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Shape Optimization of Electrostatic Capacitive Sensor

Masayoshi Satake[1], Noboru Maeda[1], Shinji Fukui[1], and Hideyuki Azegami[2]

[1] NIPPON SOKEN INC., Nishio, Japan, {masayoshi_satake, noboru_maeda, shinji_fuki}@soken1.denso.co.jp
[2] Nagoya University, Nagoya, Japan, azegami@is.nagoya-u.ac.jp

1. Abstract
This paper describes how to construct a shape optimization problem of an electrostatic capacitive sensor for detecting fingers, and how to solve the problem. We define two main problems. One is a basic electrostatic field problem consisting of sensing electrode, earth electrode and air. The other is an electrostatic field problem adding fingers to the basic electrostatic field problem. An objective cost function is defined with the negative-signed squared H^1 norm of the difference between the solutions of two main problems. The volume of sensing electrode is used as a constraint cost function. Using the solutions of the two main problems and the two adjoint problems, we present evaluation method of the shape derivative of the objective cost function. To solve the shape optimization problem to minimize the negative-signed difference norm with the volume constraint, an iterative algorithm based on the H^1 gradient method is used. A computer program to solve the shape optimization problem is developed using a commercial software as solver to the boundary value problems. Numerical examples illustrate that reasonable shapes are obtained by the present approach.
2. Keywords: shape optimization, electromagnetics, electrostatic capacitive sensor, H^1 gradient method

3. Introduction
In recent years, many electronic devices are used in vehicle. Although an alternator is typical of in-vehicle electronic device in the 1960s, many electronic devices such as car-navigation systems, dedicated short range communication, hybrid systems, intelligent transport system and many electronic sensors, are loaded on recent vehicles. Hence, performance of electronic devices became important increasingly in design of car.
To improve performance of electric devices, shapes of electric devices can be targets to be optimized. However, it is not easy that designers of an electric device find the optimum shape which satisfies many design constraints such as mounting space, weight and cost.
The present paper aims to present the applicability of the shape optimization theory to the design problems in the electromagnetic field. In the previous papers, we presented a solution of shape optimization problem of domain in which a boundary value problem of a partial differential equation is defined [1–4]. In these papers, an iterative algorithm of reshaping by using the H^1 gradient method was used to find the optimum shape.
In the present paper, as the first step to apply the solution to shape optimization problems in the electromagnetic field, we choose a design problem of an electrostatic capacitive sensor. To define the sensing performance of the electrostatic capacitive sensor, we set two main problems. One is an electrostatic field problem consisting of sensing electrode with a certain electric potential, earth electrode and air. The other is an electrostatic field problem adding fingers to the former electrostatic field problem. Using the solution of the two main problems, we put the negative-signed squared H^1 norm of the difference between the solutions of two main problems as an objective cost function. In addition, the volume of the sensing electrode is used as a constraint cost function.
In the present paper, we discuss as follows. In Section 4, we define an initial domain of an electrostatic field and choose a mapping from the initial domain to varied domain as a design variable. In the varied domain, in Section 5, we formulate the two main problems based on the Maxwell equation. In Section 6, using the solutions to the two main problems, we formulate a shape optimization problem using the negative-signed squared H^1 norm of the difference between the solutions of two main problems as the objective cost function to minimize, and the volume of the sensing electrode as a constraint cost function. The evaluation methods for the Fréchet derivatives of the cost functions with respect to the domain variation, which we call the shape derivatives of the cost functions, are shown in Section 7. Using these shape derivatives of the cost functions, we present a method to obtain the domain mappings that decrease

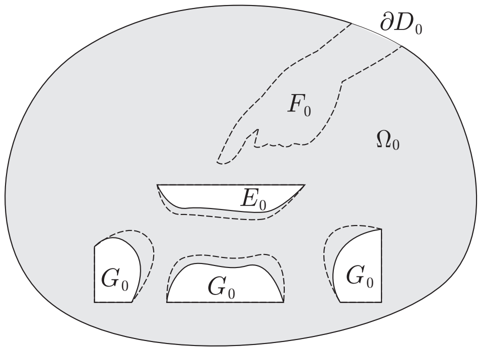


Figure 1: Initial domain $\Omega_0 = D_0 \setminus \left(\bar{E}_0 \cup \bar{G}_0\right)$ for the electrostatic field problem

the cost functions in Section 8. A scheme to solve the shape optimization problem with constraints is presented in Section 9. Finally, in Section 10, we show the numerical results for shape optimizations of electrostatic capacitive sensors.

4. Initial domain and set of domain variations

Let us define an initial domain for an electrostatic field as depicted in Fig. 1. Let D_0 be a $d \in \{2, 3\}$ dimensional bounded domain in which electrostatic fields are allowed to be designed. Moreover, let E_0, G_0 and F_0 be the subsets of D_0 showing the initial domains of sensing electrode with a certain positive electric potential, earth electrode with zero electric potential and detection object such as human fingers with zero electric potential, respectively. In the present paper, we use ∂D_0 as the boundary of D_0, and $\bar{D}_0$ as $D_0 \cup \partial D_0$. Based on the definitions, we assume $\Omega_0 = D_0 \setminus \left(\bar{E}_0 \cup \bar{G}_0\right)$ is the initial domain of electrostatic field. To define a shape optimization problem of Ω_0, its boundary $\partial\Omega_0$ is required to be at least the Lipschitz boundary, i.e., the $C^{0,1}$ class.

