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The Stiffness Spreading Method in Integrated Layout Optimization Design for Multi-component Structural Systems

Peng Wei[1], Junkui Shi[1], Haitao Ma[1]

[1] State Key Laboratory of Subtropical Building Science, School of Civil Engineering and Transportation, South China University of Technology, Guangzhou, Guangdong, China, maht@scut.edu.cn

1. Abstract
The stiffness spreading method (SSM) is applied to integrated layout optimization problems involving both movable structural components and continuum supporting structures. By solving such a problem, both of the optimal positions and orientations of movable components and the optimal topologies of the continuum can be determined at the same time. Through the use of SSM, the stiffness contribution of a movable component is expressed in terms of nodal displacements of the background mesh representing the continuum, and as a result, no re-meshing for finite element analysis is required during the optimization process. The required sensitivities can be obtained analytically and with an available optimizer, an efficient solution algorithm be implemented. Based on the proposed model, the mean compliance minimization problem has been studied, and numerical examples are presented to illustrate the feasibility and efficiency of the new approach.
2. Keywords: multi-component system; layout optimization; topology optimization; stiffness spreading method.

3. Introduction

Topology optimization is a useful tool for conceptual designs of continuum structures. Up until now, most of the research effort in this field has been focused on single-component structures. As many of the real life engineering systems consist of different components, it is meaningful to develop optimization methodologies to deal with multiple components simultaneously in one optimization model.

As early as in 1997, Chickermane and Gea[1] developed a method to model the interconnections between components. The locations of the joints and the topology of each component are optimized in the framework. Jiang and Chirehdast[2] defined the topology of connections and studied the optimal connections between different components in a system. Qian and Ananthasuresh[3] investigated the so called “embedding problem” in topology design, in which the locations of discrete components and the topology of a continuous supporting structure connected to those components are optimized simultaneously. A material interpolation function on the basis of normal distribution functions is used to establish the connection between components and supporting structure. The movements of components in the design domain are substituted with a physical variation of the material properties. The supporting structure is designed with the model of Solid Isotropic Microstructure with Penalization (SIMP)[4] and sensitivities can be calculated analytically. This problem also has been extensively studied by Zhu and his collaborators[5, 6] in recent years. They implemented density points to deal with the problem that the finite element mesh may be distorted when the positions of components are updated during the optimization process. Furthermore, they developed a finite-circle method to avoid overlapping of the components. The SIMP model is also used for the topological optimization of the supporting structure and the sensitivities required for the optimization are obtained by semi-analytic method.

Figure 1 shows an example of multi-component system design problems. To achieve the best performance of the structural system, spatial positions and orientations of the discrete components and topology of the supporting structure are optimized. In Figure 1(a), one component is fixed in the design domain and the optimal positions and orientations of the other three are to be determined. The components may be elastic bodies, bars or their assemblies. The aim of integrated layout design of such a multi-component system is to determine the optimal configurations of the components and the supporting structure simultaneously to get a design with best performance. Figure 1(b) illustrates one of the alternative designs of the problem.

In this paper, a new model based on the stiffness spreading method (SSM)[7] is developed for simultaneously determining locations of discrete components and topology of the supporting structure. In the proposed model, movable components are introduced into the design domain, and nodal displacements for these components are related to those of the background mesh for the supporting structure by radial basis functions (RBF)[8] interpolation. Through this RBF interpolation, the stiffness matrix of a movable component can be transformed to produce an equivalent stiffness matrix, which can be regarded as a spreading of the stiffness contribution of the component to the background structure. This model has been applied to the heat conduction problem[9], in which the stiffness matrix is replaced by the conductivity matrix.

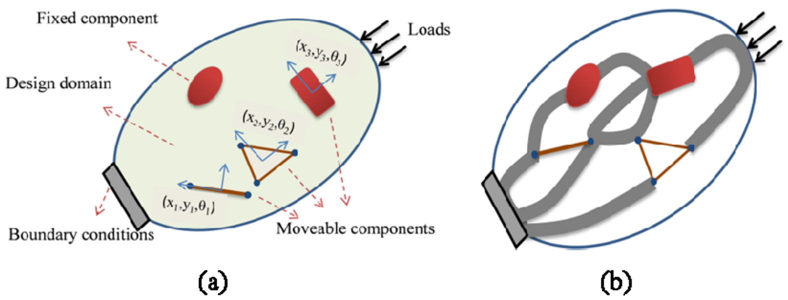

Figure 1: An example of integrated layout design of a multi-component system.
(a) Problem definition. (b) An optimized design.

