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Probabilistic Sensitivity Analysis for Novel Second-Order Reliability Method (SORM) Using Generalized Chi-Squared Distribution

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1. Abstract
Reliability-based design optimizations (RBDO) require evaluation of sensitivities of probabilistic constraints. To develop RBDO utilizing the recently proposed novel second-order reliability method (SORM) that improves the conventional SORM in terms of accuracy, the sensitivities of the probabilistic constraints at the most probable point (MPP) are required. Thus, this study presents the sensitivity analysis of the novel SORM at MPP for more accurate RBDO. During the analytic derivation in this study, it is assumed that the Hessian matrix does not change due to change of distribution parameter. The calculation of the sensitivity based on the analytic derivation requires evaluation of probability density function (PDF) of a linear combination of non-central chi-square variables, which is obtained by utilizing general chi-squared distribution. In terms of accuracy, the proposed probabilistic sensitivity analysis is compared with the finite difference method (FDM) using the Monte Carlo simulation (MCS) through numerical examples. The numerical examples demonstrate that the analytic sensitivity of the novel SORM agrees very well with the sensitivity obtained by FDM using MCS when a performance function is quadratic in U-space and input variables are normally distributed. It is further tested that sensitivity of a higher order performance function in terms of how the proposed assumption – the Hessian is constant – affects the accuracy of the sensitivity.
2. Keywords: Sensitivity Analysis, Most Probable Point (MPP), Novel Second-Order Reliability Method (SORM), General Chi-squared Distribution

3. Introduction
Reliability analysis and reliability-based design optimization (RBDO) have been successfully applied to diverse engineering applications such as structural engineering analysis [1-7]. The main goal of the reliability analysis is to evaluate probability of failure of a performance function that is used as a probabilistic constraint of RBDO with both efficiency and accuracy. Methods of obtaining the probability of failure are commonly categorized into sampling-based and most probable point (MPP)-based methods. MPP-based first-order reliability method (FORM) [8-11] very efficiently calculates the probability of failure, however, accuracy is sacrificed for highly non-linear performance functions and high-dimensional input variables. MPP-based second-order reliability method (SORM) [12-15] improves FORM in terms of accuracy, however it is computationally expensive compared to FORM since it requires the computation of the Hessian matrix. MPP-based dimension reduction method (DRM) [16-18] can be also used for approximately assessing the reliability of a system which is used as a probabilistic constraint in RBDO.
SORM that improves accuracy of FORM still contains three types of errors: (1) error due to approximating a general nonlinear limit state function by a quadratic function at the MPP in standard normal U-space, (2) error due to approximating the quadratic function in U-space by a parabolic surface, and (3) error due to calculation of the probability of failure after making the previous two approximations. On the other hand, the recently proposed novel SORM contains only type (1) error.
The general idea of the novel SORM is to compute the probability of failure using non-central or general chi-squared distribution [19]. Similarly to other MPP-based methods, the novel SORM first involves MPP search after transforming all random variables in original X-space to standard normal U-space through the Rosenblatt transformation [20]. After a quadratic approximation at MPP in U-space, the novel SORM does not use further approximation of a quadratic function. Instead, the novel SORM converts a quadratic failure function of standard normal variables to a linear combination of non-central chi-square variables by orthogonal transformation. Since every random variable in U-space is a standard normal variable, probability of failure of a quadratic function in U-space can be evaluated using a linear combination of non-central chi-square variables.
To carry out RBDO utilizing reliability analysis method, sensitivities of probabilistic constraints with respect to design variables are required, and many works have been devoted to derive the sensitivity of the probabilistic constraint [21-27]. Thus, this study presents the sensitivity analysis of the novel SORM for more accurate RBDO. Since the novel SORM performs reliability analysis at MPP, sensitivities of probabilistic constraints at MPP with

respect to distribution parameters are derived during the sensitivity analysis in this study. To calculate the sensitivity based on the sensitivity analysis, it is necessary to evaluate probability density function (PDF) of a linear combination of non-central chi-square variables, which is obtained utilizing the general chi-squared distribution in this study.

This paper is organized as follow. Section 4 reviews the reliability analysis using the novel SORM. Section 5 presents the sensitivity analysis using the novel SORM. Section 6 provides the numerical examples where analytic sensitivities are compared with the finite difference method (FDM) using Monte Carlo simulation (MCS) in terms of accuracy. Section 7 summarizes this study and discusses about the future research.

4. Review of Reliability Analysis Using Novel SORM

Reliability analysis using the novel SORM is reviewed in this section to help readers understand sensitivity analysis of the novel SORM that will be explained in Section 5 [19].

4.1 Quadratic Approximation of Performance Function

A reliability analysis involves calculation of probability of failure, denoted by P_F, which is defined using a multi-dimensional integral [8,27]

$$P_F \equiv P[G(\mathbf{X}) > 0] = \int_{G(\mathbf{X})>0} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{1}$$

where $P[\bullet]$ is a probability measure; $\mathbf{X}=\{X_1, X_2, \cdots, X_N\}^{\mathrm{T}}$ is an N-dimensional random vector where X_i are random variables and x_i are the realizations of X_i; $G(\mathbf{X})$ is the performance function such that $G(\mathbf{X}) > 0$ is defined as failure; and $f_{\mathbf{X}}(\bullet)$ is a joint probability density function (PDF) of $\mathbf{X}$.