In the present paper, we use the notation $W^{s,p}\left(\Omega_0; \mathbb{R}^d\right)$ to denote the Sobolev space for the set of functions defined in Ω_0 and having values in $\mathbb{R}^d$ that are $s \in [0, \infty]$ times differentiable and $p \in [1, \infty]$-th order Lebesgue integrable, and call its smoothness the $W^{s,p}$ class. The notation $H^s\left(\Omega_0; \mathbb{R}^d\right)$ and $C^{s,\alpha}$ for $\alpha \in (0, 1]$ are used as $W^{s,2}\left(\Omega_0; \mathbb{R}^d\right)$ and $W^{s+\alpha,\infty}\left(\Omega_0; \mathbb{R}^d\right)$.

We assume that D_0 and F_0 are fixed in the shape optimization problem. Domain variation is assumed to be given by a map $\boldsymbol{\phi} : \bar{D}_0 \to \mathbb{R}^d$ belonging to the admissible set

$$\begin{aligned} \mathcal{D} = \Big\{ \boldsymbol{\phi} \in W^{1,\infty}\left(D_0; \mathbb{R}^d\right) \;\Big|\; & \|\boldsymbol{\phi} - \boldsymbol{\phi}_0\|_{W^{1,\infty}(D_0;\, \mathbb{R}^d)} < 1,\ \boldsymbol{\phi}(\Omega_0) \subseteq D_0, \\ & \boldsymbol{\phi} = \boldsymbol{\phi}_0 \text{ on } \partial D_0 \cup \partial F_0 \cup \partial G_0 \cup \Gamma_{\mathrm{C}0} \Big\} \end{aligned} \tag{1}$$

where $\boldsymbol{\phi}_0$ denotes an identity mapping, and $\Gamma_{\mathrm{C}0} \subset \partial\Omega_0$ denotes boundary to be fixed in shape optimization problem from design demands. $\|\boldsymbol{\phi} - \boldsymbol{\phi}_0\|_{W^{1,\infty}(D_0;\, \mathbb{R}^d)} < 1$ is used so that $\boldsymbol{\phi} \in \mathcal{D}$ is a one-to-one mapping. With respect to $\boldsymbol{\phi} \in \mathcal{D}$, we denote the new domain $\{\boldsymbol{\phi}(\boldsymbol{x}) \mid \boldsymbol{x} \in \Omega_0\}$ as $\Omega(\boldsymbol{\phi})$.

5. Main problem

Using the definition of domains, let us define main problems for an electrostatic capacitive sensor. In the case of an electrostatic field or a magnetostatic field, the Maxwell's equations result in the independent Poisson equations of electric and magnetic fields, respectively. Here, we introduce the electric potential $u : D_0 \to \mathbb{R}$. Then, using the constitutive equation of electric field, the equation of electric field can be written as

$$\Delta u = \boldsymbol{\nabla} \cdot \boldsymbol{e}(u) = \frac{\rho}{\varepsilon} \quad \text{in } \Omega(\boldsymbol{\phi})$$

where $\boldsymbol{e}(u) = -\boldsymbol{\nabla} u$ is the electric field, and ρ and ε denote the charge density and the electric permittivity, respectively. The boundary conditions for main problems are defined as follows.

Here, we will define the admissible set of u for $q > d$ as

$$\mathcal{S} = \left\{ u \in W^{1,2q}\left(D_0; \mathbb{R}\right) \;\middle|\; C^1 \text{ class on } \partial E(\boldsymbol{\phi}) \cup \partial G(\boldsymbol{\phi}),\ \boldsymbol{\phi} \in \mathcal{D} \right\}. \tag{2}$$

The conditions in $\mathcal{S}$ are going to be used in the process of deriving the shape derivative of the objective cost function defined below.

Based on the definitions, we define one of the main problems as follows.

Problem 1 (Basic electrostatic field) For $\boldsymbol{\phi} \in \mathcal{D}$ and a positive constant α, find $u \in \mathcal{S}$ such that

$$\begin{aligned}
&-\boldsymbol{\nabla} \cdot \boldsymbol{e}(u) = 0 \quad \text{in } \Omega(\boldsymbol{\phi}) = D_0 \setminus (E(\boldsymbol{\phi}) \cup G(\boldsymbol{\phi})),\\
&\partial_\nu u = 0 \quad \text{on } \partial D_0(\boldsymbol{\phi}),\\
&u = \alpha \quad \text{on } \partial E(\boldsymbol{\phi}),\\
&u = 0 \quad \text{on } \partial G(\boldsymbol{\phi}).
\end{aligned}$$

Problem 1 is going to be used as a equality constraint in the shape optimization problem shown below. Then, we will define the Lagrange function for Problem 1 as

$$\begin{aligned}
\mathscr{L}_{\mathrm{N}}(\boldsymbol{\phi}, u, v) = &-\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{e}(u) \cdot \boldsymbol{e}(v)\, \mathrm{d}x + \int_{\partial E(\boldsymbol{\phi})} \{(u-\alpha)\,\partial_\nu v + v\partial_\nu u\}\, \mathrm{d}\gamma\\
&+ \int_{\partial G(\boldsymbol{\phi})} (u\partial_\nu v + v\partial_\nu u)\, \mathrm{d}\gamma
\end{aligned} \tag{3}$$

for $u \in \mathcal{S}$, $v \in \mathcal{S}$. If u is the solution of Problem 1,

$$\mathscr{L}_{\mathrm{N}}(\boldsymbol{\phi}, u, v) = 0$$

holds for all $v \in \mathcal{S}$.
As the second main problem, we define the electrostatic field with object F_0.