Locations of the movable components and topology of the supporting structure are represented with geometrical and topological design variables, respectively. The SIMP model is used for topology optimization of continuum supporting structures and the Method of Moving Asymptotes (MMA)[10] is applied as the optimizer.
This paper is organized as follows. Firstly, a brief introduction of the topology optimization and integrated layout design of multi-component systems is given. Then new model is presented with technical details of the stiffness spreading method and the interpolation scheme using radial basis functions. After that, expressions are derived for the sensitivity analysis. Finally, numerical examples are presented and some concluding remarks are made.

4. Optimization Model and Sensitivity Analysis

4.1 Optimization model
The integrated layout design problem of 2-D multi-component systems can be formulated as:

$$\begin{aligned}
&\text{find} \quad \mathbf{x}=\{\boldsymbol{\eta}_\mathbf{c},\mathbf{S}\}:\boldsymbol{\eta}_\mathbf{c}=\{\eta_{c1},\eta_{c2},\cdots,\eta_{cn},\},\mathbf{S}=\{x_1,y_1,\theta_1,x_2,y_2,\theta_2,\cdots,x_m,y_m,\theta_m\}\\
&\text{min} \quad J=J(\mathbf{x},\mathbf{u}(\mathbf{x}))\\
&\text{s.t.} \quad \mathbf{K}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})=\mathbf{F}(\mathbf{x})\\
&\qquad \sum_{e=1}^{n}V_e(\boldsymbol{\eta}_\mathbf{c})\le V_0\\
&\qquad 0<\eta_{c\min}\le\eta_{ci}\le 1,\ \ i=1,2,\cdots,n\\
&\qquad \Gamma_j(x_j,y_j,\theta_j)\subset\Gamma_D,\ \ j=1,2,\cdots,m
\end{aligned} \tag{1}$$

where $\mathbf{x}$ is a vector for all design variables; vector $\boldsymbol{\eta}_\mathbf{c}$ contains pseudo-densities of elements in the background mesh representing the supporting structure; n is the number of elements for supporting structure; vector $\mathbf{S}$ contains design variables for positions and orientation angles of movable components; m is the number of movable components; J is the objective function; $\mathbf{K}(\mathbf{x})$ is the global stiffness matrix; $\mathbf{u}(\mathbf{x})$ is the global displacement vector; $\mathbf{F}(\mathbf{x})$ is the global load vector; V_e is the volume of the eth element; V_0 is the upper bound of the volume of the supporting structure; Γ_j and Γ_D denote the domains occupied by the jth component and the permissible design region, respectively; $\eta_{c\min}$ is the lower bound of the pseudo-density variables.
Due to position changes of the movable components, re-meshing is usually required for either the whole structure or around the component boundaries[5]. With the stiffness spreading method, as shown below, this troublesome procedure can be eliminated.

4.2 Stiffness spreading using RBF
If a set of degrees of freedom (DOFs) $\boldsymbol{a}_e$ can be expressed in terms of another set of DOFs $\mathbf{u}_e$ through the following equation

$$\boldsymbol{a}_e=\mathbf{T}\cdot\mathbf{u}_e \tag{2}$$

where $\mathbf{T}$ is a transformation matrix relating $\boldsymbol{a}_e$ and $\mathbf{u}_e$, a new equivalent stiffness matrix can be derived:

$$\tilde{\mathbf{K}}_e=\mathbf{T}^T\cdot\mathbf{K}_e\cdot\mathbf{T} \tag{3}$$

where $\mathbf{K}_e$ is the original stiffness matrix with $\mathbf{u}_e$ as DOFs. Similarly, a new equivalent nodal force vector will be

$$\tilde{\mathbf{F}}_e=\mathbf{T}^T\cdot\mathbf{F}_e \tag{4}$$

where $\mathbf{F}_e$ is the original equivalent nodal load vector.