In conventional SORM, for the calculation of the probability of failure in Eq. (1), $G(\mathbf{X})$ in Eq. (1) is transformed to U-space through the Rosenblatt transformation, then it is approximated by its second-order Taylor series expansion at MPP ($\mathbf{u}^*$), which can be written as

$$\begin{aligned} G(\mathbf{X}) = g(\mathbf{U}) \cong g_Q(\mathbf{U}) &= \nabla g^{\mathrm{T}}(\mathbf{U}-\mathbf{u}^*) + \frac{1}{2}(\mathbf{U}-\mathbf{u}^*)^{\mathrm{T}}\mathbf{H}(\mathbf{U}-\mathbf{u}^*) \\ &= \left(-\nabla g^{\mathrm{T}}\mathbf{u}^* + \frac{1}{2}\mathbf{u}^{*\mathrm{T}}\mathbf{H}\mathbf{u}^*\right) - \left(-\nabla g^{\mathrm{T}} + \mathbf{u}^{*\mathrm{T}}\mathbf{H}\right)\mathbf{U} + \frac{1}{2}\mathbf{U}^{\mathrm{T}}\mathbf{H}\mathbf{U} \end{aligned} \tag{2}$$

using the gradient vector and the Hessian matrix ($\mathbf{H}$) evaluated at MPP. $\mathbf{u}^*$ in Eq. (2) is obtained by solving the following optimization problem to

$$\begin{aligned} &\text{minimize} && \|\mathbf{u}\| \\ &\text{subject to} && g(\mathbf{u}) = 0. \end{aligned} \tag{3}$$

A reliability index, denoted by β, is defined as distance from the origin to $\mathbf{u}^*$ and is given by

$$\beta = \|\mathbf{u}^*\| = \left(\mathbf{u}^{*\mathrm{T}}\mathbf{u}^*\right)^{1/2}. \tag{4}$$

Using the failure definition $G(\mathbf{X}) > 0$, $\mathbf{u}^*$ can be written as

$$\mathbf{u}^* = \beta\frac{\nabla g}{\|\nabla g\|} = \beta\boldsymbol{\alpha} \tag{5}$$

where $\boldsymbol{\alpha}$ is the normalized gradient vector at $\mathbf{u}^*$. Then, dividing Eq. (2) by $\|\nabla g\|$ and using Eq. (5) yield

$$\frac{g_Q(\mathbf{U})}{\|\nabla g\|} = \left(-\beta + \frac{\beta^2}{2}\boldsymbol{\alpha}^{\mathrm{T}}\frac{\mathbf{H}}{\|\nabla g\|}\boldsymbol{\alpha}\right) - \left(-\boldsymbol{\alpha}^{\mathrm{T}} + \beta\boldsymbol{\alpha}^{\mathrm{T}}\frac{\mathbf{H}}{\|\nabla g\|}\right)\mathbf{U} + \frac{1}{2}\mathbf{U}^{\mathrm{T}}\frac{\mathbf{H}}{\|\nabla g\|}\mathbf{U}. \tag{6}$$

4.2 Orthogonal Transformation of Quadratic Function

Consider an orthogonal transformation $\mathbf{u} = \mathbf{T}\mathbf{y}$ where $\mathbf{T}$ is the $N \times N$ matrix of the eigenvectors of $\mathbf{H}$ and $\mathbf{y} \in \mathrm{R}^N$ is an N-dimensional vector of standard normal random variables, y_i, which are statistically independent to each other since $\mathbf{u}$ is statistically independent [19]. Using the orthogonal transformation, Eq. (6) can be transformed to [14]

$$\frac{\hat{g}_Q(\mathbf{Y})}{\|\nabla g\|} = a_0 - \mathbf{a}_1^{\mathrm{T}}\mathbf{Y} + \mathbf{Y}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{Y} \tag{7}$$

where three quantities a_0, $\mathbf{a}_1^{\mathrm{T}}$, and $\hat{\mathbf{A}}$ are given by

$$a_0 = -\beta + \frac{\beta^2}{2}\boldsymbol{\alpha}^{\mathrm{T}}\frac{\mathbf{H}}{\|\nabla g\|}\boldsymbol{\alpha} \in \mathrm{R}, \tag{8}$$

$$\mathbf{a}_1^{\mathrm{T}} = \boldsymbol{\alpha}^{\mathrm{T}}\left(\mathbf{I} - \beta\frac{\mathbf{H}}{\|\nabla g\|}\right)\mathbf{T} \in \mathrm{R}^{1\times N}, \tag{9}$$

and

$$\hat{\mathbf{A}} = \frac{1}{2\|\nabla g\|}\mathrm{diag}\left[\lambda_1, \lambda_2, \cdots, \lambda_N\right] \in \mathrm{R}^{N\times N}, \tag{10}$$

respectively where λ_i is the i^{th} eigenvalue of $\mathbf{H}$.

Consequently, the probability of failure in Eq. (1) can be calculated using Eqs. (7)-(10) as

$$\begin{aligned} P_F &\cong P\left[\frac{g_Q(\mathbf{U})}{\|\nabla g\|} > 0\right] = P\left[\frac{\hat{g}_Q(\mathbf{Y})}{\|\nabla g\|} > 0\right] = P\left[2\hat{g}_Q(\mathbf{Y}) > 0\right] \\ &= P\left[2\|\nabla g\|a_0 + \sum_{k=1}^{N}\left(2\|\nabla g\|a_{1_k}Y_k + \lambda_k Y_k^2\right) > 0\right]. \end{aligned} \tag{11}$$

where a_{1_k} and Y_k are k^{th} component of $\mathbf{a}_1$ and standard normal variables, respectively. Since $\mathbf{u}$ and $\mathbf{y}$ are the independent standard normal variables, any orthogonal transformation does not change the probability of failure [19]. Thus, Eq. (11) can be rewritten as

$$P_F \cong P\left[\sum_{k=1}^{N}\lambda_k\left(Y_k + \delta_k\right)^2 > \sum_{k=1}^{N}\lambda_k\delta_k^2 - 2\|\nabla g\|a_0\right] \tag{12}$$

where the k^{th} non-centrality parameter δ_k is given by

$$\delta_k = \frac{\|\nabla g\|a_{1_k}}{\lambda_k} \tag{13}$$

Since Y_k is the standard normal variable, $\left(Y_k + \delta_k\right)^2$ is a non-central chi-square variable and $Q = \sum_{k=1}^{N}\lambda_k\left(Y_k + \delta_k\right)^2$ becomes a general chi-square variable whose PDF and CDF are available as shown in Ref. [28]. Hence, the probability of failure in Eq. (12) can be simplified as

$$P_F \cong P\left[Q > \sum_{k=1}^{N}\lambda_k\delta_k^2 - 2\|\nabla g\|a_0\right] = 1 - F_Q\left(\sum_{k=1}^{N}\lambda_k\delta_k^2 - 2\|\nabla g\|a_0\right) \tag{14}$$

where $F_Q(\bullet)$ is the CDF of Q.