Problem 2 (Electrostatic field with object) For $\boldsymbol{\phi} \in \mathcal{D}$ and positive constant α, find $u_{\mathrm{F}} \in \mathcal{S}$ such that

$$\begin{aligned}
&-\boldsymbol{\nabla} \cdot \boldsymbol{e}(u_{\mathrm{F}}) = 0 \quad \text{in } \Omega(\boldsymbol{\phi}) \setminus F_0 = D_0 \setminus (E(\boldsymbol{\phi}) \cup G(\boldsymbol{\phi}) \cup F_0),\\
&\partial_\nu u_{\mathrm{F}} = 0 \quad \text{on } \partial D_0(\boldsymbol{\phi}),\\
&u_{\mathrm{F}} = \alpha \quad \text{on } \partial E(\boldsymbol{\phi}),\\
&u_{\mathrm{F}} = 0 \quad \text{on } \partial G(\boldsymbol{\phi}) \cup \partial F_0.
\end{aligned}$$

As the same manner, we will define the Lagrange function for Problem 2 as

$$\begin{aligned}
\mathscr{L}_{\mathrm{F}}(\boldsymbol{\phi}, u_{\mathrm{F}}, v_{\mathrm{F}}) = &-\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{e}(u_{\mathrm{F}}) \cdot \boldsymbol{e}(v_{\mathrm{F}})\, \mathrm{d}x\\
&+ \int_{\partial E(\boldsymbol{\phi})} \{(u_{\mathrm{F}}-\alpha)\,\partial_\nu v_{\mathrm{F}} + v_{\mathrm{F}}\partial_\nu u_{\mathrm{F}}\}\, \mathrm{d}\gamma + \int_{\partial G(\boldsymbol{\phi})} (u_{\mathrm{F}}\partial_\nu v_{\mathrm{F}} + v_{\mathrm{F}}\partial_\nu u_{\mathrm{F}})\, \mathrm{d}\gamma
\end{aligned} \tag{4}$$

for $u_{\mathrm{F}} \in \mathcal{S}$, $v_{\mathrm{F}} \in \mathcal{S}$. If u_{F} is the solution of Problem 2,

$$\mathscr{L}_{\mathrm{F}}(\boldsymbol{\phi}, u_{\mathrm{F}}, v_{\mathrm{F}}) = 0$$

holds for all $v_{\mathrm{F}} \in \mathcal{S}$.

6. Shape optimization problem

Using the solutions of the main problems, let us construct a shape optimization problem for an electrostatic capacitive sensor. To improve the sensibility for detecting fingers, we define

$$\begin{aligned}
f_0(\boldsymbol{\phi}, u, u_{\mathrm{F}}) &= -\|u - u_{\mathrm{F}}\|^2_{H^1(\Omega(\boldsymbol{\phi})\setminus F_0(\boldsymbol{\phi});\mathbb{R})}\\
&= -\int_{\Omega(\boldsymbol{\phi})\setminus F_0} \left\{(u-u_{\mathrm{F}})^2 + (\boldsymbol{e}(u) - \boldsymbol{e}(u_{\mathrm{F}})) \cdot (\boldsymbol{e}(u) - \boldsymbol{e}(u_{\mathrm{F}}))\right\} \mathrm{d}x
\end{aligned} \tag{5}$$

as the objective cost function to be minimized. Moreover, we define

$$f_1(\boldsymbol{\phi}) = \int_{E(\boldsymbol{\phi})} \mathrm{d}x - c_1, \tag{6}$$

as a constraint cost function, where c_1 is a positive constant for which there exists some $\boldsymbol{\phi} \in \mathcal{D}$ such that $f_1(\boldsymbol{\phi}) \leq 0$.
Using these cost functions, we define the shape optimization problem as follows.

Problem 3 (Shape optimization problem) Let $\mathcal{D}$ and $\mathcal{S}$ be defined in Eq. (1) and Eq. (2). For $\boldsymbol{\phi} \in \mathcal{D}$, let $u \in \mathcal{S}$ and $u_{\mathrm{F}} \in \mathcal{S}$ be the solutions of Problem 1 and Problem 2. For f_0 and f_1 defined in Eq. (5) and Eq. (6), respectively, find $\Omega(\boldsymbol{\phi})$ such that

$$\min_{\boldsymbol{\phi} \in \mathcal{D}} \Big\{ f_0(\boldsymbol{\phi}, u, u_{\mathrm{F}}) \mid f_1(\boldsymbol{\phi}) \le 0, \\ u \in \mathcal{S}, \text{ Problem 1}, \ u_{\mathrm{F}} \in \mathcal{S}, \text{ Problem 2} \Big\}.$$

7. Shape derivatives of cost functions

To solve Problem 3, we will use an iterative algorithm of reshaping by using the H^1 gradient method. In the H^1 gradient method, the shape derivatives of cost functions are used. Then, let us derive the shape derivatives here.

Since the objective cost function $f_0(\boldsymbol{\phi}, u, u_{\mathrm{F}})$ contain u and u_{F}, we have to consider that the two main problems are equality constraints. Hence, we put