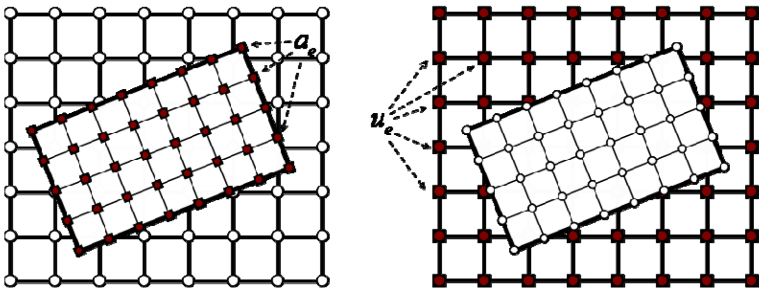

Figure 2: The transformation of DOFs.

In Figure 2, a solid 2D component is placed in a supporting structure with a regular background mesh. To model the interaction between the two parts, a conventional approach is to discretize them together and make sure that common nodes are allocated along the component boundary. Because of the change in component locations during the solution process, repeated re-meshing will be necessary, making such an approach clumsy and inefficient. With the SSM, the DOFs for displacement fields of the component ($\boldsymbol{a}_e$) are expressed in terms of the DOFs for the background mesh ($\mathbf{u}_e$) using the expression in Eq. (2). The key issue of the SSM is the establishment of relationship between the two sets of DOFs, or equivalently, the development of matrix **T** in Eq. (2). Considering that the interpolation using RBF is globally continuous and differentiable, we use the compactly supported radial basis functions (CSRBF) in the present study.

In RBF based interpolation, a displacement component can be represented by a set of coefficients and a set of basis functions. An RBF is a function of the locations of knots and the interpolation point, e.g. an RBF $g(r)$ in C^2 Wendland CSRBF is given as:

$$g(r) = \left(max(0,1-r)\right)^4 \cdot (4r+1) \tag{5}$$

where the radius of support r is given in the two dimensional case as:

$$r = \frac{\sqrt{(x-\xi_i)^2+(y-\eta_i)^2}}{d_{sp}} \tag{6}$$

where d_{sp} is a predefined constant which is the support radius of the RBF; (ξ_i, η_i) is the coordinates of the ith knot; and (x, y) is the coordinates of the interpolation point. For a given set of N knots, an interpolated displacement field can be expressed as

$$u(\mathbf{s}) = \sum_{i=1}^{N} g_i(\mathbf{s})\alpha_i \tag{7}$$

where $\mathbf{s}=\{x \ \ y\}^T$ is the coordinate vector for the interpolation point. In matrix form, we have

$$u(\mathbf{s}) = \left[g_1(\mathbf{s}) \quad g_2(\mathbf{s}) \quad \cdots \quad g_N(\mathbf{s})\right] \cdot \boldsymbol{\alpha} \tag{8}$$

where

$$\boldsymbol{\alpha} = \left\{\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_N\right\}^T \tag{9}$$

Taking all the N nodes in the background mesh as knots and using Eq. (8) for each of these nodes, we can get

$$\mathbf{u}^0 = \mathbf{A} \cdot \boldsymbol{\alpha} \tag{10}$$

where $\mathbf{u}^0=\{u_1 \ u_2 \ \cdots \ u_N\}^T$ denotes nodal displacements for the background mesh and **A** is the following matrix

$$\mathbf{A} = \begin{bmatrix} g_1(\mathbf{p}_1) & g_2(\mathbf{p}_1) & \cdots & g_N(\mathbf{p}_1) \\ g_1(\mathbf{p}_2) & g_2(\mathbf{p}_2) & \cdots & g_N(\mathbf{p}_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(\mathbf{p}_N) & g_2(\mathbf{p}_N) & \cdots & g_N(\mathbf{p}_N) \end{bmatrix} \tag{11}$$

with $\mathbf{p}_j$ being the coordinate vector of the jth knot. Thus,

$$\boldsymbol{\alpha} = \mathbf{A}^{-1} \cdot \mathbf{u}^0 \tag{12}$$

substituting which into Eq. (8) yields

$$u(\mathbf{s}) = \left[g_1(\mathbf{s}) \quad g_2(\mathbf{s}) \quad \cdots \quad g_N(\mathbf{s})\right] \cdot \mathbf{A}^{-1} \cdot \mathbf{u}^0 \tag{13}$$

By substituting the coordinates of a node in the movable components into Eq. (13), we can relate the displacement at this node to those for nodes in background mesh. Repeating this for every DOF of the nodes in movable components, the transformation matrix in Eq. (2) can be formed.
More detailed information about the stiffness spreading method can be found in [7].

