10th World Congress on Structural and Multidisciplinary Optimization
May 19 -24, 2013, Orlando, Florida, USA

Topology and Configuration Optimization of Trusses Based on Virtual Bars Concept

Dariusz Bojczuk[1], Anna Rębosz-Kurdek[2]

Faculty of Management and Computer Modelling, Kielce University of Technology, Al. Tysiąclecia Państwa Polskiego 7, 25-314 Kielce, Poland
e-mail: [1]mecdb@eden.tu.kielce.pl , [2]arebosz@tu.kielce.pl

1. Abstract
The algorithms of simultaneous optimal design of topology, configuration and cross-sectional areas of trusses are considered in this paper. In the basic case, the problem of cost minimization with constraint imposed on global stiffness is analyzed. However, the presented approach is also successfully applied to the problem of the cost minimization with constraints imposed on stresses and on buckling loads. The main feature of this approach is application of so-called virtual bars, which can join existing nodes so far non-connected.
In this paper two alternative methods of optimal design of trusses are presented. At first, the two-stage (topology and geometry) optimization procedure was presented. Initially, topology optimization is performed. Here, each virtual bar is simulated by self-equilibrated system of forces specified by single design parameter. In the next stage, configuration optimization is performed for all optimal topologies and the best solution is chosen. In the second method, optimization is carried on simultaneously with respect to topological parameters, which represent forces in virtual bars and with respect to configuration parameters, which correspond to positions of nodes. Here, identically as in the first method, cross-sectional areas of bars are determined directly depending on bar forces.
The considerations are illustrated by some numerical examples. They confirm usefulness and efficiency of the presented methods.
2. Keywords: trusses, optimal topologies, optimal configurations, algorithms of optimization, virtual bars concept.

3. Introduction
The paper is devoted to optimal design of trusses with respect to their topology, node positions and cross-sectional areas. This problem is characterized by big number of design parameters, so in order to get its solution in the easier and simpler way it is very important to reduce amount of these parameters.
In the case of topology optimization ground structure approach formulated in [8] is treated as the standard method. For prescribed number of nodes it assumes all possible bar connections between the joints. For reduction of number of used bars, concept of so-called virtual bars, which can join existing nodes so far non-connected, is applied. The concept of such bars was introduced earlier in papers by Bojczuk and Mróz (cf. [5], [11]). Here, at first initial structure should be created. It should be noted that for problems of minimum mass (or cost proportional to the mass of single bars) subjected to a single loading case and with stiffness constraints or with stress and local buckling constraints, optimal design can be always determined in the class of statically determinate structures (cf. [2], [9]). So, the initial truss can be chosen as the statically-determinate truss. Next, topological derivative is applied to select virtual bars. It is important to notice, that number of these bars is considerably smaller than the number of additional bars used in ground structure approach even in the case, where in order to reduce number of design variables only connections with neighboring nodes are taken into account (cf. [1]). Moreover, this artificial reduction may cause elimination of some connections which should appear in the optimal design. Such difficulties usually do not occur when using presented here approach.
However, further decrease of the number of design parameters can be achieved by the solution of the sub-problem with respect to cross-sectional areas of truss bars.
The problem of optimization of topology and configuration of trusses is studied in many papers. One of the first analyses were presented by Pedersen (cf. [13], [14]). He applied the optimization procedure consisting of two mutually interacted stages (topology and geometry), where topology optimization was performed using ground structure approach. The surveys of truss design problems were presented in [1], [4], [10], [15], etc. In this first paper partition of truss design problems for the two groups, namely the alternating approaches composed of geometry and topology stages, and the simultaneous approaches, where topology and configuration are treated simultaneously, was proposed and studied.
This paper extends previous considerations presented in [7]. The main feature of the formulated here algorithm of topology optimization is the possibility of simultaneous generation of many equivalent topologies. But first of all,

this paper is devoted to comparison of the alternating approach and simultaneous approach, when the concept of virtual bars is used.
In Section 4 the problem of minimum cost with stiffness constraint, topology modification conditions and algorithms of optimization will be presented. The problem of minimum cost with stress and buckling constraints will be discussed in Section 5. Section 6 will be devoted to solution of some illustrative examples. The obtained designs will be compared with the results available in literature.