$$\begin{aligned}
\mathscr{L}_0(\boldsymbol{\phi}, u, v_0, u_{\mathrm{F}}, v_{\mathrm{F}0}) &= f_0(\boldsymbol{\phi}, u, u_{\mathrm{F}}) + \mathscr{L}_{\mathrm{N}}(\boldsymbol{\phi}, u, v_0) - \mathscr{L}_{\mathrm{F}}(\boldsymbol{\phi}, u_{\mathrm{F}}, v_{\mathrm{F}0}) \\
&= \int_{\Omega(\boldsymbol{\phi}) \setminus F_0} \Big\{ -(u - u_{\mathrm{F}})^2 - (\boldsymbol{e}(u) - \boldsymbol{e}(u_{\mathrm{F}})) \cdot (\boldsymbol{e}(u) - \boldsymbol{e}(u_{\mathrm{F}})) + \boldsymbol{e}(u_{\mathrm{F}}) \cdot \boldsymbol{e}(v_{\mathrm{F}0}) \Big\} \mathrm{d}x \\
&\quad - \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{e}(u) \cdot \boldsymbol{e}(v_0) \, \mathrm{d}x \\
&\quad + \int_{\partial E(\boldsymbol{\phi})} \{ (u - \alpha) \partial_n u v_0 + v_0 \partial_\nu u - (u_{\mathrm{F}} - \alpha) \partial_\nu v_{\mathrm{F}0} - v_{\mathrm{F}0} \partial_\nu u_{\mathrm{F}} \} \, \mathrm{d}\gamma \\
&\quad + \int_{\partial G(\boldsymbol{\phi})} (u \partial_\nu v_0 + v_0 \partial_\nu u - u_{\mathrm{F}} \partial_\nu v_{\mathrm{F}0} - v_{\mathrm{F}0} \partial_\nu u_{\mathrm{F}}) \, \mathrm{d}\gamma
\end{aligned} \tag{7}$$

as the Lagrange function for $f_0(\boldsymbol{\phi}, u, u_{\mathrm{F}})$. The shape derivative of $\mathscr{L}_0$ with respect to arbitrary domain variation $\boldsymbol{\varphi} \in W^{1,\infty}(D_0; \mathbb{R}^d)$ can be obtained, by applying the formulae of shape derivatives for domain and boundary integrals [5], as

$$\begin{aligned}
\dot{\mathscr{L}}_0(\boldsymbol{\phi}, u, v_0, u_{\mathrm{F}}, v_{\mathrm{F}0})[\boldsymbol{\varphi}] &= f_{0u}(\boldsymbol{\phi}, u, u_{\mathrm{F}})[\dot{u}] + \mathscr{L}_{\mathrm{N}u}(\boldsymbol{\phi}, u, v_0)[\dot{u}] \\
&\quad + \mathscr{L}_{\mathrm{N}v_0}(\boldsymbol{\phi}, u, v_0)[\dot{v}_0] \\
&\quad + f_{0u_{\mathrm{F}}}(\boldsymbol{\phi}, u, u_{\mathrm{F}})[\dot{u}_{\mathrm{F}}] - \mathscr{L}_{\mathrm{F}u_{\mathrm{F}}}(\boldsymbol{\phi}, u_{\mathrm{F}}, v_{\mathrm{F}0})[\dot{u}_{\mathrm{F}}] \\
&\quad - \mathscr{L}_{\mathrm{F}v_{\mathrm{F}0}}(\boldsymbol{\phi}, u_{\mathrm{F}}, v_{\mathrm{F}0})[\dot{v}_{\mathrm{F}0}] \\
&\quad + \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle .
\end{aligned} \tag{8}$$

Here, the 3rd and the 6th terms of the right-hand side of Eq. (8) become 0, if u and u_{F} are the solutions of Problem 1 and Problem 2, respectively. Moreover, the 1st and the 2nd terms of the right-hand side of Eq. (8) become 0, if v_0 is the solutions of the following problem.

Problem 4 (Adjoint problem for basic electrostatic problem) For $\boldsymbol{\phi} \in \mathcal{D}$, let u be the solution of Problem 1. Find $v_0 \in \mathcal{S}$ such that

$$\begin{aligned}
&-\boldsymbol{\nabla} \cdot \boldsymbol{e}(v_0) = 2\boldsymbol{\nabla} \cdot (\boldsymbol{e}(u) - \boldsymbol{e}(u_{\mathrm{F}})) - 2(u - u_{\mathrm{F}}) \\
&\qquad \text{in } \Omega(\boldsymbol{\phi}) = D_0 \setminus (E(\boldsymbol{\phi}) \cup G(\boldsymbol{\phi}) \cup F_0), \\
&-\boldsymbol{\nabla} \cdot \boldsymbol{e}(v_0) = 0 \quad \text{in } F_0, \\
&\partial_\nu v_0 = 0 \quad \text{on } \partial D_0(\boldsymbol{\phi}), \\
&v_0 = 0 \quad \text{on } \partial E(\boldsymbol{\phi}) \cup \partial G(\boldsymbol{\phi})
\end{aligned}$$

Moreover, the 4th and the 5th terms of the right-hand side of Eq. (8) become 0, if $v_{\mathrm{F}0}$ is the solution of the following problem.

Problem 5 (Adjoint problem for electrostatic field with object) For $\boldsymbol{\phi} \in \mathcal{D}$, let u_{F} be the solution of Problem 2. Find $v_{\mathrm{F}0} \in \mathcal{S}$ such that

$$
\begin{aligned}
&-\boldsymbol{\nabla} \cdot \boldsymbol{e}\left(v_{\mathrm{F}0}\right) = 2\boldsymbol{\nabla} \cdot \left(\boldsymbol{e}\left(u\right) - \boldsymbol{e}\left(u_{\mathrm{F}}\right)\right) - 2\left(u - u_{\mathrm{F}}\right) \\
&\qquad \text{in } \Omega\left(\boldsymbol{\phi}\right) \setminus F_0 = D_0 \setminus \left(E\left(\boldsymbol{\phi}\right) \cup G\left(\boldsymbol{\phi}\right) \cup F_0\right), \\
&\partial_\nu v_{\mathrm{F}0} = 0 \quad \text{on } \partial D_0\left(\boldsymbol{\phi}\right), \\
&v_{\mathrm{F}0} = 0 \quad \text{on } \partial E\left(\boldsymbol{\phi}\right) \cup \partial G\left(\boldsymbol{\phi}\right) \cup \partial F_0.
\end{aligned}
$$