5. Sensitivity Analysis and Optimization Algorithm

The derivative of the objective function given in Eq. (1) with respect to a design variable can be calculated using the adjoint method. If the total strain energy is taken as the objective and the load vector is independent of the design variables, the derivative of objective function can be expressed as:

$$\frac{dJ}{dx} = -\frac{1}{2}\mathbf{u}^T \cdot \frac{\partial \mathbf{K}}{\partial x} \cdot \mathbf{u} \tag{14}$$

where x represents an arbitrary design variable. Because of Eq. (14), the sensitivity of the objective function can be easily calculated if the sensitivity of the stiffness matrix is available. Thus, the major part of this section will be limited to the calculation of the sensitivity of stiffness matrix. As a design variable may be either for the material pseudo-density of an element in background mesh or for the location and orientation of a movable component, two separate cases are considered.

(1) Variables for topology design of supporting structure

With the SIMP model, the element stiffness matrix can be expressed as:

$$\mathbf{K}_e = \eta_e^p \mathbf{K}_e^0 \tag{15}$$

where η_e is the pseudo-density of the element, p is the penalization factor, and $\mathbf{K}_e^0$ is the element stiffness matrix when the element is full of material. The sensitivity of the element stiffness matrix is

$$\frac{\partial \mathbf{K}_e}{\partial \eta_e} = p\eta_e^{p-1}\mathbf{K}_e^0 \tag{16}$$

(2) Variables for movable components

When the design variable x is for either the coordinate or orientation angle of the jth movable component, we have

$$\frac{\partial \mathbf{K}}{\partial x} = \frac{\partial}{\partial x}\left[\sum_{e \in E_j}\left(\mathbf{T}^T \cdot \mathbf{K}_e \cdot \mathbf{T}\right)\right] \tag{17}$$

where E_j represents the set of elements for this particular component and the summation is for all the elements in this set. Assume the design variable is for the component, Eq. (17) can be further expressed as

$$\frac{\partial \mathbf{K}}{\partial x} = \left[\sum_{e \in E_j}\left(\frac{\partial \mathbf{T}^T}{\partial x} \cdot \mathbf{K}_e \cdot \mathbf{T} + \mathbf{T}^T \cdot \mathbf{K}_e \cdot \frac{\partial \mathbf{T}}{\partial x}\right)\right] \tag{18}$$

With Eq. (18), the derivative of the transformation matrix **T** can be calculated and so is the sensitivity of the stiffness matrix.
As the sensitivity of the objective function can be calculated efficiently, a gradient based optimization algorithm can be used with the SSM based model for the integrated layout optimization problem. In this paper, the method of moving asymptotes is used for solutions of the numerical examples presented in the next section.

6. Numerical Examples

Two examples of total strain energy minimization under the volume constraint on the supporting structure are considered in this section and optimized designs are compared with other available solutions.

6.1 A short cantilever beam with a movable rectangle

The first example is the optimization of a short cantilever beam with one moveable component. The initial design and the load and boundary conditions are shown in Figure 3(a). A unit load F=1 is applied on the lower right corner. The design domain is modeled with 48×38 four-noded quadrilateral plane stress elements. The Young's moduli of the component and the supporting structure are 10 and 1, respectively. The Poisson's ratios are 0.3. An upper limit of 35% of the design domain is applied to the total volume of the supporting structure
The optimal design obtained is shown in Figure 3(b), showing that the component moves to the main force transmission path. The material of the supporting structure builds up the other part of the structure. For comparison, the optimal result without component is shown in Figure 3(c), in which the volume fraction is set to 50%. It can be seen that the presence of the movable component has certain influence on the topology and shape of the supporting structure. This example shows that, with the proposed model, the movable component and the supporting

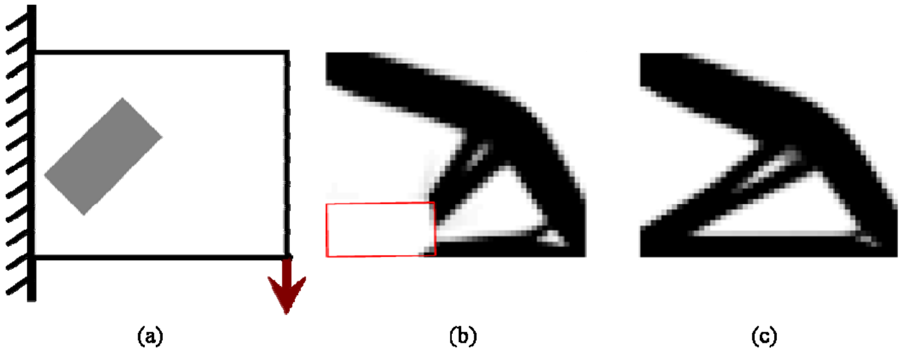

Figure 3: Optimization of a short cantilever beam. (a) Problem definition and the initial design. (b) The optimal design with a component. (c) The optimal design without component.