4. Minimum cost problem with stiffness constraint

4.1. Formulation of optimization problem
Let us formulate, analogously as in [7], the problem of optimal design of trusses with respect to minimization of material cost under constraint imposed on the global stiffness. The cost of the structure C can be expressed in the form

$$C = \sum_{i=1}^{n} c_i A_i l_i \ , \tag{1}$$

where A_i, l_i, c_i are the cross-sectional area, length, specific material cost of the *i*-th bar, and n is the current number of bars. We assume the potential energy Π as the measure of the global stiffness. In the case of problem without initial stresses or strains, it takes the form

$$\Pi = -U = -\frac{1}{2}\sum_{i=1}^{n}\frac{N_i^2 l_i}{E_i A_i} \ , \tag{2}$$

where N_i, E_i denote bar force and Young's modulus of the *i*-th bar and U is the strain energy. Now, the optimization problem can be presented in the form

$$\min_{A_i, \mathbf{x}_j, \alpha_k} C \quad \text{subject to} \quad U \le U_0 \, , \tag{3}$$

where U_0 is the allowable strain energy. Moreover, $\mathbf{x}_j(x_{1j}, x_{2j}, x_{3j})$ or $\mathbf{x}_j(x_{1j}, x_{2j})$ is the vector containing coordinates of the *j*-th node respectively for the spatial trusses and plane trusses, and α_k are the topological parameters.
Next, let us consider the optimization sub-problem with respect to cross-sectional areas of bars. Using the Lagrangian

$$C^* = C + \lambda(U - U_0) \, , \tag{4}$$

where λ $(\lambda \ge 0)$ is the Lagrange multiplier, and taking into account the sensitivity formula (cf. [6])

$$\frac{\partial U}{\partial A_i} = -\frac{N_i^2 l_i}{2 E_i A_i^2} \ , \tag{5}$$

we obtain

$$\begin{aligned} &\frac{\partial C^*}{\partial A_i} = c_i l_i - \frac{1}{2}\lambda \frac{N_i^2 l_i}{E_i A_i^2} = \left(c_i - \frac{1}{2}\lambda \varepsilon_i^2 E_i \right) l_i = 0, \\ &\lambda \left(\frac{1}{2}\sum_{i=1}^{n} \frac{N_i^2 l_i}{E_i A_i} - U_0 \right) = 0, \ \ \lambda \ge 0. \end{aligned} \tag{6}$$

Now, assuming that the stiffness condition is active, let us determine from (6), cross-sectional areas and Lagrange multiplier in function of bar forces. Then we have

$$\begin{aligned} &A_i = \frac{1}{2U_0}\frac{|N_i|}{\sqrt{c_i E_i}} \sum_{j=1}^{n} |N_j| l_j \sqrt{\frac{c_j}{E_j}} \ , \\ &\lambda = \left(\frac{1}{U_0}\sum_{i=1}^{n} |N_i| l_i \sqrt{\frac{c_i}{2E_i}} \right)^2 . \end{aligned} \tag{7}$$

Substituting (7) into (1), the cost function, both for the statically determinate and non-determinate trusses, can be rewritten in the form

$$C = \frac{1}{2U_0}\left(\sum_{i=1}^{n} |N_i| l_i \sqrt{\frac{c_i}{E_i}} \right)^2 , \tag{8}$$

When all bars are made of the same material of Young's modulus E and specific material cost c, the cost function becomes

$$C = \frac{c}{2EU_0}\left(\sum_{i=1}^{n}|N_i| l_i\right)^2 . \tag{9}$$

So, the constrained optimization problem (3), can be substituted by unconstrained problems of the form

$$\min_{\mathbf{x}_j,\alpha_k}\left(\sum_{i=1}^{n}|N_i| l_i \sqrt{\frac{c_i}{E_i}}\right), \qquad \text{or} \qquad \min_{\mathbf{x}_j,\alpha_k}\left(\sum_{i=1}^{n}|N_i| l_i\right) \tag{10}$$

corresponding respectively to (8) and (9).