4.3 General Chi-Squared Distribution

The distribution of Q in Eq. (14) can be obtained by using a general chi-squared distribution. Let $\lambda_k > 0$ for $k = 1, \cdots, \rho$, $\lambda_k < 0$ for $k = \rho+1, \cdots, \rho+\xi$, and $\lambda_k = 0$ for $k = \rho+\xi+1, \cdots, N$. Then, the linear combination Q in Eq. (14) can be expressed as

$$Q = \sum_{k=1}^{N} \lambda_k Z_k = U - V \tag{15}$$

where λ_k are distinct eigenvalues, $Z_k \sim \chi^2_{\nu_k}(\delta_k^2)$, and

$$U = \sum_{k=1}^{\rho} \lambda_k Z_k,\ V = \sum_{k=1+\rho}^{\rho+\xi} (-\lambda_k) Z_k. \tag{16}$$

The PDFs of *U* and *V* in Eqs. (15) and (16) are obtained by Ruben in Ref. [29]. Additionally, the exact PDF and CDF of the general chi-square variable using Whittaker's function [30] are obtained by Provost and Rudiuk [28] . The PDF and CDF of *Q* are not presented in this paper due to their complexities.

5. Sensitivity Analysis Using Novel SORM

The sensitivity of the probability of failure with respect to a distribution parameter *θ* can be obtained by taking the derivative of Eq. (14) as

$$\frac{dP_F}{d\theta} = -f_Q(q,\boldsymbol{\delta})\frac{dq}{d\theta} - \sum_{i=1}^{N} \frac{dF_Q(q,\boldsymbol{\delta})}{d\delta_i}\frac{d\delta_i}{d\theta}. \tag{17}$$

denoting $\sum_{k=1}^{N} \lambda_k \delta_k^2 - 2\|\nabla g\| a_0$ in Eq. (14) as *q*. In Eq. (17), $f_Q(\bullet)$ is the PDF of Q, which is obtained by using general chi-squared distribution explained in Section 4.3. Here, it should be noted that $F_Q(\bullet)$ *in Eq. (14) is not only function of q but also function of* $\boldsymbol{\delta}$, which is vector of the non-centrality parameter in Eq. (13). To calculate Eq. (17), $\frac{d\lambda_k}{d\theta}$ needs to be obtained based on the definition of *q*, and third-order derivative of performance function or derivative of the H is thus required to be obtained.

Consider the generalized eigenvalue problem of the Hessian matrix H such that $\mathbf{HT} = \boldsymbol{\lambda}\mathbf{AT}$ where $\boldsymbol{\lambda}$ and $\mathbf{T}$ are the diagonal matrix of the eigenvalues and the matrix of the eigenvectors of H, respectively, and A represents the matrix that is symmetric and positive definite. Then, according to the eigenvalue perturbation theory, $\frac{d\lambda_k}{d\theta}$ is obtained using the property of the matrix A as [31]

$$\frac{d\lambda_k}{d\theta} = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial\lambda_k}{\partial H_{ij}}\frac{\partial H_{ij}}{\partial\theta} = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial}{\partial H_{ij}}\left(\lambda_k + \mathbf{T}_k^{\mathrm{T}}(\delta\mathbf{H} - \lambda_k\varphi\mathbf{A})\mathbf{T}_k\right)\frac{\partial H_{ij}}{\partial\theta} = \sum_{i=1}^{N}\sum_{j=1}^{N} T_{k_i}T_{k_j}\left(2-\delta_{ij}\right)\frac{\partial H_{ij}}{\partial\theta} \tag{18}$$

where H_{ij} is the element of $\mathbf{H}$ at the i^{th} row and the j^{th} column; $\mathbf{T}_k$ is the k^{th} eigenvector corresponding to λ_k; T_{k_i} is the i^{th} component of $\mathbf{T}_k$; φ is perturbation size, which is chosen so that it is always much smaller than θ; and δ_{ij} is the Kronecker delta. Since the derivative of the Hessian matrix $\mathbf{H}$ is not available in SORM, it is assumed in this study that $\frac{\partial H_{ij}}{\partial\theta}$ in Eq. (18) is 0. Consequently, $\frac{d\lambda_k}{d\theta}$ in Eq. (18) becomes 0.

Using the assumption, the sensitivity of the probability of failure in Eq. (17) can be written as

$$
\begin{aligned}
\frac{dP_F}{d\theta} &= -f_Q(q,\boldsymbol{\delta})\left(\sum_{k=1}^{N}\lambda_k\frac{d\delta_k^2}{d\theta}-2\frac{d\|\nabla g\|}{d\theta}a_0-2\|\nabla g\|\frac{da_0}{d\theta}\right)-\sum_{i=1}^{N}\frac{dF_Q(q,\boldsymbol{\delta})}{d\delta_i}\left(\frac{\|\nabla g\|}{\lambda_i}\frac{da_{1_i}}{d\theta}+\frac{a_{1_i}}{\lambda_i}\frac{d\|\nabla g\|}{d\theta}\right)\\
&= -2f_Q(q,\boldsymbol{\delta})\left(\sum_{k=1}^{N}\lambda_k\delta_k\frac{d\delta_k}{d\theta}-\frac{d\|\nabla g\|}{d\theta}a_0-\|\nabla g\|\frac{da_0}{d\theta}\right)-\sum_{i=1}^{N}\frac{dF_Q(q,\boldsymbol{\delta})}{d\delta_i}\left(\frac{\|\nabla g\|}{\lambda_i}\frac{da_{1_i}}{d\theta}+\frac{a_{1_i}}{\lambda_i}\frac{d\|\nabla g\|}{d\theta}\right).
\end{aligned} \tag{19}
$$

Equation (19) can be then rewritten based on the definition of δ stated in Eq. (13) as

$$
\frac{dP_F}{d\theta} = -2f_Q(q,\boldsymbol{\delta})\left(\|\nabla g\|\boldsymbol{\delta}^T\frac{d\mathbf{a_1}}{d\theta}+\frac{d\|\nabla g\|}{d\theta}\boldsymbol{\delta}^T\mathbf{a_1}-\frac{d\|\nabla g\|}{d\theta}a_0-\|\nabla g\|\frac{da_0}{d\theta}\right)-\sum_{i=1}^{N}\frac{dF_Q(q,\boldsymbol{\delta})}{d\delta_i}\left(\frac{\|\nabla g\|}{\lambda_i}\frac{da_{1_i}}{d\theta}+\frac{a_{1_i}}{\lambda_i}\frac{d\|\nabla g\|}{d\theta}\right). \tag{20}
$$

