\mathbf{X}) d\mathbf{x} \tag{1}$$

It is extremely difficult to evaluate P_f directly using Eq. (1) as it requires numerical evaluation of multidimensional integrations over failure region. Let $RF=\{x, \mathrm{G}(x) <0\}$ denotes the failure region, thus the probability of failure can be expressed as

$$P_f = P(\mathbf{X} \in RF) = \int_{R^{nr}} I_{RF}(\mathbf{X}) f_{\mathbf{X}}(\mathbf{X}) d\mathbf{x} = E[I_{RF}(\mathbf{X})] \tag{2}$$

where P(.) represents a probability measure; E[.] denotes the expectation operator; $\mathrm{I}_{\mathrm{RF}}(\mathbf{X})$ is an indicator function and defined as

$$I_{RF}(\mathbf{X}) = \begin{cases} 1, & \mathbf{X} \in RF \\ 0, & otherwise \end{cases} \tag{3}$$

According to Eq.(2), the probability of failure at a particular design point can be calculated using statistical sampling methods such as MCS, although in general direct sampling is computationally prohibited due to a large number of function evaluations required. To alleviate this difficulty, Kriging technique can be generally used to develop a surrogate model so that the sampling based methods can be used to estimate true reliability.

Kriging is a nonparametric interpolation model in which the training sample points are involved in approximating the model parameters and forecasting the unknown response of a new point. Considering a limit state function with k random input variables, a Kriging model can be developed with n sample points denoted by $(\mathbf{x}_i, G_i)$, in which $\mathbf{x}_i =$

($x_i^1 \ldots x_i^k$) (i = 1…n) are sample inputs and G_i is the value of limit state function for given $\mathbf{x}_i$. In the kriging model, limit state function is assumed to be generated from the model:

$$G_K(\mathbf{x}) = f(\mathbf{x}) + S(\mathbf{x}) \tag{4}$$

where $f(\mathbf{x})$ is a polynomial term of $\mathbf{x}$ that interpolates the input sample points; $S(\mathbf{x})$ is a Gaussian stochastic process with zero mean and variance. $G_K(\mathbf{x})$ represents the estimation of limit state function at point $\mathbf{x}$ using Kriging model. The polynomial term $f(\mathbf{x})$ is substituted by a constant. Thus, the kriging model in Eq. (4) can be expressed as

$$G_K(\mathbf{x}) = \mu + S(\mathbf{x}) \tag{5}$$

In this kriging model, the correlation of $S(\mathbf{x})$ is given by

$$Corr[S(\mathbf{x}_i), S(\mathbf{x}_j)] = \sigma^2 R(\mathbf{x}_i, \mathbf{x}_j) \tag{6}$$

where $R(\mathbf{x}_i, \mathbf{x}_j)$ represents the (i, j) entry of an $n \times n$ symmetric correlation matrix. The $R(\mathbf{x}_i, \mathbf{x}_j)$ in the correlation matrix is defined by the distance between two sample points $\mathbf{x}_i$ and $\mathbf{x}_j$, which is expressed as

$$d(\mathbf{x}_i, \mathbf{x}_j) = \sum_{p=1}^{k} a_p \left| x_i^p - x_j^p \right|^{b_p} \tag{7}$$

where a_p and b_p are parameters of the Kriging model. In this equation, a_p is a positive weight factor related to each design variable and b_p is non-negative power factor with a value usually within the range [0, 2]. With d ($\mathbf{x}_i$, $\mathbf{x}_j$) shown in Eq. (7), $R(\mathbf{x}_i, \mathbf{x}_j)$ can be obtained accordingly as

$$R(\mathbf{x}_i, \mathbf{x}_j) = \exp\left[-d(\mathbf{x}_i, \mathbf{x}_j)\right] \tag{8}$$

With the n number of sample points ($\mathbf{x}_i$, G_i) for the kriging model, the likelihood function of the model parameters can be given as

$$Likelihood = -\frac{1}{2}[n\ln(2\pi) + n\ln\sigma^2 + \ln|R| + \frac{1}{2\sigma^2}(\mathbf{G} - \mathbf{A}\mu)^T R^{-1}(\mathbf{G} - \mathbf{A}\mu)] \tag{9}$$

where $\mathbf{G}$ is response vector ($G_1 \ldots G_n$); $\mathbf{A}$ is an $n \times 1$ unit vector. In this equation, we can solve for the values of m and s^2 by maximizing the likelihood function in closed form as:

$$\mu = \left[\mathbf{A}^T R^{-1} \mathbf{A}\right]^{-1} \mathbf{A}^T R^{-1} \mathbf{G} \tag{10}$$

$$\sigma^2 = \frac{(\mathbf{G} - \mathbf{A}\mu)^T R^{-1} (\mathbf{G} - \mathbf{A}\mu)}{n} \tag{11}$$