4.2. Condition of topology modification and specification of topological parameters

Using the topological derivative approach (cf. [11], [12]), the condition of topology modification by introduction of a new (n+1)-th bar of Young's modulus E_{n+1} and specific cost c_{n+1} to the truss with optimal cross-sectional areas determined in (7), can be expressed as follows

$$\left.\frac{\partial C^*}{\partial A_{n+1}}\right|_{A_{n+1}=0} = \left(c_{n+1} - \frac{1}{2}\lambda\varepsilon_{n+1}^2 E_{n+1}\right) l_{n+1} < 0, \;\; \text{or} \;\; |\varepsilon_{n+1}| > \varepsilon_{\lim}^{(n+1)}, \;\; \text{where} \;\; \varepsilon_{\lim}^{(n+1)} = \sqrt{\frac{2c_{n+1}}{\lambda E_{n+1}}} . \tag{11}$$

Here, ε_{n+1} denotes virtual strain value along the line connecting the respective nodes and

$$\varepsilon_{n+1} = \frac{u_2^{(n+1)} - u_1^{(n+1)}}{l_{n+1}}, \tag{12}$$

where $u_1^{(n+1)}$, $u_2^{(n+1)}$ are displacements of these nodes along the line direction. When the condition (11) is satisfied after introduction of a new virtual bar, decrease of the cost function occurs.

In the case of all bars made of the same material, the condition (11) takes the form

$$|\varepsilon_{n+1}| > \varepsilon_{\lim}, \quad \text{where } \varepsilon_{\lim} = |\varepsilon_1| = |\varepsilon_2| = \ldots = |\varepsilon_n| . \tag{13}$$

Now, we assume that new bars can be introduced in all lines, where the condition (11) or (13) is satisfied. The introduction of a new k-th bar is simulated by two opposite forces $\alpha_k \hat{N}_k$ applied at the nodes of this bar, where $\hat{N}_k$ denotes the force of unit value. Moreover, α_k is the load factor, which will play role of the topological parameter. The system of two forces $\alpha_k \hat{N}_k$ induces force $\alpha_k N_i^{(k)}$ in i-th bar. Finally, we obtain statically admissible field composed of total forces in bars of the initial truss

$$N_i = N_i^{(init)} + \sum_{k=1}^{K}\alpha_k N_i^{(k)} \tag{14}$$

and forces $\alpha_k \hat{N}_k$ in the virtual bars. Here, $N_i^{(init)}$ are the primary bar forces in the initial design induced by external loading and K is the number of virtual bars.

Now, the general optimization problem $(10)_1$ can be rewritten in the form

$$\min_{\mathbf{x}_j,\alpha_k}\left(\sum_{i=1}^{n_0}\left|N_i^{(init)} + \sum_{k=1}^{K}\alpha_k N_i^{(k)}\right| l_i \sqrt{\frac{c_i}{E_i}} + \sum_{k=1}^{K}\left|\alpha_k \hat{N}_k\right| l_k \sqrt{\frac{c_k}{E_k}}\right), \tag{15}$$

where n_0 is the number of bars of the initial design.

4.3. Algorithms of truss optimization

Let us assume initial design as a statically determinate truss of required level of complexity. The optimal design can contain the same number of bars or some redundant bars can be removed. Now, two alternative algorithms of structure optimization will be presented.