As shown in Eq. (20), $\frac{d\mathbf{a_1}}{d\theta}$, $\frac{d\|\nabla g\|}{d\theta}$, and $\frac{da_0}{d\theta}$ need to be derived for the sensitivity analysis of the probability of failure. From Eq. (9) and the assumption made in the previous paragraph, $\frac{d\mathbf{a_1}}{d\theta}$ in Eq. (20) becomes

$$
\frac{d\mathbf{a_1}}{d\theta} \cong \mathbf{T}^T\left(\mathbf{I}-\beta\frac{\mathbf{H}}{\|\nabla g\|}\right)\frac{d\boldsymbol{\alpha}}{d\theta}-\mathbf{T}^T\mathbf{H}\boldsymbol{\alpha}\frac{d}{d\theta}\left(\frac{\beta}{\|\nabla g\|}\right)=\mathbf{T}^T\left(\mathbf{I}-\beta\frac{\mathbf{H}}{\|\nabla g\|}\right)\frac{d\boldsymbol{\alpha}}{d\theta}-\frac{\mathbf{T}^T\mathbf{H}\boldsymbol{\alpha}}{\|\nabla g\|}\frac{d\beta}{d\theta}+\frac{\beta\mathbf{T}^T\mathbf{H}\boldsymbol{\alpha}}{\|\nabla g\|^2}\frac{d\|\nabla g\|}{d\theta} \tag{21}
$$

where $\frac{d\boldsymbol{\alpha}}{d\theta}$ can be derived based on the definition of $\boldsymbol{\alpha}$ stated in Eq. (5) as

$$
\frac{d\boldsymbol{\alpha}}{d\theta}=\frac{d}{d\theta}\left(\frac{\nabla g}{\|\nabla g\|}\right)=\frac{1}{\|\nabla g\|}\frac{d\nabla g}{d\theta}-\frac{\nabla g}{\|\nabla g\|^2}\frac{d\|\nabla g\|}{d\theta}=\frac{1}{\|\nabla g\|}\frac{d\nabla g}{d\theta}-\frac{\boldsymbol{\alpha}}{\|\nabla g\|}\frac{d\|\nabla g\|}{d\theta}. \tag{22}
$$

Utilizing the chain rule, $\frac{d\nabla g}{d\theta}$ in Eq. (22) can be written as

$$
\frac{d\nabla g}{d\theta}=\frac{\partial\nabla g}{\partial\theta}+\mathbf{H}\frac{d\mathbf{u}^*}{d\theta} \tag{23}
$$

where $\frac{d\mathbf{u}^*}{d\theta}$ can be obtained by taking derivative of $\mathbf{u}^*$ in Eq. (5) with respect to θ as

$$
\frac{d\mathbf{u}^*}{d\theta}=\frac{d\beta}{d\theta}\boldsymbol{\alpha}+\beta\frac{d\boldsymbol{\alpha}}{d\theta}. \tag{24}
$$

$\frac{d\|\nabla g\|}{d\theta}$ in Eq. (22) can be obtained as

$$
\frac{d\|\nabla g\|}{d\theta}=\boldsymbol{\alpha}^T\frac{d\nabla g}{d\theta}. \tag{25}
$$

Then, using Eqs. (23), (24) and (25), Eq. (22) can be rewritten as

$$\frac{d\boldsymbol{\alpha}}{d\theta} = \frac{\mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}^T}{\|\nabla g\|}\left[\frac{\partial \nabla g}{\partial \theta} + \mathbf{H}\left(\frac{d\beta}{d\theta}\boldsymbol{\alpha} + \beta\frac{d\boldsymbol{\alpha}}{d\theta}\right)\right]. \tag{26}$$

Thus, it is obtained

$$\frac{d\boldsymbol{\alpha}}{d\theta} = \left[\mathbf{I} - \frac{\beta\left(\mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}^T\right)\mathbf{H}}{\|\nabla g\|}\right]^{-1}\frac{\left(\mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}^T\right)}{\|\nabla g\|}\left(\frac{\partial \nabla g}{\partial \theta} + \frac{d\beta}{d\theta}\mathbf{H}\boldsymbol{\alpha}\right). \tag{27}$$

The limit-state function in the original X-space is expressed as [21]

$$g\left(\mathbf{u};\theta\right) = G\left(\mathbf{x}\right). \tag{28}$$

Due to the fact that distribution parameter θ has no influence on the limit state function expressed in Eq. (28), derivative of the left hand side of Eq. (28) with respect to θ is zero, which implies

$$\frac{dg\left(\mathbf{u};\theta\right)}{d\theta} = \frac{\partial g}{\partial \theta} + \nabla^T g\frac{d\mathbf{u}}{d\theta} = 0. \tag{29}$$

Thus, it is derived using Eq. (29) as

$$\frac{\partial \nabla g}{\partial \theta} = \nabla\frac{\partial g}{\partial \theta} = -\nabla\left(\nabla^T g\frac{d\mathbf{u}}{d\theta}\right). \tag{30}$$

By inserting Eq. (30) into Eq. (27), it is obtained

$$\frac{d\boldsymbol{\alpha}}{d\theta} = \left[\mathbf{I} - \frac{\beta\left(\mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}^T\right)\mathbf{H}}{\|\nabla g\|}\right]^{-1}\frac{\left(\mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}^T\right)}{\|\nabla g\|}\left[-\nabla\left(\nabla^T g\frac{d\mathbf{u}}{d\theta}\right) + \frac{d\beta}{d\theta}\mathbf{H}\boldsymbol{\alpha}\right]. \tag{31}$$