Substituting Eqs. (10) and (11) into Eq. (9), the likelihood function is transformed to a concentrated likelihood function which depends only upon the parameters a_p and b_p for any p within [1, k]. The parameters a_p and b_p can be obtained through maximizing the concentrated likelihood function and thereafter the correlation matrix R can be computed. With the Kriging model, the response for any given new point $\mathbf{x}$' can be estimated as

$$G_K(\mathbf{x}') = \mu + \mathbf{r}^T R^{-1} (\mathbf{G} - \mathbf{A}\mu) \tag{12}$$

where $\mathbf{r}$ is the correlation vector between $\mathbf{x}$' and the sampled points $\mathbf{x}_1 \sim \mathbf{x}_n$, of which the i^{th} element of $\mathbf{r}$ is given by $\mathbf{r}_i = Corr(S(\mathbf{x}'), S(\mathbf{x}_i))$. The mean square error $e(\mathbf{x}')$ can be computed by

$$e(\mathbf{x}') = \sigma^2 \left[1 - \mathbf{r}^T R^{-1} \mathbf{r} + \frac{(1 - \mathbf{A}^T R^{-1} \mathbf{r})^2}{\mathbf{A}^T R^{-1} \mathbf{A}}\right] \tag{13}$$

Thus, the estimation at new point $\mathbf{x}$' using Kriging model can be treated as a random variable that follows normal distributions $G(\mathbf{x}')$~Normal($G_K(\mathbf{x}')$, $e(\mathbf{x}')$).

4.2. Cumulative Confidence Level of Reliability Estimation

Assuming that n samples generated initially by Latin hypercube sampling or grid sampling methods within the design space are evaluated for a limit state function, a Kriging model can then be developed as discussed in subsection 2.1. With the developed Kriging model, MCS can be used directly for reliability analysis. Let's denote the initial data set by $\mathbf{D} = \{(\mathbf{x}_i, G_i), i=1,\ldots,n\}$, where $\mathbf{x}_i$ is the i^{th} sample and G_i is its corresponding limit state value,

and let **M** represents the constructed Kriging model using data set **D**.
To calculate reliability using MCS based on the Kriging model, N_m samples are generated according to the randomness of input variables. Let $\mathbf{X}_m$ denotes the N_m samples, in which $\mathbf{x}^m{}_i$ represents the i^{th} samples (i=1…N_m). According subsection 2.1, the estimated response of the limit state function at sample point $\mathbf{x}^m{}_i$ is a normally distributed variable given by $G(\mathbf{x}^m{}_i) \sim Normal(G_k(\mathbf{x}^m{}_i), \mathrm{e}(\mathbf{x}^m{}_i))$. Thus the sample point $\mathbf{x}^m{}_i$ can be bluntly classified as failure ("0") or safe ("1") by

$$I_{RF}(\mathbf{x}_i^m) = \begin{cases} 1, & G_k(\mathbf{x}_i^m) < 0 \\ 0, & otherwise \end{cases} \tag{14}$$

The confidence level of the classification for point $\mathbf{x}^m{}_i$ can be conveniently estimated by the probability that $G(\mathbf{x}^m{}_i)$ is larger than zero if $G_k(\mathbf{x}^m{}_i)$ is positive, or the probability that $G(\mathbf{x}^m{}_i)$ is less than zero if $G_k(\mathbf{x}^m{}_i)$ is negative, which can be expressed as

$$CL(\mathbf{x}_i^m) = \begin{cases} \Pr(G(\mathbf{x}_i^m) < 0), & G_k(\mathbf{x}_i^m) < 0 \\ \Pr(G(\mathbf{x}_i^m) > 0), & G_k(\mathbf{x}_i^m) > 0 \end{cases} = \Phi\left(\frac{|G_k(\mathbf{x}_i^m)|}{\sqrt{e(\mathbf{x}_i^m)}}\right) \tag{15}$$

where Φ is standard normal cumulative distribution function; |.| is the absolute operator. Note that $CL(.)$ is a positive value within (0.5, 1). After estimating responses for all the N_{MCS} samples in MCS, the reliability is calculated by

$$R = 1 - P_f = 1 - \frac{\sum_{i=1}^{N_{MCS}} I_{RF}(\mathbf{x}_i^m)}{N_{MCS}} \tag{16}$$