structure can be optimized simultaneously to yield a reasonable design. This methodology can help the designer to decide where to put the component with predefined shape and how to distribute material of the supporting structure.

6.2 A simply supported beam with four movable rectangles

A simply supported beam with four moveable components is considered. For simplicity, only half of the structure is modeled, as shown in Figure 4(a). A downward load F=1.8×10^4 is applied on the top of the right corner. A mesh with 60×30 four-noded quadrilateral plane stress elements is used over the design domain. The Young's moduli of movable components and supporting structure are 2×10^{11} and 7×10^{10}, respectively. The Poisson's ratios are 0.3. The upper limit on volume of the supporting structure is set to 35% of the design domain.

Figures 4(b), (c) and (d) present some of the designs at the different stages of optimization process. Figure 4(d) shows that the two components have moved to their final locations at convergence, connecting with the supporting structure to resist the applied load. It can be seen that there is little material overlapping between the supporting structure and the movable components, even though no special scheme is implemented to avoid possible overlaps. Figure 4(e) shows the result in [5] for the same problem. While the two results are similar, the differences are still obvious. It seems very hard, if not impossible, to get the global optimum for an integrated layout design problem of a multi-component system. The result of optimal design with no component is shown in Figure 4(f). Again, the presence of movable components has a significant influence on the configuration of the supporting structure.

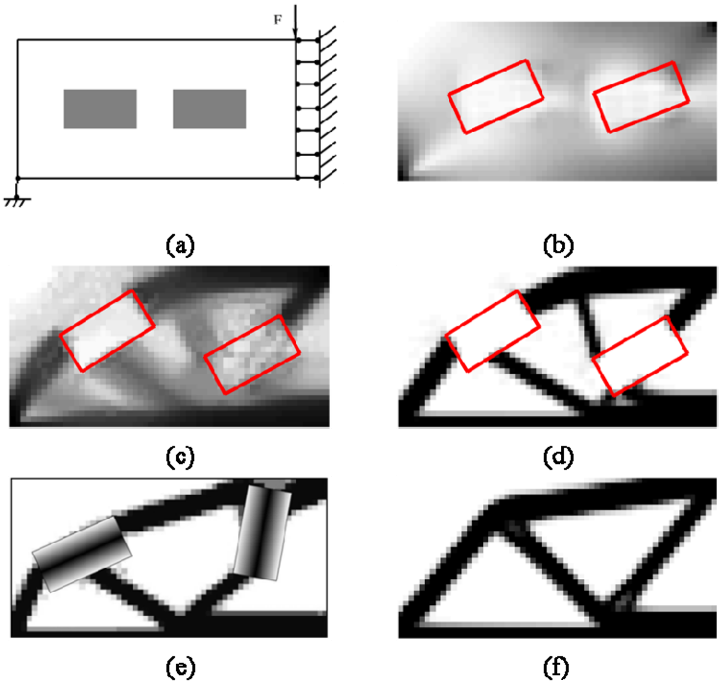

Figure 4: Optimization of a simply supported beam with movable components. (a) Problem definition. (b) Step 4. (c) Step 30. (d) Optimal design. (e) Optimal design of [5]. (f) Optimal design without discrete components.

7. Conclusions

A new approach based on the stiffness spreading method is proposed for the integrated layout optimization of multi-component systems. Both the layout of movable components and the topology of a continuum supporting structure are considered in a single optimization model. Using the stiffness spreading concept, displacement DOFs for a movable component are related to those for the supporting structure. The establishment of such a relationship makes it simple to combine different components in one finite element model for both structural analysis and design optimization. Numerical examples are presented and the obtained designs are compared with the results published in the literature and from topology optimization only, confirming the validity of the new model and the effectiveness of the solution algorithm.

8. Acknowledgements

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