The reliability index β can be expressed based on Eq. (5) as

$$\beta = \boldsymbol{\alpha}^T\mathbf{u}^*. \tag{32}$$

Then, by taking derivative of Eq. (32) with respect to θ yields

$$\frac{d\beta}{d\theta} = \frac{d\boldsymbol{\alpha}^T}{d\theta}\mathbf{u}^* + \boldsymbol{\alpha}^T\frac{d\mathbf{u}^*}{d\theta} = \boldsymbol{\alpha}^T\frac{d\mathbf{u}^*}{d\theta} \tag{33}$$

owing to the fact that $\frac{d\boldsymbol{\alpha}^T}{d\theta}$ and $\boldsymbol{\alpha}$ are mutually orthogonal and $\mathbf{u}^* = \beta\boldsymbol{\alpha}$. The orthogonality can be verified directly by differentiation of $\boldsymbol{\alpha}^T\boldsymbol{\alpha} = 1$ or by

$$\begin{aligned}\frac{d\boldsymbol{\alpha}^T}{d\theta}\mathbf{u}^* &= \frac{d}{d\theta}\left[\nabla g\left(\nabla^T g\nabla g\right)^{-1/2}\right]^T\mathbf{u}^* = \left\{\frac{1}{\|\nabla g\|}\frac{d}{d\theta}\left(\nabla g\right) - \frac{\nabla g}{\|\nabla g\|^3}\left[\nabla^T g\frac{d}{d\theta}\left(\nabla g\right)\right]\right\}^T\mathbf{u}^* \\ &= \frac{\beta}{\|\nabla g\|^2}\frac{d}{d\theta}\left(\nabla g\right)^T\nabla g - \frac{\beta}{\|\nabla g\|^2}\left[\nabla^T g\frac{d}{d\theta}\left(\nabla g\right)\right] = 0.\end{aligned} \tag{34}$$

The limit-state function at MPP in U-space is given by $g\left(\mathbf{u}^*;\theta\right) = 0$ and the differentiation of it gives

$$\frac{dg}{d\theta} = \frac{\partial g}{\partial \theta} + \nabla^T g \frac{d\mathbf{u}^*}{d\theta} = 0. \tag{35}$$

Then, dividing both sides of Eq. (35) by $\|\nabla g\|$ gives

$$\frac{1}{\|\nabla g\|}\left[\frac{\partial g}{\partial \theta} + \nabla^T g \frac{d\mathbf{u}^*}{d\theta}\right] = \frac{1}{\|\nabla g\|}\frac{\partial g}{\partial \theta} + \frac{\nabla^T g}{\|\nabla g\|}\frac{d\mathbf{u}^*}{d\theta} = \frac{1}{\|\nabla g\|}\frac{\partial g}{\partial \theta} + \boldsymbol{\alpha}^T \frac{d\mathbf{u}^*}{d\theta} = 0. \tag{36}$$

Using Eqs. (29) and (36), Eq. (33) can be rewritten as

$$\frac{d\beta}{d\theta} = -\frac{1}{\|\nabla g\|}\frac{\partial g}{\partial \theta} = \left[\boldsymbol{\alpha}^T \frac{d\mathbf{u}}{d\theta}\right]_{\mathbf{u}=\mathbf{u}^*}. \tag{37}$$

Finally, $\frac{da_0}{d\theta}$ in Eq. (20) can be obtained using Eq. (8) as

$$\begin{aligned}\frac{da_0}{d\theta} &\cong -\frac{d\beta}{d\theta} + \frac{\beta}{\|\nabla g\|}\frac{d\beta}{d\theta}\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H}\boldsymbol{\alpha} + \frac{\beta^2}{2}\frac{\frac{d\left(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H}\boldsymbol{\alpha}\right)}{d\theta}\|\nabla g\| - \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H}\boldsymbol{\alpha}\frac{d\|\nabla g\|}{d\theta}}{\|\nabla g\|^2} \\ &= -\frac{d\beta}{d\theta} + \frac{\beta}{\|\nabla g\|}\frac{d\beta}{d\theta}\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H}\boldsymbol{\alpha} + \frac{\beta^2}{\|\nabla g\|}\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H}\frac{d\boldsymbol{\alpha}}{d\theta} - \frac{\beta^2\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H}\boldsymbol{\alpha}}{2\|\nabla g\|^2}\frac{d\|\nabla g\|}{d\theta}.\end{aligned} \tag{38}$$

Then, using Eqs. (22) and (25), Eq. (38) can be expressed as

$$\frac{da_0}{d\theta} \cong -\frac{d\beta}{d\theta} + \frac{\beta}{\|\nabla g\|}\frac{d\beta}{d\theta}\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H}\boldsymbol{\alpha} + \frac{\beta^2}{\|\nabla g\|}\left[\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H} - \frac{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{H}\boldsymbol{\alpha}}{2}\boldsymbol{\alpha}^{\mathrm{T}}\left(\mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}^T\right)^{-1}\right]\frac{d\boldsymbol{\alpha}}{d\theta}. \tag{39}$$

The sensitivity of the probability of failure with respect to θ is therefore given by

$$\begin{aligned}\frac{dP_F}{d\theta} &= -2f_Q(q,\boldsymbol{\delta})\left(\|\nabla g\|\boldsymbol{\delta}^T\frac{d\mathbf{a_1}}{d\theta} + \frac{d\|\nabla g\|}{d\theta}\boldsymbol{\delta}^T\mathbf{a_1} - \frac{d\|\nabla g\|}{d\theta}a_0 - \|\nabla g\|\frac{da_0}{d\theta}\right) - \sum_{i=1}^{N}\frac{dF_Q(q,\boldsymbol{\delta})}{d\delta_i}\left(\frac{\|\nabla g\|}{\lambda_i}\frac{da_{1_i}}{d\theta} + \frac{a_{1_i}}{\lambda_i}\frac{d\|\nabla g\|}{d\theta}\right) \\ &= -2f_Q(q)\left(\|\nabla g\|\boldsymbol{\delta}^T\frac{d\mathbf{a_1}}{d\theta} + \boldsymbol{\alpha}^T\frac{d\nabla g}{d\theta}\left(\boldsymbol{\delta}^T\mathbf{a_1} - a_0\right) - \|\nabla g\|\frac{da_0}{d\theta}\right) - \sum_{i=1}^{N}\frac{dF_Q(q,\boldsymbol{\delta})}{d\delta_i}\left(\frac{\|\nabla g\|}{\lambda_i}\frac{da_{1_i}}{d\theta} + \frac{a_{1_i}}{\lambda_i}\frac{d\|\nabla g\|}{d\theta}\right)\end{aligned} \tag{40}$$