Due to the reliability prediction variability induced by Kriging models, a cumulative confidence level (CCL) of the reliability estimation in Eq. (16) can be defined as

$$CCL_{MCS}(\mathbf{M}, \mathbf{X}_m) = \frac{\sum_{i=1}^{N_{MCS}} CL(\mathbf{x}_i^m)}{N_{MCS}} \tag{17}$$

where $CL(\mathbf{x}^m{}_i)$ represents the confidence level of classification for sample $\mathbf{x}^m{}_i$. It is worth to notice that CCL_{MCS} will always be greater than 0.5 and less than 1.

4.3. Maximum Confidence Enhancement Based Sequential Sampling Scheme
As MCS is employed for reliability analysis based on a Kriging model for any particular design during the iterative RBDO process, the Kriging model used should be accurately developed to represent the system limit state function. In section 2.2, a CCL measure is introduced to qualify the performance of reliability estimation using MCS based on the Kriging model. The insight of the developed sequential sampling method is to select new samples that can maximize the enhancement of the cumulative confidence level measure for reliability estimation.
For a given input sample **x***, the evaluated limit state function value is denoted as G^*, and then the improvement of the CCL of reliability estimation can be defined as

$$I(\mathbf{x}^*) = CCL_{MCS}(\mathbf{M}^*, \mathbf{X}_m) - CCL_{MCS}(\mathbf{M}, \mathbf{X}_m) \tag{18}$$

where **M*** is the updated Kriging model that is built using updated data set **D*** by adding new sample point (**x***, G^*); **M** is the original Kriging model built upon the sample set **D**. With the defined CCL improvement, a sample point that maximizes the CCL improvement should be selected as new sample point in the updating process. However, we can hardly know the CCL improvement of **x*** in Eq. (18) if the true limit state value G^* is not available. Thus, this paper proposes a new criterion for the estimation of the CCL improvement for any given new input sample **x***, referred to as estimated improvement (EI) sampling criterion, that is defined by

$$EI(\mathbf{x}^*) = (1 - CL(\mathbf{x}^*)) \times f_{\mathbf{x}}(\mathbf{x}^*) \times \sqrt{e(\mathbf{x}^*)} \tag{19}$$

where $f_{\mathbf{x}}(\mathbf{x}^*)$ is the probability density function value at the new input sample point **x***; $CL(\mathbf{x}^*)$ denotes the confidence level of classification at **x*** using current Kriging model; $e(\mathbf{x}^*)$ is the mean square error of the estimation using current Kriging model at **x***. As shown in Eq. (19), the estimated improvement sampling criterion is composed with three parts: the first term (1-$CL(\mathbf{x}^*)$) represents the potential prediction confidence level improvement if the sample **x*** is evaluated; the second term is the probability density function value at **x*** and the multiplication of the first and second term represents how likely we can obtain the potential improvement

(1-CL(**x***)). It is also known that the accuracy of Kriging model near **x*** will be improved if current Kriging model is updated by adding and evaluating **x*** as a new input sample, thus the third term $e(\mathbf{x}^*)^{0.5}$ is added to consider this effect since the bigger is the estimation variance at **x*** the larger improvement can be projected.

Start from initial sample set **D**, an initial Kriging model **M** can be built and MCS can be employed with **M** using N_{MCS} samples $\mathbf{X}_m$, then the reliability R and cumulative confidence level of reliability estimation $CCL(\mathbf{M}, \mathbf{X}_m)$ can be calculated using Eq.(16) and (17) respectively. By checking the $CCL(\mathbf{M}, \mathbf{X}_m)$, **M** will be used for reliability estimation if the confidence level is greater than a predefined critical threshold; otherwise, the estimated improvements $EI(\mathbf{X}_m)$ are computed for all the samples $\mathbf{X}_m$, and the sample **x*** with the largest estimated CCL improvement, Max($EI(\mathbf{X}_m)$), will be selected as a new sample to update the Kriging model M. This updating process is repeated until the termination rule is satisfied. To balance the accuracy and efficiency of the MCE based sequential sampling scheme, convergence of reliability and confidence level are used to validate the constructed Kriging models. Let ΔR represent the difference between reliabilities obtained using the original Kriging model and the updated one, a Kriging model is considered to be a valid one if ΔR is less than a critical threshold value $C_{\Delta R}$ and cumulative confidence level, CCL, is greater than a confidence level target, CCL_T. Table 1 summarizes the procedure of the developed MCE-based sequential sampling scheme.