Then, if u, u_{F}, v_0 and $v_{\mathrm{F}0}$ are the solutions of Problem 1, Problem 2, Problem 4 and Problem 5, respectively, $\mathscr{L}_0\left(\boldsymbol{\phi}, u, v_0, u_{\mathrm{F}}, v_{\mathrm{F}0}\right)[\boldsymbol{\varphi}]$ in Eq. (8) becomes

$$
\begin{aligned}
\left\langle \boldsymbol{g}_0, \boldsymbol{\varphi} \right\rangle = &\int_{\Omega(\boldsymbol{\phi})\setminus F_0} \Big[2\left(\boldsymbol{e}\left(u\right) - \boldsymbol{e}\left(u_{\mathrm{F}}\right)\right) \cdot \left\{ \boldsymbol{\nabla}\boldsymbol{\varphi}^{\mathrm{T}} \left(\boldsymbol{e}\left(u\right) - \boldsymbol{e}\left(u_{\mathrm{F}}\right)\right) \right\} \\
&- \boldsymbol{e}\left(u_{\mathrm{F}}\right) \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{e}\left(v_{\mathrm{F}0}\right)\right) - \boldsymbol{e}\left(v_{\mathrm{F}0}\right) \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{e}\left(u_{\mathrm{F}}\right)\right) \\
&+ \Big\{ -\left(u - u_{\mathrm{F}}\right)^2 - \left(\boldsymbol{e}\left(u\right) - \boldsymbol{e}\left(u_{\mathrm{F}}\right)\right) \cdot \left(\boldsymbol{e}\left(u\right) - \boldsymbol{e}\left(u_{\mathrm{F}}\right)\right) + \boldsymbol{e}\left(u_{\mathrm{F}}\right) \cdot \boldsymbol{e}\left(v_{\mathrm{F}0}\right) \Big\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \Big] \mathrm{d}x \\
&+ \int_{\Omega(\boldsymbol{\phi})} \Big\{ \boldsymbol{e}\left(u\right) \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{e}\left(v_0\right)\right) + \boldsymbol{e}\left(v_0\right) \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{e}\left(u\right)\right) - \boldsymbol{e}\left(u\right) \cdot \boldsymbol{e}\left(v_0\right) \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \Big\} \mathrm{d}x. \qquad (9)
\end{aligned}
$$

Here, we used the Diriclet conditions in Problem 1, Problem 2, Problem 4 and Problem 5.
On the other hand, for $f_1\left(\boldsymbol{\phi}\right)$,

$$
f_1'\left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}] = \int_{E(\boldsymbol{\phi})} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \ \mathrm{d}x = \left\langle \boldsymbol{g}_1, \boldsymbol{\varphi} \right\rangle \qquad (10)
$$

is obtained.
We call $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$ the shape derivatives of f_0 and f_1, respectively.

8. The H^1 gradient method

The H^1 gradient method is proposed as a method for finding the variation of the design variable, such as the domain mapping or the density parameter that decreases a cost function, as a solution to a boundary value problem of an elliptic partial differential equation [2,3,6]. In the case that a shape derivative $\boldsymbol{g}_i$ of a cost function $f_i\left(\boldsymbol{\phi}\right)$ for $i \in \{0, 1\}$, the H^1 gradient method can be described as follows.

Problem 6 (H^1 gradient method for shape optimization) Let X be a Hilbert space of $H^1\left(D_0; \mathbb{R}^d\right)$, and let $a : X \times X \to \mathbb{R}$ be a coercive bilinear form on X such that there exists $\beta > 0$ that satisfies

$$
a\left(\boldsymbol{w}, \boldsymbol{w}\right) \geq \beta \left\|\boldsymbol{w}\right\|_X^2
$$

for all $\boldsymbol{w} \in X$. For $\boldsymbol{g}_i \in X'$ (dual space of X), which is a Fréchet derivative of cost function $f\left(\boldsymbol{\phi}\right)$ at $\boldsymbol{\phi} \in X$, find $\boldsymbol{\varphi}_{gi} \in X$ such that

$$
a\left(\boldsymbol{\varphi}_{gi}, \boldsymbol{w}\right) = -\left\langle \boldsymbol{g}_i, \boldsymbol{w} \right\rangle \qquad (11)
$$

for all $\boldsymbol{w} \in X$.

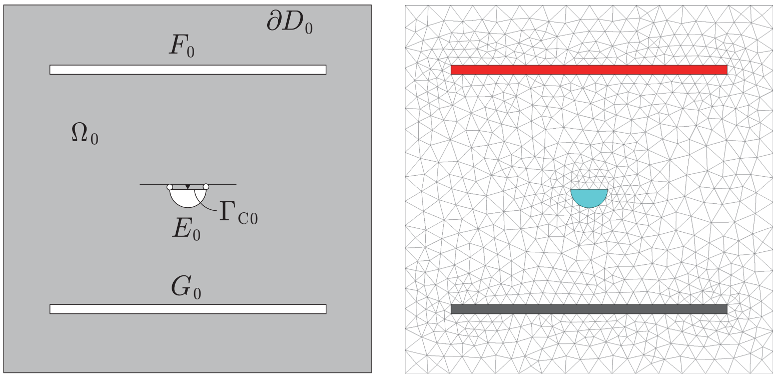
Problem 6 can be solved numerically with the standard finite element method by considering that Eq. (11) is a weak form of a boundary value problem of an elliptic partial differential equation. In the present paper, we use