At first, the optimization procedure composed of two mutually interacted stages (topology and geometry) called also alternating approach (cf. [1]) was presented. Initially, topology optimization is performed and it corresponds to solution of the unconstrained problem of the form analogous to (15), namely

$$\min_{\alpha_k}\left(\sum_{i=1}^{n_0}\left|N_i^{(init)} + \sum_{k=1}^{K}\alpha_k N_i^{(k)}\right| l_i \sqrt{\frac{c_i}{E_i}} + \sum_{k=1}^{K}\left|\alpha_k \hat{N}_k\right| l_k \sqrt{\frac{c_k}{E_k}}\right) . \tag{16}$$

Let us notice that this problem is equivalent to the linear programming problem with respect to topological parameters α_k.

However, for some problems of truss topology optimization big number of equivalent topologies appears, especially for problems with symmetric (antisymmetric) loading systems. In order to ensure simultaneous generation of many equivalent topologies, special procedure called bar exchange method, described in [7], can be used. Here, we have to find which existing bar can be substituted by one of the virtual bars. For this purpose all values of parameters $\alpha_k, k = 1,2,..., K$ corresponding to disappearance of normal forces in a certain bar i.e.

$$N_i = N_i^{(init)} + \alpha_k N_i^{(k)} = 0 \quad \text{for} \quad k = 1,2,\ldots,K \quad \text{and} \quad i = 1,2,..., n, \tag{17}$$

are determined and these values related to the biggest decrease of the objective function are chosen. This procedure is continued until reduction of the objective function appears.

In the next stage, configuration optimization problem of the form

$$\min_{\mathbf{x}_j} \left(\sum_{i=1}^{n} |N_i| l_i \sqrt{\frac{c_i}{E_i}} \right) \tag{18}$$

should be solved for all optimal topologies and the best solution should be chosen, where n denotes number of bars in analyzed at present topology.

In the second method, which can be called simultaneous approach, optimization is carried on simultaneously with respect to topological parameters, which represent forces in virtual bars and with respect to configuration parameters, which are positions of nodes. So, it corresponds to the solution of the problem (15). Here, analogously as in the first method, cross-sectional areas of bars are determined directly depending on bar forces.

5. Minimum cost problem with stress and buckling constraints

The problem of optimal design of trusses with respect to minimization of material cost under stress, buckling and cross-sectional areas constraints, is formulated in the form (cf. [7])

$$\min_{A_i, \mathbf{x}_j, \alpha_k} C = \sum_{i=1}^{n} c_i A_i l_i \quad \text{subject to} \quad |\sigma_i| \le \sigma_{ai}, \; A_i \ge A_{i\min}, \tag{19}$$

where the notation is the same as in Section 4. Moreover,

$$\sigma_i = \frac{N_i}{A_i} \tag{20}$$

are the bar stresses and

$$\sigma_{ai} = \begin{cases} R_{ei} & \text{for} \quad \sigma_i \ge 0 \\ \dfrac{\pi^2 E_i}{s_i^2} & \text{for} \quad \sigma_i < 0, \quad s_i \ge s_i^{cr} \\ R_{ei} - \frac{1}{2} R_{ei} (s_i / s_i^{cr})^2 & \text{for} \quad \sigma_i < 0, \quad s_i < s_i^{cr} \end{cases} \tag{21}$$

denote the admissible stresses in the *i*-th bar and R_{ei} is the conventional yield limit. The last expression of (21) corresponds to the Johnson-Ostenfeld formula for the critical buckling stress in the elasto-plastic range (cf. [3]). Here, the slenderness and critical slenderness are of the form

$$s_i = l_i \sqrt{\frac{A_i}{I_i}}, \qquad s_i^{cr} = \pi \sqrt{\frac{2E_i}{R_{ei}}}, \quad \text{where} \qquad I_i = \xi_i A_i^m, \quad m = 1,2,3, \tag{22}$$

is the minimal moment of inertia of the bar cross-section and ξ_i is the constant parameter. For further considerations $m = 2$ is assumed. It corresponds to the proportional change of the cross-sectional dimensions.