Table 1**:** Procedure of the MCE-based sequential sampling scheme

Steps	Procedure
Step 1:	Identify initial design points, and evaluate the responses; obtain date set $\mathbf{D} = [(\mathbf{x}_i, G_i), i=1\ldots n]$; Initialize the confidence target CCL_T, critical threshold value $C_{\Delta R}$; set ΔR=1;
Step 2:	Develop a Kriging model **M** using existing data set **D**; generate N_{MCS} samples $\mathbf{X}_m$ according the input randomness;
Step 3:	Compute the reliability R and confidence level $CCL(\mathbf{M},\mathbf{X}_m)$ using Kriging; compute ΔR if possible
Step 4:	Convergence check: If $\Delta R< C_{\Delta R}$ and $CCL(\mathbf{M},\mathbf{X}_m) > CCL_T$, stop; otherwise, go to Step 5;
Step 5	Compute $EI(\mathbf{X}_m)$ and search the sample **x*** in $\mathbf{X}_m$ that has Max($EI(\mathbf{X}_m)$); evaluate G(**x***) and update **D** by adding new data (**x***, G(**x***)); go to Step 2;

5. RBDO Using MCE-Based Sequential Sampling Scheme

This section presents the sampling-based RBDO using the developed MCE-based sequential sampling approach. Subsection 3.1 overviews the RBDO formulation, while subsection 3.2 presents the developed new smooth stochastic sensitivity analysis approach using Kriging. The overall procedure of using sequential sampling approach is summarized in subsection 3.3.

5.1. RBDO Formulation

Reliability based design optimization (RBDO) aims to find optimum system designs that compromise between cost and reliability by taking into account of system input uncertainties.

In general, RBDO can be formulated as:

$$\begin{aligned} &Minimize: \quad Cost(\mathbf{X},\mathbf{d}) \\ &subject\ to: \quad P_f\,(G_i(\mathbf{X},\mathbf{d})<0) \le 1-\Phi(\beta_{t_i}), \quad i=1,....,n \\ &\qquad\qquad \mathbf{d}^{\mathbf{L}} \le \mathbf{d} \le \mathbf{d}^{\mathbf{U}}, \quad \mathbf{d}\in R^{nd} \ \ and\ \ \mathbf{X}\in R^{nr} \end{aligned} \tag{20}$$

where $P_f(G_i(\mathbf{X}, \mathbf{d}) < 0)$ is the probability of failure; β_t is its prescribed reliability target; $Cost(\mathbf{X}, \mathbf{d})$ is the objective function, **d** is a vector of the mean values of design variables **X**, and n, nd, and nr are the numbers of probabilistic constraints, design variables, and random variables, respectively. The objective here is to minimize the cost $Cost$ (**X**, **d**) while maintaining the target reliability level is satisfied.

5.2. Stochastic Sensitivity Analysis Using Kriging

Design sensitivity information of reliability with respect to random design variables is essential in the iterative design process, as it not only affects the efficiency but also determines the convergence of the design process to an optimum design. In RBDO while finite different method (FDM) is adopted for design sensitivity analysis, reliability analysis needs to be performed nd times for the perturbed design points where nd is number of design variables.

For design sensitivity analysis using sampling based methods, taking the partial derivative of probability of failure with respect to the i^{th} design variable d_i yields

$$\frac{\partial P_f}{\partial d_i} = \frac{\partial}{\partial d_i} \int_{R^{nr}} I_{RF}(\mathbf{X}) f_{\mathbf{X}}(\mathbf{X}) d\mathbf{x} \tag{21}$$

According to the Leibniz's rule of differentiation, the differential and integral operators in the Eq. (21) can be interchanged and thus yields

$$\frac{\partial P_f}{\partial d_i} = \int_{R^{nr}} I_{RF}(\mathbf{X}) \frac{\partial f_{\mathbf{X}}(\mathbf{X})}{\partial d_i} d\mathbf{x} = \int_{R^{nr}} I_{RF}(\mathbf{X}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{X})}{\partial d_i} f_{\mathbf{X}}(\mathbf{X}) d\mathbf{x} = E[I_{RF}(\mathbf{X}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{X})}{\partial d_i}] \tag{22}$$