$$
a\left(\boldsymbol{\varphi}, \boldsymbol{\psi}\right) = c_a \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{E}\left(\boldsymbol{\varphi}\right) \cdot \boldsymbol{E}\left(\boldsymbol{\psi}\right) + c_b \boldsymbol{\varphi} \cdot \boldsymbol{\psi}\right) \mathrm{d}x \qquad (12)
$$

for $\boldsymbol{\varphi} \in X$ and $\boldsymbol{\psi} \in X$, where

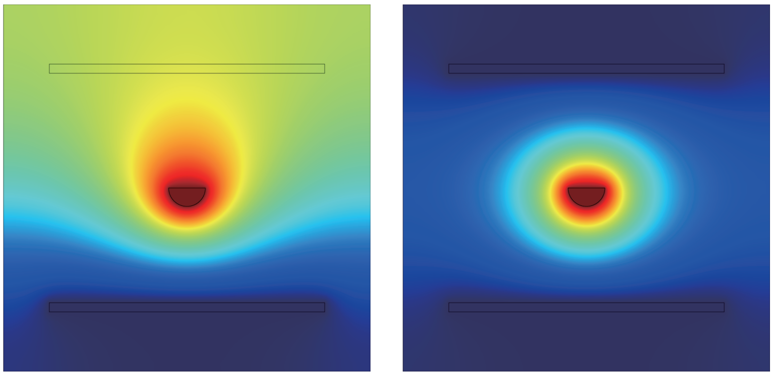
$$
\boldsymbol{E}\left(\boldsymbol{\varphi}\right) = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\mathrm{T}} + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\mathrm{T}}\right)^{\mathrm{T}} \right),
$$

and c_a and c_b are positive constants.
If u and u_{F} are included in $\mathcal{S}$, we can confirm that the solution $\boldsymbol{\varphi}_{g0}$ of Problem 6 belongs to $W^{1,\infty}\left(D_0; \mathbb{R}^d\right)$.



(a) Problem setting (b) Finite element model

Figure 2: Example 1: Two dimensional electrostatic field with parallel electrodes



(a) u (without object F_0) (b) u_F (with object F_0)

Figure 3: Example 1: Electric potentials in the initial domain

9. Solution to the shape optimization problem

We use an iterative method based on sequential quadratic programming to solve Problem 3. To determine the domain variation decreasing $f_0(\boldsymbol{\phi}, u, u_F)$ while satisfying $f_1(\boldsymbol{\phi}) \le 0$, we used the solution of the following problem. In this section, we denote $f_0(\boldsymbol{\phi}, u, u_F)$ as $f_0(\boldsymbol{\phi})$ and its shape derivative as $\boldsymbol{g}_0$.

Problem 7 (SQ approximation) For $\boldsymbol{\phi} \in \mathcal{D}$, let $\boldsymbol{g}_i$ be the shape derivatives of $f_i(\boldsymbol{\phi})$ for $i \in \{0, 1\}$, and let $f_1(\boldsymbol{\phi}) \le 0$. Let $a(\,\cdot\,,\,\cdot\,)$ be given as in Eq. (11). Find $\boldsymbol{\varphi}$ such that

$$\min_{\boldsymbol{\varphi} \in W^{1,\infty}(D_0;\mathbb{R}^d)} \left\{ q(\boldsymbol{\varphi}) = \frac{1}{2} a(\boldsymbol{\varphi}, \boldsymbol{\varphi}) + \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle \;\middle|\; f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle \le 0 \right\}.$$

The Lagrange function of Problem 8 is defined as

$$\mathscr{L}_{\mathrm{SQ}}(\boldsymbol{\varphi}, \lambda_1) = q(\boldsymbol{\varphi}) + \lambda_1 (f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle)$$

where $\lambda_1 \in \mathbb{R}$ is the Lagrange multiplier for the constraint $f_1(\boldsymbol{\varphi}) \le 0$. The Karush–Kuhn–Tucker conditions for Problem 8 are given as

$$a(\boldsymbol{\varphi}, \boldsymbol{\varphi}) + \langle \boldsymbol{g}_0 + \lambda_1 \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle = 0, \tag{13}$$

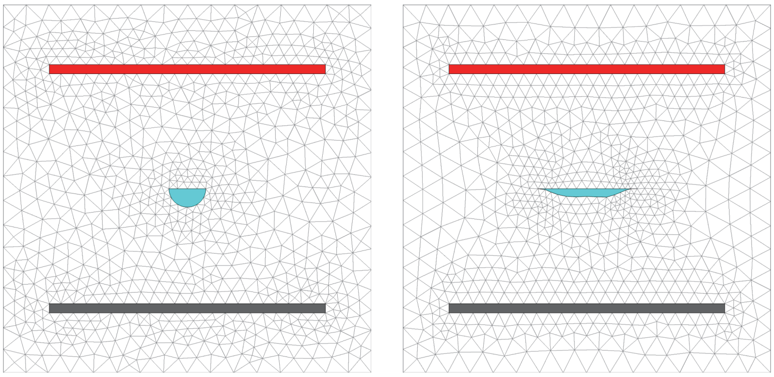
$$f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle \le 0, \tag{14}$$

$$\lambda_1 (f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle) = 0, \tag{15}$$

