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Interval uncertain optimization of structures using Chebyshev meta-models

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Abstract
This paper proposes a new design optimization method for structures subject to uncertainty. Interval model is used to account for uncertainties of uncertain-but-bounded parameters. It only requires the determination of lower and upper bounds of an uncertain parameter, without necessarily knowing its precise probability distribution. The interval uncertain optimization problem will be formulated as a nested double loop procedure, in which both the design variables and structural parameters are regarded as interval numbers. In practice, the nested double loop optimisation will be computationally prohibitive, so the linear optimization model is widely used. However, the linear optimization model induces large error for strong nonlinear model. To improve the accuracy and without increasing computation cost much, the interval arithmetic is applied to the inner loop to directly evaluate the bounds of interval functions, replacing the linear model. The interval arithmetic is easily subject to overestimation due to its intrinsic wrapping effect, so the Taylor inclusion function will be introduced to compress the overestimation in interval computations. Since it is hard to evaluate the high order coefficients in the Taylor inclusion function, a Chebyshev meta-model is proposed to approximate the Taylor inclusion function. A typical 25-bar space truss structure optimization problem with interval uncertainties are used to demonstrate the effectiveness of the proposed method in the uncertain design optimization of structures.
Keywords: structure optimization; interval uncertainty; Chebyshev meta-models; Taylor inclusion functions.

1. Introduction
Design optimization of structures has experienced considerable development over the past two decades with a wide range of engineering applications. However, the majority of these works are focused on the investigation of the deterministic optimization. In engineering, there are many uncertain factors inevitably related to material properties, geometry dimensions, loads and tolerance in the whole life cycle of design, manufacturing, service and aging of the structure [1], due to the inherent uncertain nature of the real-world systems. Hence, there is an increasing demand to consider the impact of uncertainties quantitatively in the optimization of structures in spite of unavoidable variability and uncertainty, to enhance structural safety and avoid failure in extreme working conditions. To incorporate uncertainties in the design optimization, the deterministic design problem should be suitably modified and enhanced.

The optimization with uncertainty mainly contains two paradigms, which are the reliable-based optimization (RBO) [2] and the robust design optimization (RDO) [3]. RDO aims at determining a robust design to optimize the deterministic performance about a mean value, while making it insensitivity with respect to uncertain variations by minimizing the performance variance. RBO focuses on a risk-based solution taking into account the feasibility of design target at expected probabilistic levels, in which the failure probabilities and expected values are used to quantitatively express the effects of uncertainties. In fact, RDO and RBO can be represented in the uniform theory framework. For instance, Beyer and et al [4] also indicated that the RBO can be regarded as a specific case of the RDO. In RDO and RBO methods, uncertain parameters are mostly treated as random variables, with precise probability distributions to be predefined based on the availability of complete information. However, it is generally a time-consuming and even an impossible process to achieve sufficient uncertain information to determine probability distributions, due to the complexity of practical problems [5]. Furthermore, Ben-Haim and Elishakoff [6] have shown that even small variations deviating from real values may cause relatively large errors of the probability distributions in the feasible region. Hence, probabilistic methods may experience difficulty for engineering problems. At the same time, there are a large number of design problems involved uncertain-but-bounded parameters in engineering.

The interval model has attracted much attention recently in the optimization of structures [7]. In interval models, the interval number is used to measure the uncertainty, because the representation of intervals only requires bounds of uncertain variables, which can be determined easier than the probability distribution. The corresponding bounds of an interval function are the minimal and maximal responses of the uncertain objective and constraints. The interval method has been successfully applied to the design of structures involving uncertain-but-bounded parameters [7]. Most convex models involve a nested double loop procedure. For instance, a nested double-loop optimization method using ellipsoid models was proposed to structural optimization [8], which included the method of moving asymptotes in the outer loop and a sequential quadratic programming in the inner loop. Although the nested double loop optimization is applicable, the computational cost is still prohibitive, as each individual outer loop consists of an inner loop minimization. To reduce the computational cost of the nested optimization, the first-order Taylor series expansion has been applied to approximate the maximum or minimum values of the bounds in the inner loop, instead of using the optimization algorithm. Kang and Luo [8] shown that a Taylor series-based linearization approach was more efficient than a double loop optimization method, since it can avoid expensive iterations in the inner loop. However, this method requires that the variability of interval variations is relatively small or moderate. Chakraborty and et al [9] applied the matrix perturbation theory via a first order Taylor series expansion to obtain a conservative dynamic response of interval functions, also under the assumption of a small level of uncertainty. Chen and et al [10] used the first-order Taylor series expansion to analyse the robust response of interval vibration control systems. In fact, the linear model optimization is actually a type of degenerative double loop optimization, in which the inner optimization is replaced by the first-order Taylor series expansion (linear model). However, the linear model with the low-order Taylor approximation has a lower numerical accuracy and then the approximation optimization may lead to a solution in unfeasible regions.