Introducing nonnegative Lagrange multipliers $\mu_i \ge 0$, $\eta_i \ge 0$, the augmented objective function can be expressed as

$$C^* = C + \sum_{i=1}^{n} \mu_i (|\sigma_i| - \sigma_{ai}) + \sum_{i=1}^{n} \eta_i (A_{\min} - A_i). \tag{23}$$

Next, limiting considerations to the class of statically determinate trusses, solving sub-problem with respect to cross-sectional areas and substituting them to the (16), the problem (19) and the cost of the structure take the form (cf. [7])

$$\min_{\mathbf{x}_j, \alpha_k} C = \sum_{(1)} \frac{c_i}{R_{ei}} N_i l_i + \sum_{(2)} \frac{c_i}{\pi \sqrt{\xi_i E_i}} \sqrt{|N_i| l_i^2} + \sum_{(3)} \frac{c_i}{R_{ei}} \left[|N_i| l_i + \frac{R_{ei}^2 l_i^3}{4 E_i \pi^2 \xi_i} \right] + \sum_{(4)} c_i A_{i\min} l_i, \tag{24}$$

where $\sum_{(i)}$ denotes the summation with respect to all bars belonging to group i. Four groups were distinguished, namely:

- group (1) containing tensile bars of cross-sectional areas greater than $A_{\min}$;
- group (2) containing compressive bars of slenderness not less than the critical slenderness s_i^{cr} and of cross-sectional areas greater than $A_{\min}$;
- group (3) containing compressive bars of slenderness less than the critical slenderness s_i^{cr} and of cross-sectional areas greater than $A_{\min}$;
- group (4) of bars of cross-sectional areas $A = A_{\min}$.

The condition of topology modification by introduction of (n+1)-th bar in tension takes the form (11) or (13), where here $\varepsilon_{\lim}^{(n+1)} = R_{e(n+1)} / E_{n+1}$. In the case of bar in compression the condition can be presented as

$$|\varepsilon_{n+1}| > \varepsilon_{0_{n+1}} \text{, where } \varepsilon_{0_{n+1}} = \frac{\pi^2 \xi_{n+1} A_{(n+1)\min}}{l_{n+1}^2} \tag{25}$$

denotes strain appearing in the virtual bar in compression of the minimal cross-sectional area.
For the analyzed problem also two procedures namely alternating approach and simultaneous approach described in Subsection 4.3 can be used for topology and configuration optimization. However, here the objective function is expressed by (24), but new values of bar forces after topology modification can be determined, as previously, from (14) and (17).

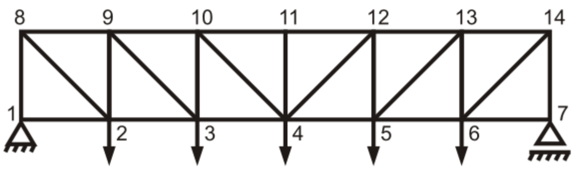
6. Numerical examples

The theoretical considerations are illustrated by some numerical examples. They confirm usefulness and efficiency of the presented algorithms. In particular, decrease of number of design parameters significantly reduces computation time required for generation of optimal designs. The results obtained for both methods namely alternating approach and simultaneous approach, were compared with the results available in literature, especially in [1].

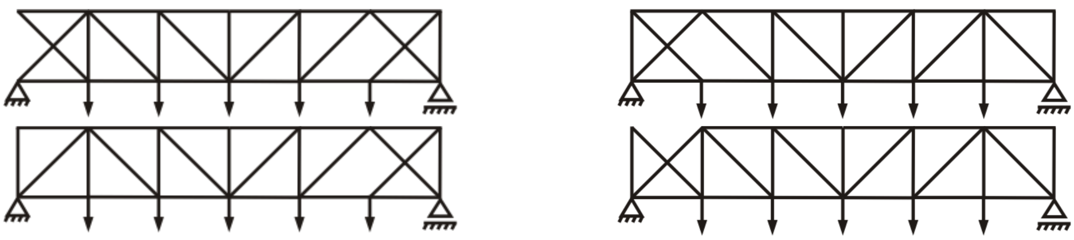
6.1. Optimal design of topology and geometry of bridge truss for minimum cost under stiffness constraint