Although the analytical form for the sensitivity of reliability can be derived, it cannot be used for the cases when all the samples in MCS are identified as safe or failure. If I_{RF} equals either 0 or 1 for all the N_{MCS} samples, Eq. (22) becomes

$$\frac{\partial P_f}{\partial d_i} = \frac{\partial}{\partial d_i} \int_{R^{nr}} I_{RF}(\mathbf{X}) f_{\mathbf{X}}(\mathbf{X}) d\mathbf{x} = I_{RF}(\mathbf{X}) \times \frac{\partial}{\partial d_i} \int_{R^{nr}} f_{\mathbf{X}}(\mathbf{X}) d\mathbf{x} = 0 \tag{23}$$

Apparently from Eq. (23), the sensitivity information of reliability cannot be estimated appropriately if the reliability for a design is zero or one, thus under this circumstance the optimum design can hardly be obtained in RBDO. To alleviate such a difficulty, this paper presents a new way to calculate sensitivity of reliability without extra computational cost. Defined as the integration of probability density function of system input variables over the safe region (G($\mathbf{X}$)>0), reliability has direct proportional relationship to the mean and variance of limit state function, which can be expressed as

$$R = P(G(\mathbf{X}) > 0) = \int .. \int_{G(\mathbf{X})>0} f_{\mathbf{X}}(\mathbf{X}) d\mathbf{x} \approx \Phi(\frac{Mean_G(\mathbf{x})}{Var_G(\mathbf{x})^{0.5}}) \propto \frac{Mean_G(\mathbf{x})}{Var_G(\mathbf{x})^{0.5}} = \frac{\int_{R^{nr}} G(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}}{\sqrt{\int_{R^{nr}} (G(\mathbf{x}) - Mean_G(\mathbf{x}))^2 f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}}} \tag{24}$$

where $Mean_G(\mathbf{x})$ and $Var_G(\mathbf{x})$ are the mean and variance of limit state function $G(\mathbf{x})$. The sensitivity of reliability with respect to the i^{th} design variable d_i can then be approximated as

$$\frac{\partial R}{\partial d_i} \propto \frac{\partial}{\partial d_i} \left[\frac{Mean_G(\mathbf{x})}{Var_G(\mathbf{x})^{0.5}} \right] \tag{25}$$

Note that the right part of Eq.(25) only provides an estimated sensitivity vector that is proportional to the true design sensitivity. Thus the sensitivity information in Eq.(25) can be normalized and derived as

$$\frac{\partial R}{\partial d_i} \approx \alpha \times \frac{\partial}{\partial d_i} \left[\frac{Mean_G(\mathbf{x})}{Var_G(\mathbf{x})^{0.5}} \right] \Bigg/ \left\| \frac{\partial}{\partial d_i} \left[\frac{Mean_G(\mathbf{x})}{Var_G(\mathbf{x})^{0.5}} \right] \right\| \tag{26}$$

where a is a proportional coefficient; |•| is norm operation. To calculate the sensitivity of reliability, the derivative term in the left side of Eq. (26) can be derived as

$$\frac{\partial}{\partial d_i} \left[\frac{Mean_G(\mathbf{x})}{Var_G(\mathbf{x})^{0.5}} \right] = \frac{\partial [Mean_G(\mathbf{x})]}{\partial d_i} \times \frac{1}{Var_G(\mathbf{x})^{0.5}} - \frac{1}{2} Mean_G(\mathbf{x}) \times \frac{1}{Var_G(\mathbf{x})^{1.5}} \times \frac{\partial [Var_G(\mathbf{x})]}{\partial d_i} \tag{27}$$

As defined previously, **M** denotes the Kriging model built for limit state function $G(\mathbf{x})$ using data **D**; $\mathbf{X}_m$ represents the generated N_{MCS} samples in MCS for the reliability analysis. For the i^{th} sample $\mathbf{x}^{\boldsymbol{m}}_{\boldsymbol{i}}$ in $\mathbf{X}_m$, the response $G(\mathbf{x}^{\boldsymbol{m}}_{i})$ and the first derivative of $G(\mathbf{x}^{\boldsymbol{m}}_{i})$ with respect to design variables $g_v(i)=(g_v^1,\ldots,g_v^{nd})$ can be obtained using Kriging model where nd is the number of design variables. Thus the $Mean_G(\mathbf{x})$ and $Var_G(\mathbf{x})$ can be easily estimated by **M**, and the derivative of $Mean_G(\mathbf{x})$ and $Var_G(\mathbf{x})$ with respect to d_i can be calculated by