In this research, the interval arithmetic, which defines the fundamental arithmetic operators, is introduced into the inner optimization of the nested double loop process to evaluate the maximum and minimum values of an interval function, as the interval arithmetic can easily obtain the bounds of a design function with interval parameters. However, the range of an interval function will be enlarged in the numerical implementation, due to the inherent wrapping effect of the interval arithmetic. To control the overestimation, the Taylor inclusion function [11] is utilized to evaluate the bounds of interval function, and then the interval arithmetic is used to calculate the range of the polynomial function. However, the coefficients, a set of high-order derivatives, in the polynomial function is hard to be obtained even for functions with explicit expressions. To this end, the Chebyshev series [12] are used to approximate coefficients of the Taylor inclusion (polynomial) function, so as to develop a Chebyshev meta-model. This meta-model can be constructed by evaluating function values at some specified interpolation points rather than the high-order derivatives, to improve computational efficiency [12]. After obtaining the Chebyshev approximation, the interval arithmetic can be used to calculate the bounds of the Taylor inclusion function in the inner loop.

2. Design optimization with interval uncertainties

A general deterministic optimization model for the design of structures is given by

$$\begin{cases} \min\limits_{\mathbf{x}} & f(\mathbf{x},\mathbf{y}) \\ \text{s.t.} & g_i(\mathbf{x},\mathbf{y}) \le 0, \quad i=1,2,...,n \\ & \mathbf{x}^l \le \mathbf{x} \le \mathbf{x}^u \end{cases} \tag{1}$$

The above mathematical model is used to minimize the objective f subject to constraints g_i. $\mathbf{x} \in R^k$ is the vector including deterministic design variables, and $\mathbf{y} \in R^q$ is the vector of consisting of deterministic parameters. To describe uncertainties in the design, interval numbers [11] are introduced to express the variations induced by the uncertainty. Any interval $[x]$ can be expressed as

$$[x]=[\underline{x},\overline{x}]=x_c+[\Delta x]=x_c+[-\mathrm{rad}([x]),\mathrm{rad}([x])] \tag{2}$$

where $\underline{x}$ and $\overline{x}$ denotes the lower and upper bounds of $[x]$, respectively, $x_c=(\overline{x}+\underline{x})/2$ denotes the midpoint of $[x]$, and $[\Delta x]$ denotes the symmetric interval of $[x]$, $\mathrm{rad}([x])=(\overline{x}-\underline{x})/2$ denotes the radius reflecting the uncertain degree of $[x]$.

Consider the uncertainties, the deterministic optimization model (1) can be re-defined as follows:

$$\begin{cases} \min\limits_{[x]} & f([\mathbf{x}],[\mathbf{y}]) \\ \text{s.t.} & g_i([\mathbf{x}],[\mathbf{y}]) \le 0, \; i=1,2,...,n \\ & \mathbf{x}^l \le [\mathbf{x}] \le \mathbf{x}^u \end{cases} \tag{3}$$

Here, the ranges for the interval parameters $[y]$ will in general be pre-determined. Since the radius of an interval design variable $[x]$ is also pre-given as ξ, any interval design variable can be expressed as

$$[\mathbf{x}]=\mathbf{x}_c+[-\xi,\ \xi] \tag{4}$$

The responses of the objective and constraints would also be interval numbers, denoted by $[f]$ and $[g]$, respectively, because the design variables and parameters are interval vectors,

$$[f]=\left[\underline{f},\overline{f}\right],[g]=\left[\underline{g},\overline{g}\right] \tag{5}$$

The above minimization problem is to minimize both the average value and the width of the uncertain objective function, to ensure the "robustness" of the design. The minimization of the width will lead to the decrease of the variance of the objective function, to make the uncertain objective function insensitive to the variation due to the uncertainty. It is noted that the midpoint value and radius are functionally similar to the probabilistic counterparts in the conventional robust design optimization, which is a standard technique to minimize both the mean value and the standard deviation of the objective function.