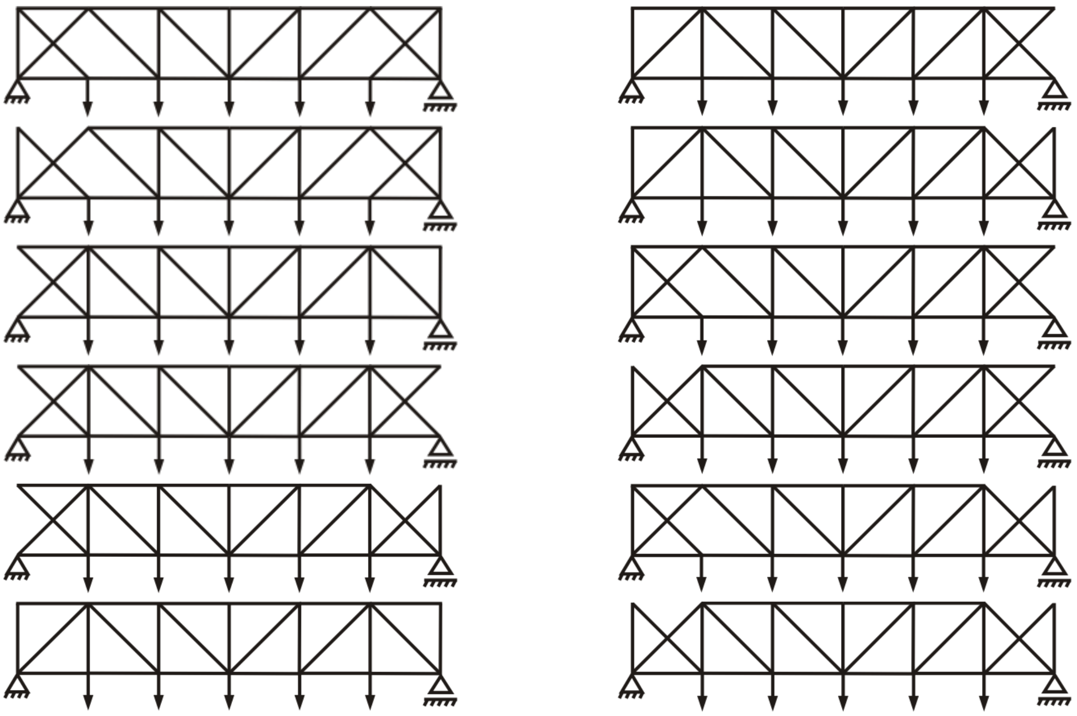
Let us consider the cost minimization problem of the form expressed by (3) for the truss presented in Fig. 1a. Five lower nodes of the truss are loaded by vertical forces $P = 10^3 N$. The truss is composed of 25 bars. It is made of linearly elastic material of Young's modulus value $E = 2.1 \cdot 10^{11} N/m^2$. The specific material costs c_i for all bars are the same. The length of all horizontal and vertical bars is $l = 1m$, while the length of inclined bars equals $\sqrt{2}l$. The optimization will be performed using two methods and here 12 virtual bars (or 6 pairs of symmetrically located virtual bars) will be used.
At first, the two-stage (alternating) approach is applied. The topology optimization is performed by bars exchanges and it gives 16 equivalent designs (Fig. 1b). For each design the objective function reaches the same value, so the solution is non-unique. The removal of zero bars system from optimal topologies, reduces their number to 4 (Fig. 1c). Next, for each optimal topology, optimal configuration is determined and the best solution presented in Fig. 1d is chosen. Now, the ratio of costs of the initial design corresponding to the truss presented in Fig. 1a with the optimal cross-sectional areas and the optimal design equals $C^{(init)} / C^{(opt)} = 1.77210$.

a)