$$\lambda_1 \ge 0 \tag{16}$$

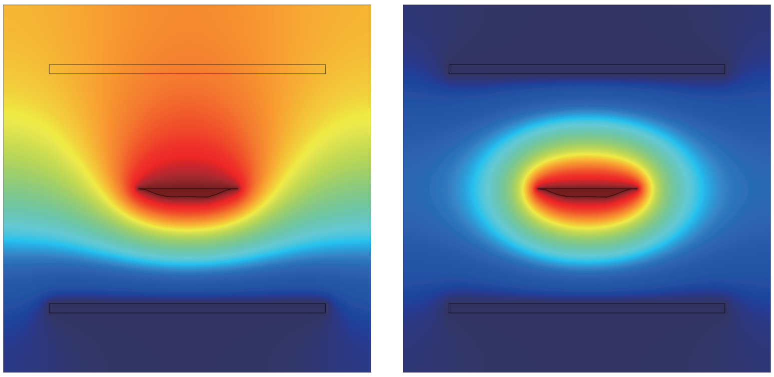
for all $\boldsymbol{\varphi} \in W^{1,\infty}\left(D_0; \mathbb{R}^d\right)$. Here, let $\boldsymbol{\varphi}_{gi}$ for $i \in \{0, 1\}$ be the solutions to Problem 6, and set

$$\boldsymbol{\varphi}_g = \boldsymbol{\varphi}_{g0} + \lambda_1 \boldsymbol{\varphi}_{g1}. \tag{17}$$



(a) Initial (b) Optimized

Figure 4: Example 1: Shapes before and after domain variation



(a) u (without object F_0) (b) u_{F} (with object F_0)

Figure 5: Example 1: Electric potentials after optimized

Then, by substituting $\boldsymbol{\varphi}_g$ of Eq. (17) for $\boldsymbol{\varphi}$ in Eq. (13), Eq. (13) holds. If the constraint in Eq. (14) is active, i.e., Eq. (14) holds with the equality, we have

$$\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g1} \rangle \lambda_1 = -f_1(\boldsymbol{\phi}) + \langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g0} \rangle . \tag{18}$$

Equation (18) has a unique solution of λ_1. Moreover, if $f_1(\boldsymbol{\phi}) = 0$, we have

$$\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g1} \rangle \lambda_1 = -\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g0} \rangle . \tag{19}$$

Since Eq. (19) is independent of the magnitude of $\boldsymbol{\varphi}_{g0}$ and $\boldsymbol{\varphi}_{g1}$ to determine λ_1, Eq. (19) is used in the numerical scheme for the initial domain Ω_0 in which we assume $f_1(\boldsymbol{\phi}) = 0$ is satisfied. If $\lambda_1 < 0$ in the solution λ_1 to Eq. (18) or Eq. (19), by putting $\lambda_1 = 0$, we have λ_1 satisfying Eq. (13) to Eq. (16). The detail of the numerical scheme is shown in the previous paper [3].

10. Numerical examples

In the present paper, we developed a computer program by JAVA API using a commercial software "COMSOL Multiphysics" as solver to the boundary value problems.
The problem setting and the finite element mesh of Example 1 is shown in Fig. 2. Here, we assume that the domain variation is fixed in the normal direction on $\Gamma_{\mathrm{C}0}$ and perfectly fixed at the center point of $\Gamma_{\mathrm{C}0}$. Figure 3 shows electric potentials u and u_{F} for the initial domain. From the problem setting, we can consider that a flatter shape decreases the cost function more. Actually, the optimized shape shown in Figure 4 (b) is flatter than the initial shape (a). From the comparison of the electric potentials after the domain variation shown in Fig. 5 with the initial electric potentials in Fig. 3, we can observe that the difference between u and u_{F} increased. The iteration history of the cost functions with respect to number of reshaping shown in Fig. 6 shows the quantitative data of decreasing f_0 under constant f_1.

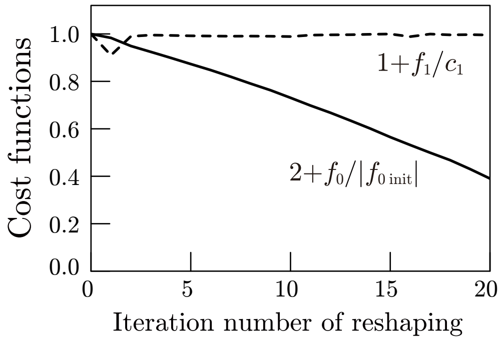
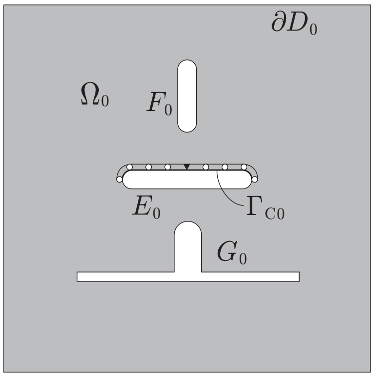
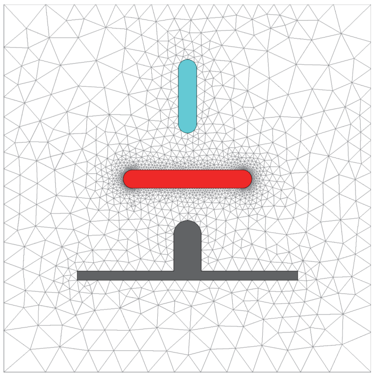