To optimize the objective, both the midpoint and radius of the objective should be minimized, which can actually be regarded as a type of robust designs. Thus, the new objective f_{obj} can be specified as

$$f_{obj}=\alpha f_c+\beta \mathrm{rad}([f]) \tag{1}$$

where α and β denotes the weighting coefficients, and we set both α and β as 1 in this study. Then the objective can be re-defined as follows:

$$f_{obj}=f_c+\mathrm{rad}([f])=\overline{f} \tag{2}$$

Then the objective would be the upper bound of interval [*f*], which is the maximum value of *f* under the uncertainty.

For the interval constraints, there are three cases in the design space: $0\le\underline{g}$, $\underline{g}\le 0\le\overline{g}$ and. The first case violates the constraint, and the second case contains the possibility of violating the constraint. Only the last case can guarantee the design points in the feasible region, which denotes a 100% reliability index. So the upper bounds should be used to meet the constraints

$$\overline{g}_i([\mathbf{x}],[\mathbf{y}])\le 0,\ i=1,2,...,n \tag{3}$$

The upper bounds of the objective and constraints can be calculated through maximizing the value in the range of uncertainty. Consider Eqs. (3), (7) and (8), the optimization model can be finally expressed as

$$\begin{cases} \min\limits_{\mathbf{x}_c} \quad \max\limits_{\mathbf{x}\in[\mathbf{x}],\mathbf{y}\in[\mathbf{y}]} f(\mathbf{x,y}) \\ \text{s.t.} \quad \max\limits_{\mathbf{x}\in[\mathbf{x}],\mathbf{y}\in[\mathbf{y}]} g_i(\mathbf{x,y})\le 0,\ i=1,2,...,n \\ \qquad [\mathbf{x}]=\mathbf{x}_c+[-\xi,\ \xi],\ [\mathbf{y}]=[\underline{\mathbf{y}},\ \overline{\mathbf{y}}] \\ \qquad \mathbf{x}^l+\xi\le\mathbf{x}_c\le\mathbf{x}^u-\xi \end{cases} \tag{4}$$

3. Linear optimization model

The optimization model in Eq. (9) involves a nested double loop optimization process. The outer loop searches the optimal midpoint of interval design variables to minimize the objective, while the inner loop finds the maximum values (or minimum values) of the objective and constraints within the ranges of interval variables and parameters. The two optimization loops should use different optimization strategy to balance numerical accuracy and computational efficiency, as the different characteristics are possessed by the optimization models at two different layers.

To seek the global optimal solution and avoid multiple local minima, some heuristic techniques with strong global searching ability have to be used. This study employs the Multi-Island Genetic Algorithm (MIGA) [13] to solve the outer loop optimization problem. The MIGA is similar to the general GA, which consists of two processes: the first process is the selection of individuals for the production of the next generation, and the second process is the manipulation of the selected individuals to produce the next generation by crossover and mutation techniques. However, in MIGA, the population is divided into several sub-populations and the migration operation is added. Each sub-population evolves independently for optimizing the same objective function. The migration occurs every M generation, and copies of the individuals which are the best N% of the island populations are allowed to migrate. M is called the interval of the migration and N% is called the rate of the migration [13]. In the migration, the top N% strings in the sub-population A may be copied to another sub-population B, and the least N% strings of the sub-population B will be eliminated. Similarly, the sub-population A will receive the top strings from other sub-population and eliminates its least strings. This operation repeats until each sub-population achieves top strings from another sub-population. Although MIGA has higher possibility to search the global optimum, its convergence ratio is slower than traditional gradient-based algorithms, especially within the neighbourhood of the optimal solution. To improve the efficiency, the Sequential Quadratic Programming (SQP) is included to search the optimal point after MIGA, which means the optimal point of MIGA is used as the initial point of SQP. In this case, the number of generations in MIGA can be reduced, because only a limited number of points near the global optimal solution are required, which will greatly decrease the calculation time of MIGA.