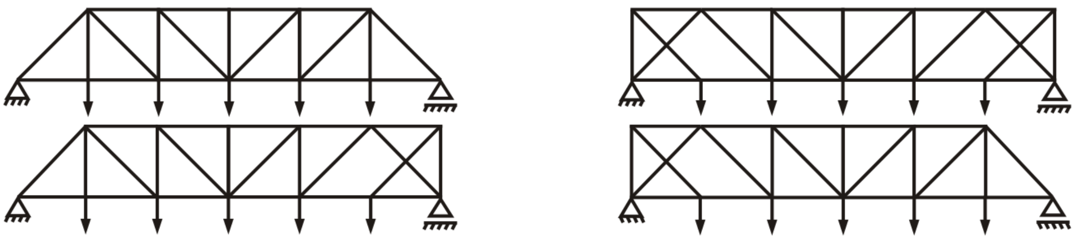


b)





c)



d)

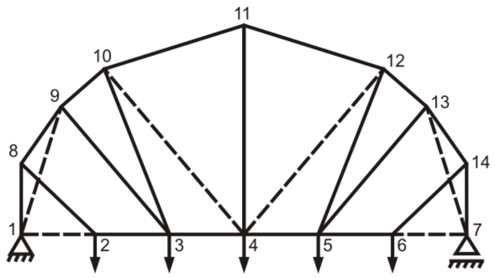


Fig.1. Alternating optimization: a) The initial design, b) The optimal topologies, c) The optimal topologies after removal of zero-bar systems, d) The optimal design

In the second approach the optimization problem (15) is solved simultaneously with respect to topological parameters and configurational parameters. It leads to two different optimal designs presented In Fig. 2a ($C^{(init)}/C^{(opt)} = 1.77245$) and in Fig. 2b ($C^{(init)}/C^{(opt)} = 1.77259$). The optimal designs presented in Fig. 1d and Fig. 2a have the same layout of non-zero bars, however they differ a little because of different systems of zero-bars and some numerical errors.

Moreover, let us consider simultaneous optimization of this truss presented in [1]. Here, the initial design (Fig. 3a) is obtained by modified ground structure approach, where only connections with neighboring nodes are taken into account. The optimal design presented in Fig. 3b is exceptionally non-symmetric and the ratio of the costs of the initial design from the Fig. 1a and this optimal design equals $C^{(init)}/C^{(opt)} = 1.77236$.

It is important to notice that the cost for four optimal designs differs very insignificantly (less than 0.028%), however the best solution is presented in Fig. 2b. The node coordinates for all optimal designs are assembled in Table 1.