where

$$\frac{d\mathbf{a_1}}{d\theta} = \mathbf{T}^T\left(\mathbf{I} - \beta\frac{\mathbf{H}}{\|\nabla g\|}\right)\frac{d\boldsymbol{\alpha}}{d\theta} - \frac{\mathbf{T}^T\mathbf{H}\boldsymbol{\alpha}}{\|\nabla g\|}\frac{d\beta}{d\theta} + \frac{\beta\mathbf{T}^T\mathbf{H}\boldsymbol{\alpha}\boldsymbol{\alpha}^T}{\|\nabla g\|^2}\frac{d\nabla g}{d\theta} \tag{41}$$

$$\frac{d\nabla g}{d\theta} = -\nabla\left(\nabla^T g\frac{d\mathbf{u}}{d\theta}\right) + \mathbf{H}\left(\frac{d\beta}{d\theta}\boldsymbol{\alpha} + \beta\frac{d\boldsymbol{\alpha}}{d\theta}\right) \tag{42}$$

and $\frac{d\|\nabla g\|}{d\theta}$, $\frac{d\beta}{d\theta}$, $\frac{d\boldsymbol{\alpha}}{d\theta}$, and $\frac{da_0}{d\theta}$ are obtained from Eqs. (25), (37), (31) and (39), respectively. $\frac{d\mathbf{u}}{d\theta}$ in Eqs. (37) and (42) is obtained by the Rosenblatt transformation, whose i^{th} component is obtained as

$$\frac{du_i}{d\theta} = \frac{d\Phi^{-1}\left[F_{X_i}\left(x_i;\theta\right)\right]}{d\theta}. \tag{43}$$

For normally distributed independent random variables and when distribution parameter is j^{th} mean of random variable, $\frac{du_i}{d\theta}$ in Eq. (43) becomes

$$\frac{du_i}{d\mu_j} = -\delta_{ij}\frac{1}{\sigma_j} \tag{44}$$

where δ_{ij} is the Kronecker delta.

To calculate Eq. (40), $\frac{dF_Q\left(q,\boldsymbol{\delta}\right)}{d\delta_i}$ is also required to be calculated, which is obtained using FDM as

$$\frac{dF_Q\left(q,\boldsymbol{\delta}\right)}{d\delta_i} = \frac{F_Q\left(q,\delta_i^{'}\right) - F_Q\left(q,\delta_i\right)}{\delta_i^{'} - \delta_i} \tag{45}$$

where $\delta_i^{'}$ is perturbed value of δ_i. To calculate $\frac{dF_Q\left(q,\boldsymbol{\delta}\right)}{d\delta_i}$ in Eq. (45), $F_Q\left(q,\delta_i\right)$ is first calculated using general chi-squared distribution in Section 4.3. $F_Q\left(q,\delta_i^{'}\right)$ is then calculated after setting the appropriate value for $\delta_i^{'}$. During the calculation of $F_Q\left(q,\delta_i^{'}\right)$, another MPP search, which is iterative algorithm and thus can significantly affect efficiency of the calculation, is not involved and all other parameters except δ_i, which are used to calculate $F_Q\left(q,\delta_i\right)$, remain constant. Thus, efficiency is not lost during the calculation of $\frac{dF_Q\left(q,\boldsymbol{\delta}\right)}{d\delta_i}$ in Eq. (45). Additionally, accuracy is not lost either by setting very small perturbation size for $\delta_i^{'}$ due to the fact that $F_Q\left(q,\delta_i\right)$ and $F_Q\left(q,\delta_i^{'}\right)$ are exactly calculated.

6. Numerical Examples

The first two numerical studies are carried out in this section to verify the proposed sensitivity analysis for both low-dimensional and high-dimensional cases. The last numerical study is carried out to test that the sensitivity of higher-order performance function in terms of how the proposed assumption that the Hessian does not change with respect to distribution parameter affects the accuracy of the sensitivity calculation.

6.1 Sensitivity Using Novel SORM for Two-Dimensional Inputs

In this numerical example, the sensitivity of the probability of failure with respect to the design point is obtained using the analytic derivation in Section 5 for the two-dimensional performance function. The means of the random variables are used as design point.

Consider the following 2D performance function given in X-space as

$$G(\mathbf{X}) = X_1^2 + 2X_1 + X_2^2 + 2X_2 - 0.5X_1X_2 - 13, \tag{46}$$

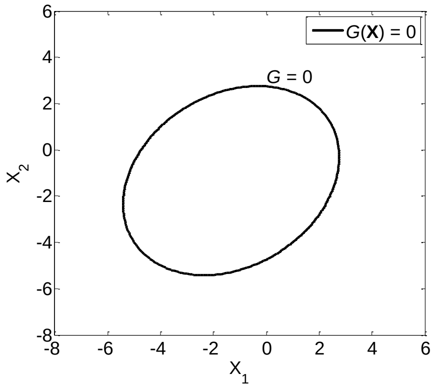

Figure 1. Performance Function

where $X_i \sim N(0,1)$ and they are statistically independent to each other, which is also shown in Fig. 1. The performance function in Eq. (46) in U-space becomes

$$g(\mathbf{U}) = (U_1-\mu_1)^2 + 2(U_1-\mu_1) + (U_2-\mu_2)^2 + 2(U_2-\mu_2) - 0.5(U_1-\mu_1)(U_2-\mu_2) - 13 \tag{47}$$

using the Rosenblatt transformation. Eq. (47) is further transformed to χ^2-space as [19]

$$\hat{g}_L(\mathbf{Z}) = \frac{3}{2}Z_1 + \frac{5}{2}Z_2 - \frac{94}{3} \tag{48}$$

where $Z_1 \sim \chi_1^2\left(\frac{32}{9}\right)$ and $Z_2 \sim \chi_1^2(0)$. The probability of failure then can be calculated using Eq. (48) and numerical integration as [19]

$$\begin{aligned} P_F &= \int_{\hat{g}_L(\mathbf{Z})>0} f_{\mathbf{Z}}(\mathbf{z})d\mathbf{z} = \int_0^\infty \left(\int_{\frac{188}{15}-0.6z_1}^\infty f_{Z_2}(z_2)dz_2 \right) f_{Z_1}(z_1)dz_1 \\ &= 1 - \int_0^\infty F_{Z_2}\left(\frac{188}{15} - 0.6z_1\right) f_{Z_1}(z_1)dz_1 = 1.0650\%. \end{aligned} \tag{49}$$