The design space of the inner loop optimization is relatively narrow, so the inner loop can be approximated by linear model. That is, the objective can be expressed with respect to the design variables and parameters at the midpoints via the first-order Taylor series

$$f(\mathbf{x},\mathbf{y})=f(\mathbf{x}_c,\mathbf{y}_c)+\left.\frac{\partial f}{\partial \mathbf{x}}\right|_{(\mathbf{x}_c,\mathbf{y}_c)}(\mathbf{x}-\mathbf{x}_c)+\left.\frac{\partial f}{\partial \mathbf{y}}\right|_{(\mathbf{x}_c,\mathbf{y}_c)}(\mathbf{y}-\mathbf{y}_c)+\mathrm{O}(2) \tag{10}$$

If the higher order terms are ignored, the maximum value can be determined as

$$f_{\max}(\mathbf{x},\mathbf{y})\approx f(\mathbf{x}_c,\mathbf{y}_c)+\left|\left.\frac{\partial f}{\partial \mathbf{x}}\right|_{(\mathbf{x}_c,\mathbf{y}_c)}\right|\mathrm{rad}([\mathbf{x}])+\left|\left.\frac{\partial f}{\partial \mathbf{y}}\right|_{(\mathbf{x}_c,\mathbf{y}_c)}\right|\mathrm{rad}([\mathbf{y}]) \tag{11}$$

where $\mathrm{rad}([\mathbf{x}])$ and $\mathrm{rad}([\mathbf{y}])$ denote the radius of interval variables [**x**] and parameters [**y**], respectively. Similarly, the maximum value of constraint functions g_i can also be calculated via the linear model as

$$g_{\max}(\mathbf{x},\mathbf{y})\approx \mathrm{g}(\mathbf{x}_c,\mathbf{y}_c)+\left|\left.\frac{\partial g}{\partial \mathbf{x}}\right|_{(\mathbf{x}_c,\mathbf{y}_c)}\right|\mathrm{rad}([\mathbf{x}])+\left|\left.\frac{\partial g}{\partial \mathbf{y}}\right|_{(\mathbf{x}_c,\mathbf{y}_c)}\right|\mathrm{rad}([\mathbf{y}]) \tag{12}$$

The truncated errors exit in this linear model, due to the neglect of the higher-order terms in Eq. (10). For high nonlinear problems, the truncation error cannot be ignored. The linear model can improve computational efficiency of the double loop optimization. However, it may result in a poor numerical accuracy due to the truncated error. To reduce the computational time without sacrificing numerical accuracy, a new optimization strategy will be proposed based on the interval arithmetic.

The flowcharts for the linearized optimization using the first-order Taylor series is shown in Fig 1. The lower-order Taylor series expansion is used to replace the inner loop, so as to avoid the computationally expensive double loop process. This study will employ an alternative method to achieve a balanced performance of the efficiency and numerical accuracy in the numerical implementation.

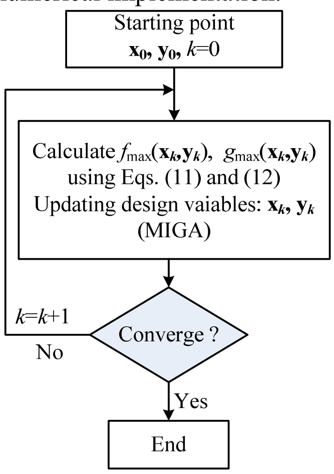

Figure 1: Flowchart of linear optimization model

4. Interval optimization using Chebyshev meta-models

In this section, the proposed methodology includes three parts: (1) the interval arithmetic is introduced to calculate the bounds of interval functions, to eliminate the inner optimization, (2) the Taylor inclusion function with higher-order series is used to reduce the overestimation triggered by the wrapping effect, which is intrinsic in interval arithmetic. However, higher-order derivatives are involved as coefficients in the calculation of the Taylor inclusion function, which again weights the computational cost, and (3) to this end, a Chebyshev meta-model is proposed to approximate the Taylor inclusion function, to improve the efficiency and accuracy.

4.1 Taylor inclusion function in the interval arithmetic

The notation of interval numbers has been introduced in Section 2. The interval arithmetic defines some basic arithmetic operations between two different interval numbers. Consider two interval variables [x] and [y], and the basic arithmetic operations [11] between them can be defined as follows:

$$\begin{cases}[x]+[y]=\left[\underline{x}+\underline{y},\overline{x}+\overline{y}\right],\\ [x]-[y]=\left[\underline{x}-\overline{y},\overline{x}-\underline{y}\right],\\ [x]\times[y]=\left[\min\left(\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\right),\max\left(\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\right)\right],\\ [x]\div[y]=\left[\min\left(\underline{x}/\underline{y},\underline{x}/\overline{y},\overline{x}/\underline{y},\overline{x}/\overline{y}\right),\max\left(\underline{x}/\underline{y},\underline{x}/\overline{y},\overline{x}/\underline{y},\overline{x}/\overline{y}\right)\right],\ if\ \ 0\notin[y]\end{cases} \tag{13}$$