Figure 6: Example 1: Iteration history of cost functions for reshaping

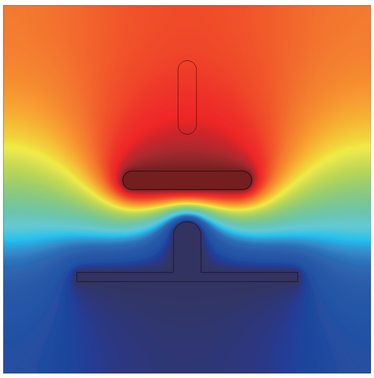


(a) Problem setting

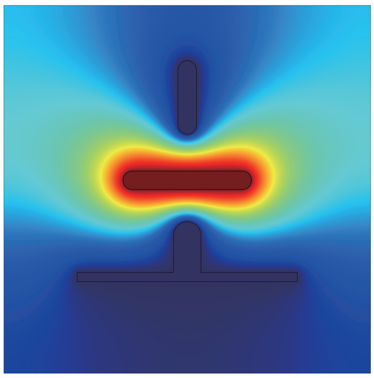


(b) Finite element model

Figure 7: Example 2: Two dimensional electrostatic field with protrusion earth electrode



(a) u (without object F_0)

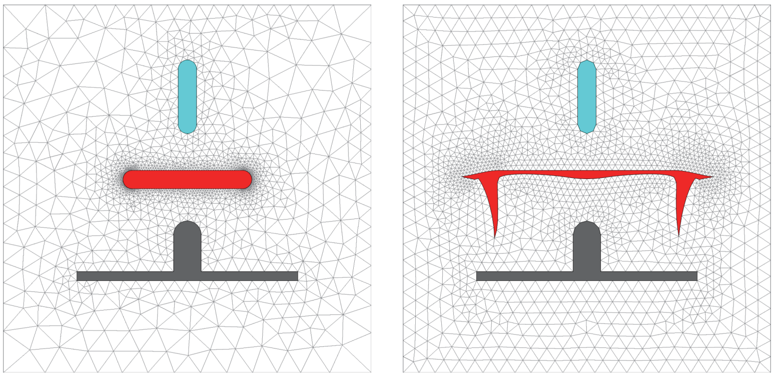


(b) u_{F} (with object F_0)

Figure 8: Example 2: Electric potentials in the initial domain

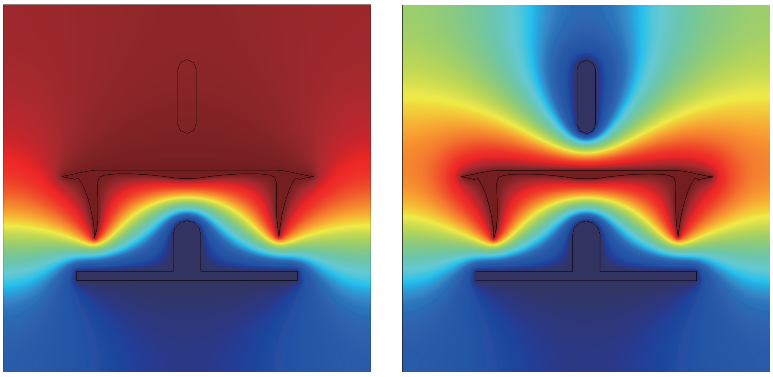
However, after the 20th iteration of reshaping, we encountered the trouble of mesh distortion. In order to advance the analysis, some re-meshing method is needed.

Moreover, we examined Example 2 as shown in Fig. 7. Here, we assumed that the domain variation was fixed in the normal direction on $\Gamma_{\mathrm{C}0}$ and fixed perfectly at the center point of $\Gamma_{\mathrm{C}0}$. Figure 8 shows electric potentials u and u_{F} for the initial domain. The shape obtained by the present method is shown in Figure 9 (b). The electric potentials after the domain variation is shown in Fig. 10. From the comparison between Fig. 8 and Fig. 10, we can observe the difference between u and u_{F} increased by the domain variation. The iteration history of the cost functions for reshaping is shown in Fig. 11. This result shows that f_0 surely decreases under satisfying the constraint condition $f_1 \leq 0$. The trouble of mesh distortion generated also in Example 2 after the 8th iteration of reshaping. Re-meshing is needed to continue the



(a) Initial (b) Optimized

Figure 9: Example 2: Shapes before and after domain variation



(a) u (without object F_0) (b) u_{F} (with object F_0)

Figure 10: Example 2: Electric potentials after optimized

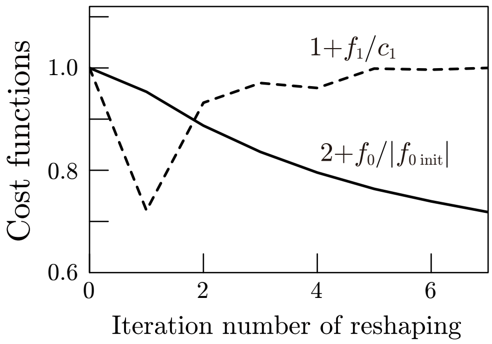


Figure 11: Example 2: Iteration history of cost functions for reshaping

analysis.

11. Conclusions

In the present paper, we formulated a shape optimization problem of an electrostatic field concerning a capacitive sensor for detecting fingers. The sensibility for detecting fingers is defined with the cost function of the negative-signed squared H^1 norm of the difference between the solutions of two main problems. One is a basic electrostatic field problem consisting of sensing electrode, earth electrode and air. The other is an electrostatic field problem adding fingers to the basic electrostatic field problem. The volume of sensing electrode is used as a constraint cost function. The shape derivative of the objective cost function was evaluated with the solutions of the two main problems and the two adjoint problems.

To solve the shape optimization problem to minimize the negative-signed difference norm with the volume constraint, an iterative algorithm based on the H^1 gradient method was used. A computer program was developed by JAVA API using a commercial software "COMSOL Multiphysics" as solver to the boundary value problems. With the computer program, we illustrated that reasonable shapes of sensing electrodes were obtained by the present approach. However, in the present shape optimization problem, end shape deforms toward a crack. Therefore, some re-meshing technique will be needed to obtain the shape having more sharp edges.

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