$$\begin{aligned} \frac{\partial [Mean_G(\mathbf{x})]}{\partial d_i} &= \frac{\partial \left[\int_{R^{nr}} G(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \right]}{\partial d_i} \approx \sum_{i=1}^{N_{MCS}} g_v(i) \\ \frac{\partial [Var_G(\mathbf{x})]}{\partial d_i} &= \frac{\partial \left[\int_{R^{nr}} (G(\mathbf{x}) - Mean_G(\mathbf{x}))^2 f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \right]}{\partial d_i} \approx \sum_{i=1}^{N_{MCS}} \left[2 \times G\left(\mathbf{x}^m{}_i\right) \times g_v(i) - 2 \times Mean_G(\mathbf{x}) \times g_v(i) \right] \end{aligned} \tag{28}$$

To calculate the sensitivity of reliability appropriately, the proportional coefficient a should be determined. In this paper, a is set to one initially at the first iteration and will be updated based on the reliabilities of current and previous designs. Let R_i and R_{i+1} represent the reliabilities for the ith and (i+1)th iterations respectively, d_i and d_{i+1} be the designs at the ith and (i+1)th iterations, and SR_i denote the sensitivity of reliability calculated using Eq.(26). Also let a_i and a_{i+1} be the proportional coefficients for ith and (i+1)th iterations. With these notations, the proportional coefficient a_{i+1} can be updated by

$$\alpha_{i+1} = \begin{cases} -\dfrac{R_{i+1} - R_i}{(d_{i+1} - d_i) \bullet (SR_i)}, & if\ |R_{i+1} - R_i| > C_a \\ a_i, & otherwise \end{cases} \tag{29}$$

where C_a is a predefined critical threshold within 10^{-1} to 10^{-4}. In this study, C_a is set to 10^{-3} for all case studies.

5.3. Numerical Procedure

For a given engineering design application, the first step of employing the developed MCE-based sequential sampling scheme with the developed sensitivity analysis method is to build an initial Kriging model for every limit state function. For a limit state G_i(**X**,**d**) (i=1,…n), Latin hypercube or grid sampling method can be used to generate N_I number of initial samples $\mathbf{x}_j$ (j=1,…N_I) in the predefined design space. To balance the accuracy and efficiency, 10*(nd) samples are suggested as initial sample size N_I. Since the same procedure of the proposed MCE-based sampling approach for each limit state will be used, G_i(**X**,**d**) is denoted in general as G(**X**,**d**) in the rest of paper. The limit state values of G(**X**,**d**) at N_I number of initial samples are evaluated and denoted by G_j (j=1,…,N_I), then the data D={(x_j,G_j), j=1,…N_I} is obtained and used to construct a Kriging model **M** as discussed in subsection 2.1.

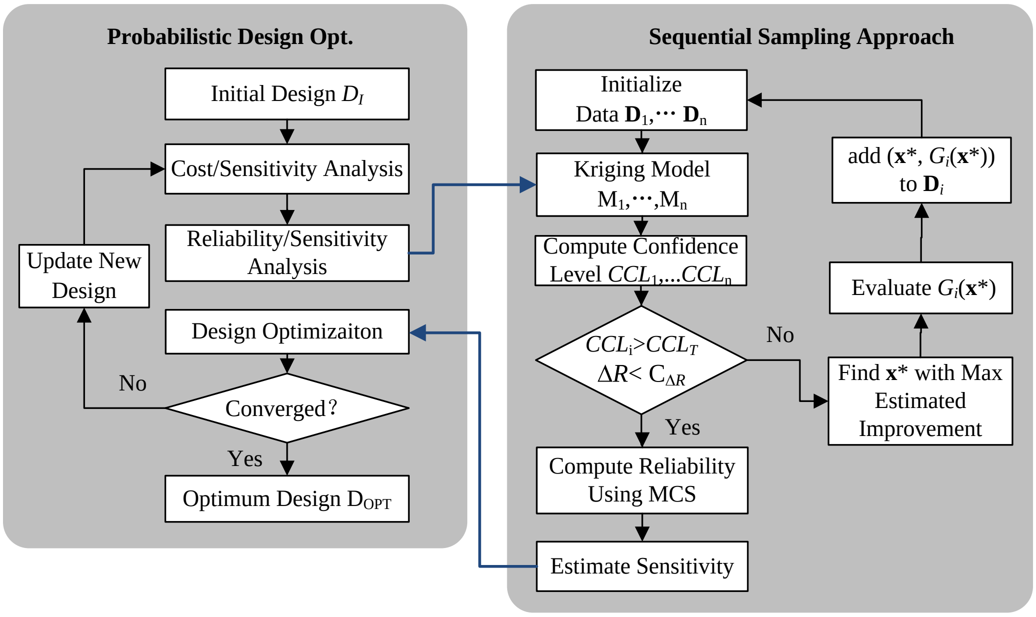

Figure 1: Flowchart of the sampling-based RBDO using Kriging

With the constructed Kriging model M, deterministic design optimization is first solved to derive a deterministic optimum design. The samples evaluated during the deterministic design optimization process are combined with initial samples together for the updating of the initial Kriging model **M**.