From Eq. (14), it can be found that the interval arithmetic only depends on the bound of interval variables, which can obviously improve the computational efficiency of the optimization problem. However, interval arithmetic will lead to a large overestimation in the optimization, because of the dependence between interval variables.
The interval function [*f*] is an inclusion function of the function *f* if

$$\forall[x]\in IR^n,\ f([x])\subset[f]([x]) \tag{14}$$

For a large class of functions *f*, one of the most important objectives for the interval analysis is to provide inclusion functions [*f*] for *f*. This is required to be evaluated reasonably, such that the result is not too large. The natural inclusion function, which is the product by directly applying interval arithmetic to evaluate interval functions, will often result in relatively large overestimation due to the wrapping effect of intervals. To make the result sharper, the high-order Taylor series expansions of functions are usually used.
If the function f is (n+1) times partially differentiable on an opening set containing the interval [x], the nth-order Taylor inclusion function can be expressed as

$$\left[f_{T_n}\right]([x])=f(x_c)+f'(x_c)[\Delta x]+\ldots+\frac{1}{n!}f^{(n)}(x_c)[\Delta x]^n+\frac{1}{(n+1)!}\left[f^{(n+1)}([x])\right][\Delta x]^{n+1} \tag{15}$$

The front n+1 terms at the right side of Eq. (15) are the truncated Taylor series expansion of $f(x)$. The Eq. (15) calculates the rigorous enclosure for the function $f(x)$. In general, the last term at the right hand side of Eq. (15) can be neglected to obtain the approximate enclosure of $f(x)$.
Extending Eq. (15) to k-dimension problem and neglecting the remainder terms, we can obtain the multi-dimensional Taylor inclusion function

$$[f]([\mathbf{x}])\approx\sum_{0\le i_1+\ldots+i_k\le n}\beta_{i_1\ldots i_k}[\Delta x_1]^{i_1}\ldots[\Delta x_k]^{i_k} \tag{16}$$

where $\beta_{i_1\ldots i_k}$ denote the coefficients which are related with the partial derivatives of f with respect to **x,** and the total number of coefficients is $N_T=(n+k)!/n!k!$. In most cases, the Taylor inclusion function will produce a narrower interval than the interval calculated directly by interval arithmetic. However, a major problem of the higher-order Taylor inclusion function is that a set of high-order partial derivatives, acting as the coefficients of the evaluation function, are required to be calculated. Since the high-order derivatives are hard to calculate, another numerical method will be applied to evaluate these coefficients, which will lead to a meta-model for the approximation of the high-order Taylor inclusion function.

4.2 Chebyshev meta-model

The Chebyshev series can also be used to expand the continuous function, which may produce higher accuracy than Taylor series expansion. Wu and et al [12] has shown that the Chebyshev polynomials have higher approximation accuracy than the Taylor polynomials under the same orders. To simplify the problem but without losing any generality, we consider a variable $\mathbf{x}\in[-1,1]^k$. The continuous function $f(\mathbf{x})$ can be approximated by

$$f(\mathbf{x})\approx\sum_{i_1=0}^{n}\cdots\sum_{i_k=0}^{n}\left(\frac{1}{2}\right)^p f_{i_1\ldots i_k}C_{i_1\ldots i_k}(\mathbf{x}) \tag{17}$$

where p denotes the total number of zero(s) to be occurred in the subscripts $i_1,\ldots,i_k$. $C_{i_1\ldots i_k}(\mathbf{x})$ is the k-dimensional Chebyshev polynomials [12], $f_{i_1\ldots i_k}$ is a kth-order tensor with $(n+1)^k$ elements. Each coefficient of the Chebyshev polynomials can be calculated using the following integral formula [12]:

$$f_{i_1\ldots i_k}=\left(\frac{2}{\pi}\right)^k\int_{-1}^{1}\cdots\int_{-1}^{1}\frac{f(\mathbf{x})C_{i_1\ldots i_k}(\mathbf{x})}{\sqrt{1-x_1^2}\ldots\sqrt{1-x_k^2}}dx_1\ldots dx_k\approx\left(\frac{2}{m}\right)^k\sum_{j_1=1}^{m}\cdots\sum_{j_k=1}^{m}f(x_{j_1},\ldots,x_{j_k})C_{i_1\ldots i_k}(x_{j_1},\ldots,x_{j_k}) \tag{18}$$

where m denotes the order of numerical integral formula (m=n+1 in this study), x_j are the interpolation points of numerical integral formula. The interpolation points in each dimension are the zeros of $(n+1)$*th* order Chebyshev polynomial, to be determined by

$$x_j=\cos\theta_j,\ \text{where}\ \theta_j=\frac{2j-1}{n+1}\frac{\pi}{2},\ j=1,2,\ldots,n+1 \tag{19}$$

Thus, the number of interpolation points for a k-dimensional problem would be $N_s=(n+1)^k$.