Table 1. Numerical Values of Parameters

Parameter	Numerical Value
$f_Q(q,\boldsymbol{\delta})$	0.002066
$dq/d\mu_1$	0
δ_i	$(-4\sqrt{2}/3, 0)$
$dF_Q(q,\boldsymbol{\delta})/d\delta_i$	$(0.022996, 0)$
$d\delta_i/d\theta$	$(-0.70711, -0.70711)$

The sensitivity of the probability of failure in Eq. (49) with respect to the means of the random variables is then calculated using Eq. (17), the general chi-squared distribution in Section 4.3, and the analytic derivation in Section 5. The parameters, which are necessary to calculate $\frac{dP_F}{d\mu_1}$, are obtained based on the derivation in Section 5 and they are shown Table 1. Using the result, $\frac{dP_F}{d\mu_1}$ is calculated as 0.016261. Likewise, $\frac{dP_F}{d\mu_2}$ is calculated as 0.016261.

Table 2. Comparison of Sensitivity Calculation

	Proposed Sensitivity	FDM using MCS (10 M)
$dP_F / d\mu_1$	0.016261	0.016252
$dP_F / d\mu_2$	0.016261	0.016253

The calculated sensitivity is then compared with the sensitivity obtained by FDM using MCS, which is shown in Table 2. According to Table 2, the sensitivity calculated based on the sensitivity analysis in Section 5 is very close to the sensitivity obtained by FDM using MCS, which is owing to the fact that the performance function is perfectly quadratic.

6.2 Sensitivity Using Novel SORM for High-Dimensional Inputs

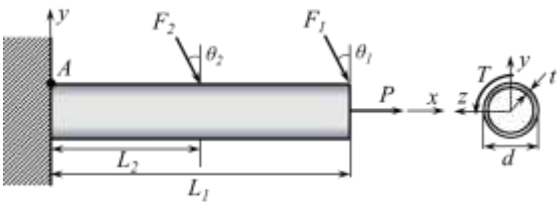

Figure 2. Schematic Diagram of Cantilever Tube

In this numerical example, the sensitivity of the probability of failure with respect to the design point is obtained using the analytic derivation in Section 5 for the high-dimensional performance function. Consider the cantilever tube shown in Fig. 2 subjected to external forces F_1, F_2, and P, and torsion T [32]. The high-dimensional performance function is defined as the difference between the yield strength of $190\,\text{MPa}$ and the maximum stress $\sigma_{\max}$, which is given as

$$G(\mathbf{X}) = \sigma_{\max} - 190\,\text{MPa} \tag{50}$$

where $\sigma_{\max}$ is the maximum von Mises stress on the top surface of the tube at the origin, which is given by

$$\sigma_{\max} = \sqrt{\sigma_x^2 + 3\tau_{zx}^2} \tag{51}$$

where the normal stress σ_x can be obtained as

$$\sigma_x = \frac{P + F_1 \sin\theta_1 + F_2 \sin\theta_2}{\frac{\pi}{4}\left[d^2 - (d-2t)^2\right]} + \frac{\left(F_1 L_1 \cos\theta_1 + F_2 L_2 \cos\theta_2\right)d}{2 \times \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]} \tag{52}$$

and the shear stress τ_{xz} can be obtained as

$$\tau_{xz} = \frac{Td}{4 \times \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]}, \tag{53}$$

Table 3. Properties of Random Variables

Variables	Mean	Standard Deviation	Distribution Type
$X_1(t)$	4 mm	0.04 mm	Normal
$X_2(d)$	40 mm	0.4 mm	Normal
$X_3(L_1)$	120 mm	$\sqrt{3}$ mm	Normal
$X_4(L_2)$	60 mm	$\sqrt{3}/2$ mm	Normal

$X_5(F_1)$	3.0 kN	0.3 kN	Normal
$X_6(F_2)$	3.0 kN	0.3 kN	Normal
$X_7(P)$	12.0 kN	1.2 kN	Normal
$X_8(T)$	90.0 $\text{N}\cdot\text{m}$	9.0 $\text{N}\cdot\text{m}$	Normal

respectively. The properties of the random variables used in Eqs. (53)-(56) are given in Table 3 and the means of random variables are used as design point. Also, they are all statistically independent to each other. Two angles are assumed to be fixed at $\theta_1 = 5^\circ$ and $\theta_2 = 10^\circ$.

Table 4. Comparison of Sensitivity Calculation

	Proposed Sensitivity	FDM using MCS (10 M)	Error (%)
$dP_F / d\mu_1$	-4.8394×10^{2}	-4.6396×10^{2}	4.3062
$dP_F / d\mu_2$	-1.3928×10^{2}	-1.3669×10^{2}	1.8993
$dP_F / d\mu_3$	1.2205×10^{1}	1.2084×10^{1}	9.9947×10^{-1}
$dP_F / d\mu_4$	1.1379×10^{1}	1.1372×10^{1}	6.1062×10^{-2}
$dP_F / d\mu_5$	4.3886×10^{-4}	4.4037×10^{-4}	3.4255×10^{-1}
$dP_F / d\mu_6$	2.2138×10^{-4}	2.2123×10^{-4}	4.2100×10^{-2}
$dP_F / d\mu_7$	2.9765×10^{-5}	2.9818×10^{-5}	1.7643×10^{-1}
$dP_F / d\mu_8$	3.4008×10^{-4}	3.5081×10^{-4}	3.0581

Using the above information and the analytic derivation in Section 5, the sensitivity is calculated, and the obtained sensitivity is then compared with the sensitivity obtained by FDM using MCS. Based on the mostly small percent errors shown in Table 4, it can be concluded that the proposed sensitivity analysis accurately calculates sensitivity. For a few of the random variables, relatively large errors compared to the previous example are generated, which is owing to the fact that the performance function is not quadratic in this example.