In the next step, RBDO will be performed using the latest Kriging model constructed in the previous step. Starting with the deterministic optimum obtained from the deterministic optimization as an initial design, the cost function and the sensitivity of cost with respect to design variables are computed first; then developed MCE-based sequential sampling approach is used to estimate the reliabilities and sensitivities for all limit states. The cost functions, reliabilities and corresponding design sensitivities will be provided to an optimizer to generate a new design point or otherwise determine the optimum if converged. The iterative design process will be repeated until an optimum design is achieved. Figure 1 shows the flowchart of sampling-based RBDO using the developed MCE-based sequential sampling scheme.

6. Case Study

A mathematical problem is used to demonstrate the RBDO with the proposed MCE approach in this section. In this example, two random design variables X_1 and X_2 are considered. Both random variables are normally distributed as: $X_1 \sim Normal\ (\mu_1, 0.6)$ and $X_2 \sim Normal\ (\mu_2, 0.6)$, where the design variable $\mathbf{d} = [d_1, d_2]^T = [\mu_1 = \mu(X_1), \mu_2 = \mu(X_2)]^T$. The RBDO problem is formulated as

$$\begin{aligned}
&\text{Minimize}: \ Cost = 10 - d_1 + d_2 \\
&s.t. \ \ P_f[G_i(\mathbf{X}, \mathbf{d}) < 0] \le 1 - R_t, \ \ i = 1 \sim 3, 0 \le d_1 \,\&\, d_2 \le 10 \\
&G_1 = \frac{1 - X_1^{\,2} X_2}{20}; G_2 = 1 - \frac{(X_1 + X_2 - 5)^2}{30} - \frac{(X_1 - X_2 - 12)^2}{120}; G_3 = 1 - \frac{80}{(X_1^{\,2} + 8X_2 - 5)}
\end{aligned} \tag{30}$$

The target reliability level is set to $R_t = 0.99865$; the proposed approach will be tested by setting a cumulative confidence level target of CCL_T=0.80 and a convergence criterion of $C_{\Delta R}$=5e-4. Figure 2 shows the contours of the limit state functions in the design space.

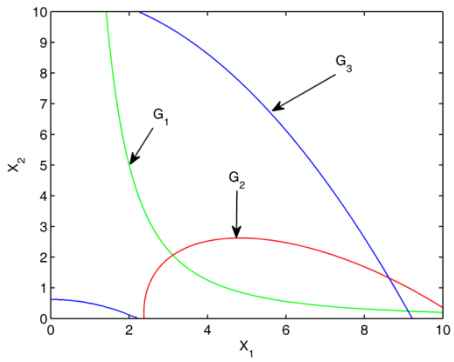

Figure 2: Limit state functions

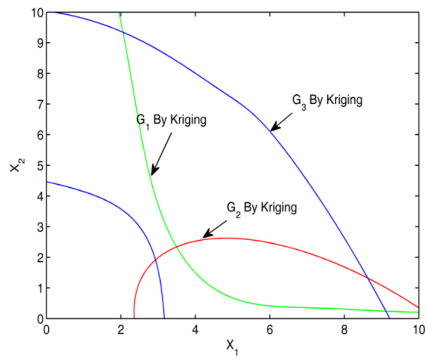

Figure 3: Estimated limit state using Kriging models

The deterministic design optimization process starts with an initial design $\mathbf{d}_i$ = [5, 5]. To construct the surrogate model for constraints, one Kriging model is built for each constraint with nine initial samples which are generated in the design space using the grid sampling technique. Following the procedure outlined in subsection 3.3, the deterministic optimum design is obtained as $\mathbf{d}_{opt}$ = [8.6296, 1.3202] and the initial sample size is enlarged to 13 samples for each constraint. Three Kriging models $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ are developed using the 13 sets of initial samples. By setting d_{opt} as the initial design for RBDO, the probabilistic optimum design is obtained after 8 iterations. During the RBDO process, 2, 6 and 11 samples are used for the updating for Kriging models $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ respectively.