From Eqs. (17) to (19), it can be found that the process of constructing the Chebyshev approximant is similar to the response surface methodology (RSM), which obtains the data at sampling points (or interpolation points in this study) and then produces the coefficients based on these data. Equation (17) can be transformed to a polynomial based on the power function

$$f(\mathbf{x}) \approx \sum_{i_1=0}^{n} \dots \sum_{i_k=0}^{n} \left(\frac{1}{2}\right)^p f_{i_1 \dots i_k} C_{i_1 \dots i_k}(\mathbf{x}) = \sum_{i_1=0}^{n} \dots \sum_{i_k=0}^{n} F_{i_1 \dots i_k} x_1^{i_1} \dots x_k^{i_k} \tag{20}$$

where $F_{i_1 \dots i_k}$ denotes the coefficients after the transformation. In Eq. (20), replacing the variable $\mathbf{x}$ with the interval variable $[\mathbf{x}]$, we can obtain the Taylor inclusion function. However, Eq. (20) contains $(n+1)^k$ terms, while the number of items in the Taylor inclusion function (Eq. (16)) is $N_T = (n+k)!/n!k!$ which is usually smaller than $(n+1)^k$.

Thus, if the Chebyshev polynomials are used to approximate the Taylor inclusion function, some higher order items will not be necessary. At the same time, the number of interpolation points for constructing Chebyshev polynomial equals to the number of items in Eq. (16), which is still computationally expensive, especially for the high dimensional problems. To further save the computational cost, only a part of the interpolation points will be used to build the Chebyshev polynomials, which is termed Chebyshev meta-model. Removing the items with orders higher than n in Eq. (20), the meta-model can be expressed by

$$f(\mathbf{x}) \approx \sum_{0 \le i_1 + \dots + i_k \le n} \left(\frac{1}{2}\right)^p f_{i_1 \dots i_k} C_{i_1 \dots i_k}(\mathbf{x}) = \sum_{0 \le i_1 + \dots + i_k \le n} F_{i_1 \dots i_k} x_1^{i_1} \dots x_k^{i_k} \tag{21}$$

Since only a part of interpolation points are used to construct the Chebyshev meta-model, the Eq. (18) cannot be used to calculate the coefficients. To reduce the error between the meta-model and evaluation function, the least squares method (LSM) can be employed to produce the coefficients. The number of coefficients in Eq. (21) is N_T, so the number of sampling points from the interpolation should not less than N_T. At the same time, the number of interpolation points is $N_s=(n+1)^k$, which is larger than N_T when $k>1$. Therefore, the number of sampling points can be chosen as any number in the interval $[N_T, N_s]$. The larger number of sampling points, the smaller error of the approximation, but lower efficiency. Some studies [14] show there will be a good balance between the accuracy and efficiency, when the number of the sampling points is twice of the number of the coefficients. Thus, when $N_s > 2N_T$, the $2N_T$ interpolation points are chosen as the sampling points randomly. Otherwise, all the interpolation points are chosen as the sampling points. After the set of sampling data is obtained, the LSM is used to calculate the coefficients and establish the meta-model.

After the Chebyshev meta-model is obtained, it can then be combined with the outer loop optimization (MIGA+SQP) to implement the uncertain optimization. The major advantage of the interval arithmetic is that the maximum and minimum values of a function are contained in the interval results, which provide rigorous constraints for the outer loop to guarantee the outer loop optimal solution is in the feasible region. The optimal design of the interval arithmetic may be more conservative than that of the double loop optimization, but it is more reliable than the double loop optimization. The flowchart in Fig. 2 illustrates the numerical process of the proposed interval optimization strategy, which use the Chebyshev meta-model to replace the linear model.