6.3 Sensitivity Using Novel SORM for Higher-Order Performance Function

In this numerical example, the sensitivities of the probability of failures with respect to the three design points are obtained using the analytic derivation in Section 5 for the higher-order performance function. The higher-order or highly non-linear performance function in X-space is given as

$$G(\mathbf{X}) = 15 + (Y-8)^2 + (Y-8)^3 + YZ \tag{54}$$

where $\begin{Bmatrix} Y \\ Z \end{Bmatrix} = \begin{bmatrix} 0.9063 & 0.4226 \\ 0.4226 & -0.9036 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$. The properties of the random variables at three design points are also

Table 5. Properties of Random Variables at Three Design Points

	X_1			X_2		
	Distribution Type	μ_1	σ_1	Distribution Type	μ_2	σ_2
Design Point 1	Normal	3	0.8	Normal	2	0.8
Design Point 2	Normal	4	0.8	Normal	5	0.8
Design Point 3	Normal	7	0.8	Normal	10	0.8

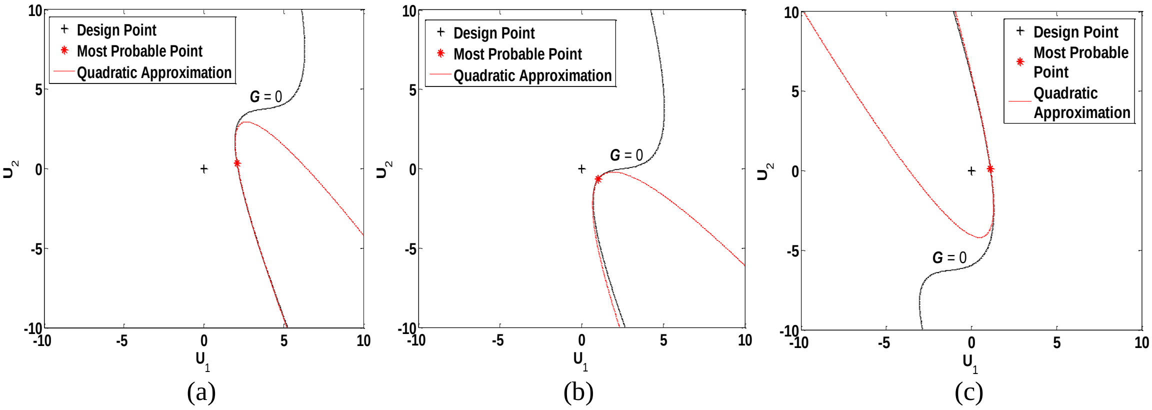

Figure 3. Quadratic Approximations in U-space at (a) Design Point 1, (b) Design Point 2, and (c) Design Point 3

shown in Table 5 and the means of the random variables are the design points. They are also all statistically independent to each other. In Table 5, standard deviations for both random variables at every design point are intentionally set to be large to generate noticeable amount of error in this example. As previously stated, the highly non-linear performance function given in Eq. (57) is not quadratic, and quadratic approximations performed at three design points in U-space are shown in Fig. 3.

Table 6. Comparison of Sensitivity Calculation at Three Design Points

		Novel SORM	FDM using MCS (10 M)	Error (%)
Design Point 1	$dP_F / d\mu_1$	5.0510×10^{-2}	5.0277×10^{-2}	1.4034
	$dP_F / d\mu_2$	6.4421×10^{-3}	7.1123×10^{-3}	9.4235
Design Point 2	$dP_F / d\mu_1$	1.3181×10^{-1}	1.3826×10^{-1}	4.6621
	$dP_F / d\mu_2$	-1.0629×10^{-1}	-9.9598×10^{-2}	6.7183
Design Point 3	$dP_F / d\mu_1$	2.6453×10^{-1}	2.7133×10^{-1}	2.5035
	$dP_F / d\mu_2$	3.3058×10^{-2}	3.3988×10^{-2}	2.7387

Using the above information and the analytic derivation in Section 5, the sensitivity is calculated at each design point, and it is then compared with the sensitivity obtained by FDM using MCS, which is shown in Table 6. As shown in Table 6, the generated errors are not large and they are within range from 1 to 10%.

7. Conclusions

In this study, the sensitivity of the probability of failure with respect to distribution parameter using the novel SORM has been derived through the sensitivity analysis. During the derivation of the sensitivity, sensitivity of eigenvalue of the Hessian matrix is first required. Based on the eigenvalue perturbation theory, sensitivity of eigenvalue with respect to distribution parameter is obtained. However, since it is inherent in SORM that the third-order derivative of performance function is not available, it is assumed that sensitivity of eigenvalue with respect to distribution parameter is zero. With the assumption, the sensitivity of the probability of failure with respect to distribution parameter in addition to all the necessary parameters are derived through sensitivity analysis. The calculation of the derived sensitivity includes calculation of the sensitivity of CDF of linear combination of non-central chi-square variables with respect to non-centrality parameter by FDM, which, however, does not require any iterative algorithm and repeating calculation of the parameters except the non-centrality parameter after perturbation. Thus, efficiency is not lost. Accuracy is not lost either since CDF of linear combination of non-central chi-square variables before and after perturbation are exactly calculated in novel SORM. The calculation of the derived sensitivity also requires probability density function (PDF) of a linear combination of non-central chi-square variables, which is obtained by utilizing general chi-squared distribution. In numerical examples, the derived sensitivity is applied to calculate sensitivity in both low- and high- dimensional examples. For the low-dimensional example that is perfectly quadratic and the high-dimensional example that is not quadratic, the obtained sensitivities based on the proposed sensitivity analysis are very close to those obtained by FDM using MCS. To further test the assumption how the Hessian matrix does not change affects the accuracy of the calculation of the sensitivity, the last numerical example is carried out with higher-order or highly non-linear performance function. The generated errors are not large and they are within acceptable ranges with the largest one below 10%. In conclusion, the sensitivity calculated by the proposed sensitivity analysis is efficient and accurate, and the assumption that the eigenvalue of the Hessian matrix does not change with respect to distribution parameter does not significantly affect the accuracy. In future research, RBDO utilizing the sensitivity analysis in this study will be performed. Also, the problem detected while doing research for this study that probability of failure and its sensitivity are not obtainable when one or some of the eigenvalues for the Hessian matrix is/are zero are necessarily to be resolved.

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