During the RBDO process, the design history of reliabilities of three constraints, cost function values, iterative design points and sensitivity coefficients is detailed in Table 2. As shown in the Figure 3, the estimated limit state using updated Kriging models are also compared with true limit states as shown previously in Figure 1.

Table 2: Design history of RBDO

	Design Variables		Reliabilities			Sensitivity Coefficients			Cost
Iter.	X_1	X_2	R_1	R_2	R_3	a_1	a_2	a_3	$Cost$
0	8.627	1.321	0.999	0.505	0.497	1	1	1	2.694
1	8.232	1.766	1	0.707	0.688	1	0.408	0.380	3.534
2	7.559	2.436	1	0.918	0.916	1	0.296	0.279	4.877
3	7.261	2.687	1	0.959	0.965	1	0.149	0.166	5.425
4	7.013	2.904	1	0.981	0.984	1	0.083	0.096	5.890
5	6.836	3.063	1	0.991	0.991	1	0.051	0.047	6.227
6	6.670	3.177	1	0.995	0.995	1	0.024	0.025	6.507
7	6.518	3.289	1	0.997	0.997	1	0.015	0.018	6.771
8	6.444	3.351	1	0.998	0.998	1	0.014	0.019	6.906

The overall design optimization process including both deterministic optimization and the RBDO is shown in Figure 4, in which black cycles are initial samples used for constructing the initial Kriging models; black pentagrams represent new samples added during the deterministic design process; green stars, red squares and blue triangles are new samples employed for the updating of Kriging models $\mathbf{M}_1$ and $\mathbf{M}_2$ and $\mathbf{M}_3$ during the RBDO process; black plus denotes the iterative design during RBDO. The MCS is employed to verify the accuracy of the proposed approach at the optimum design point. As the results shown in Table 3, the proposed approach is also compared with other existing methods such as the first order reliability method (FORM) and constraint boundary sampling method (CBS). With 1000,000 samples, the MCS provides the reliabilities for the three probabilistic constraints based on the optimum designs obtained through different algorithms respectively. It is observed that the proposed MCE-based sequential sampling approach for the RBDO generates accurate optimum designs and outperforms other existing methods.

Table 3: Comparison of SSA and other approaches

	Optimum	R_1	R_2	R_3	*Cost*	*No. of Eva.*
FORM	(6.400,3.442)	1	0.9988	0.9987	7.0422	402
CBS	(5.858,3.423)	1	0.9957	1	7.5645	99
MCE	(6.444,3.351)	1	0.9975	0.9987	6.9069	48

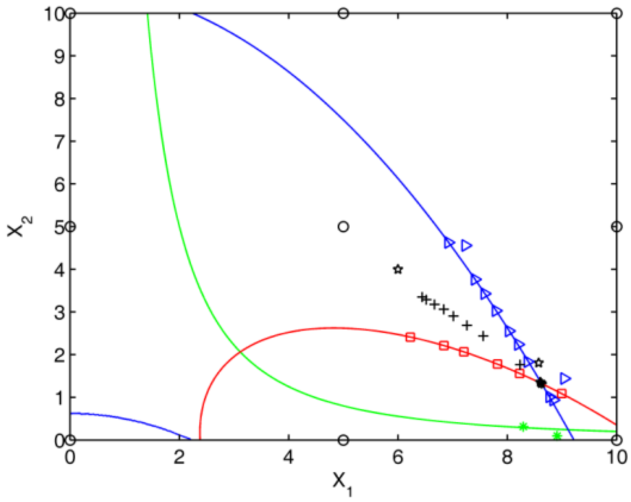

Figure 4: Overall process of design optimization

7. Conclusion

This paper presented a maximum confidence enhancement (MCE) based sequential sampling approach for simulation-based design under uncertainty. To sequentially validate and update the Kriging models constructed for each design constraint, cumulative confidence level (CCL) measure is developed to quantify the accuracy of reliability estimation when MCS is used to proximate reliability based upon Kriging models. A sequential sampling scheme is then developed based on the defined CCL measure to update the Kriging models efficiently. To employ the developed MCE-based sequential sampling approach RBDO, a new smooth design sensitivity analysis approach is developed which estimates smooth design sensitivities without inducing any extra function evaluations. Thus the efficiency and robustness of the proposed approach are greatly improved. The design case study indicated that the proposed approach outperforms other existing approaches.

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