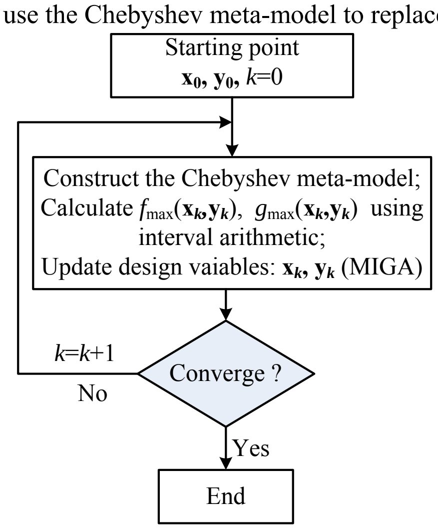

Figure 2: The flowchart of interval optimization strategy

5. Numerical examples

Figure 3 shows the 18-bar the 25-bar truss structure for transmission towers [15] has been widely studied, mostly under the assumption that all the variables are deterministic. In this study, we consider the cross-sectional areas of truss members as interval variables, and the interval width is 0.1in^2. The density of material is $\rho = 0.1$ lb/in^3, and the elasticity modulus is 10^7 lb/in^2. The objective is to minimize the total weight of the space truss. This space truss is subjected to two loading conditions, which are shown in Table 1. The structure is required to be doubly symmetric about x- and y-axis, and so the truss members can be grouped as Table 2, which also shows the stress limitations of each group. At the same time, the maximum displacements of nodes in each direction are limited to ±0.35in. The cross-sectional area of all members is changed in the range of 0.01~10in^2.

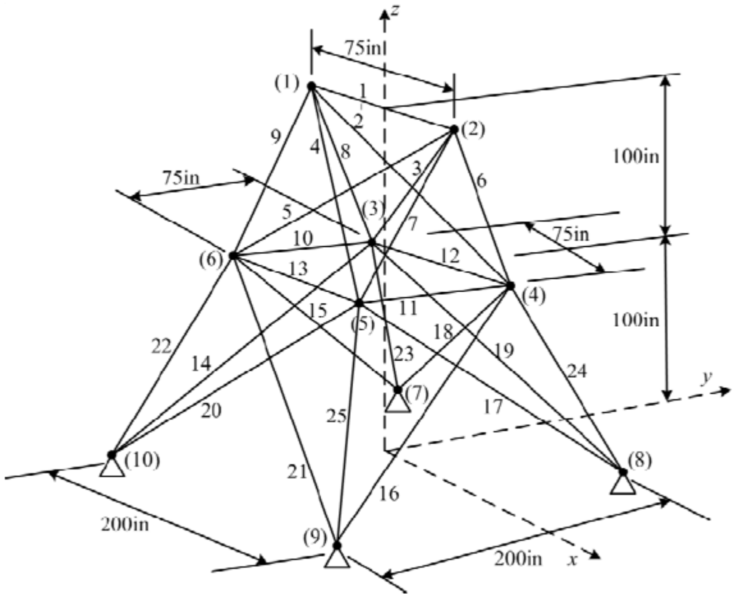

Figure 3: 25-bar space truss structure

The uncertain optimization model can be defined as

$$
\begin{aligned}
&\min_{\mathbf{x}_c} \quad \max_{\mathbf{x}\in[\mathbf{x}]} \; w = \sum_{i=1}^{25} A_i L_i \rho \\
&\text{s.t.} \quad g_1 = \max_{i=1,\ldots,25}\left(\max_{\mathbf{x}\in[\mathbf{x}]}(\sigma_i)\right) \le 40000; \; g_2 = \min_{i=1,10,\ldots,13}\left(\min_{\mathbf{x}\in[\mathbf{x}]}(\sigma_i)\right) \ge -35092; \\
&\qquad g_3 = \min_{i=2,\ldots,5}\left(\min_{\mathbf{x}\in[\mathbf{x}]}(\sigma_i)\right) \ge -11590; \; g_4 = \min_{i=6,\ldots,9}\left(\min_{\mathbf{x}\in[\mathbf{x}]}(\sigma_i)\right) \ge -17305; \\
&\qquad g_5 = \min_{i=14,\ldots,17}\left(\min_{\mathbf{x}\in[\mathbf{x}]}(\sigma_i)\right) \ge -6759; \; g_6 = \min_{i=18,\ldots,21}\left(\min_{\mathbf{x}\in[\mathbf{x}]}(\sigma_i)\right) \ge -6959; \\
&\qquad g_7 = \min_{i=22,\ldots,25}\left(\min_{\mathbf{x}\in[\mathbf{x}]}(\sigma_i)\right) \ge -11082; \; g_8 = \max_{i=1,\ldots,30}\left(\max_{\mathbf{x}\in[\mathbf{x}]}(|d_i|)\right) \le 0.35; \\
&\qquad [\mathbf{x}] = \mathbf{x}_c + [-\xi, \xi] \quad \xi = [0.1 \;\; \ldots \;\; 0.1]^{\mathrm{T}}_{1\times 8} \\
&\qquad [0.01 \;\; \ldots \;\; 0.01]^{\mathrm{T}}_{1\times 4} + \xi \le \mathbf{x}_c \le [10 \;\; \ldots \;\; 10]^{\mathrm{T}}_{1\times 4} - \xi
\end{aligned}
\tag{22}
$$

where σ_i denotes the stress of ith member, and d_i is the displacement of each node in each direction.

Table 1: Loading conditions

Node	Condition 1			Condition 2		
	P_x (lb)	P_y (lb)	P_z (lb)	P_x (lb)	P_y (lb)	P_z (lb)
1	0	20000	5000	1000	10000	-5000
2	0	-20000	-5000	0	10000	-5000
3	0	0	0	500	0	0
6	0	0	0	500	0	0

The linear model optimization and interval optimization methods are applied to solve this problem, respectively. The results in in Table 3 show that the two methods can result in similar objective function values (ranging from 613 to 614).

Table 2: Member stress limitations

variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	A_1	A_2 ~A_5	A_6 ~A_9	A_{10} ~A_{11}	A_{12} ~A_{13}	A_{14} ~A_{17}	A_{18} ~A_{21}	A_{22} ~A_{25}
Compressive stress limitations (lb/in2)	-35092	-11590	-17305	-35092	-35092	-6759	-6959	-11082
Tensile stress limitations (lb/in^2)	40000	40000	40000	40000	40000	40000	40000	40000

Table 3: The optimization results

	x_1(in^2)	x_2(in^2)	x_3(in^2)	x_4(in^2)	x_5(in^2)	x_6(in^2)
Linear optimization	0.1332	2.1719	2.9970	0.1260	0.1260	0.8413
Interval optimization	0.11	1.8747	3.2984	0.11	0.11	0.8214
	x_7(in^2)	x_8(in^2)	g_6(lb/in^2)	g_8(lb)	W(in)	Time
Linear optimization	1.7822	2.7062	-6959(-6974)	0.3492(0.3514)	613.83	3'10''
Interval optimization	1.8877	2.6446	-6709(-6695)	0.3500(0.3499)	613.44	13'30''

However, according to the validated values of the constraints shown in the brackets, it can be found that the linear optimization violates the two constraints, which are g_6=-6974 lb/in^2 and g_8=0.3514in. The values marked by red colour with underlines are used to denote the violation of the constraints. The interval optimization is located in a safer position for both of the constraints (the rest 6 constraints, which are not listed in this table, satisfy the given conditions). Thus, the interval method would keep the optimization result lie in a feasible region but the linear method often generates more risk results. For the calculation time, the interval optimization method takes 13'30'', which is acceptable relative to the linear optimization method 3'30''.

6. Conclusions

This research has proposed a new uncertain optimization method for the design of structures involving uncertain-but-bounded parameters. The interval model is used to describe uncertainties of the bounded parameters, which only requires the lower and upper bounds of an interval number. The proposed interval uncertain optimization model has the characteristics of both the robust design and reliability based optimization. The interval optimization commonly leads to the nested double loop process. In the linear optimization procedure, the outer loop is usually used to update the design variables to seek the optimal solution while the linear model (inner loop) is to calculate the bounds of the interval objective and constraints. The linear optimization model has higher efficiency than traditional nested double loop optimization process, but lower accuracy.

To improve the accuracy of the linear optimization model, the interval arithmetic has been introduced into the inner loop to directly evaluate the bounds of interval design functions, so as to replace the linear model. Furthermore, to reduce the overestimation in the interval arithmetic, the high-order Taylor inclusion function is utilized to calculate the bounds of the interval design functions. However, the calculation of the high-order derivatives in the inclusion function is not easy. Hence, the Chebyshev meta-model is incorporated in the inclusion function to approximate the high-order derivatives, so that a Chebyshev model is developed, which can provide higher approximation accuracy than the truncated Taylor series. Typical numerical examples are used to demonstrate the effectiveness of the proposed interval optimization methodology. Compared to the linearized optimization, the method in this paper can improve numerical accuracy and reliability without increasing computation cost largely.

7. References

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