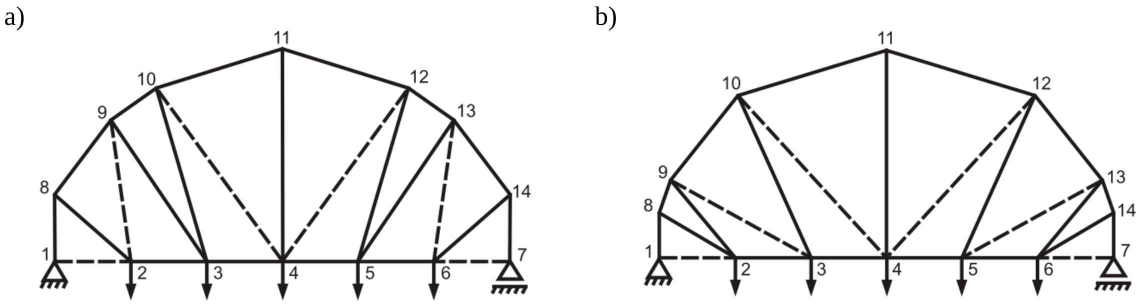


Fig. 2. Simultaneous optimization: a-b) Selected optimal designs

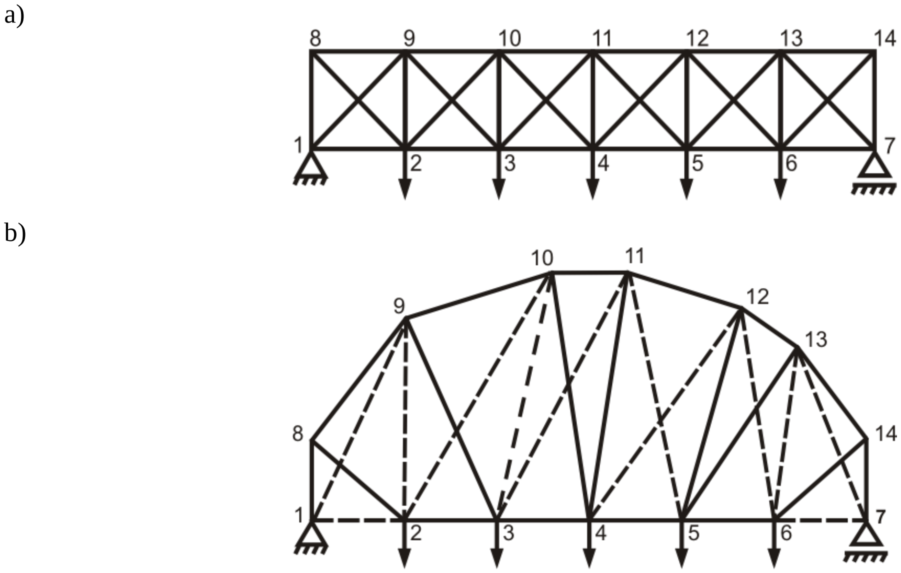


Fig. 3. Simultaneous optimization (solution from the paper [1]): a) The initial design, b) The optimal design

Table 1: Node coordinates for the optimal designs

Nr of node	Design from Fig.1d		Design from Fig.2a		Design from Fig.2b		Design from Fig. 3b	
	horizontal coordinate	vertical coordinate	horizontal coordinate	vertical coordinate	horizontal coordinate	vertical coordinate	horizontal coordinate	vertical coordinate
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
3	2.00	0.00	2.00	0.00	2.00	0.00	2.00	0.00
4	3.00	0.00	3.00	0.00	3.00	0.00	3.00	0.00
5	4.00	0.00	4.00	0.00	4.00	0.00	4.00	0.00
6	5.00	0.00	5.00	0.00	5.00	0.00	5.00	0.00
7	6.00	0.00	6.00	0.00	6.00	0.00	6.00	0.00
8	0.00	0.94	0.00	0.88	0.00	0.60	0.00	0.87
9	0.54	1.71	0.74	1.86	0.15	1.03	1.02	2.20
10	1.12	2.22	1.34	2.28	1.04	2.15	2.60	2.69
11	3.00	2.81	3.00	2.80	3.00	2.75	3.42	2.69
12	4.88	2.22	4.66	2.28	4.96	2.15	4.65	2.31
13	5.46	1.71	5.26	1.86	5.85	1.03	5.26	1.88
14	6.00	0.94	6.00	0.88	6.00	0.60	6.00	0.89

6.2. Optimal design of topology and geometry of cantilever truss for minimum cost under stiffness constraint

Let us consider the cost minimization problem of the form expressed by (3) for the truss presented in Fig. 4a. The truss is composed of 21 bars and loaded by single vertical force $P = 10^3 N$. It is made of linearly elastic material of

Young's modulus value $E = 2.1 \cdot 10^{11} N/m^2$. The specific material costs c_i for all bars are the same. The length of all horizontal and vertical bars is $l = 1m$, while the length of inclined bars equals $\sqrt{2}l$. The optimization will be performed using two methods and here 22 virtual bars (or 11 pairs of symmetrically located virtual bars) is used.
At first, the alternating approach is applied. The topology optimization is done by bars exchanges and removal of zero bars system. It provides 5 equivalent designs of the same value of the objective function (Fig. 4b), which is 1.5 times smaller than for the truss presented in Fig. 4a with the optimal cross-sectional areas. Next, for each optimal topology, optimal configuration is determined and the best solution (Fig. 4c), is chosen. Now, the ratio of costs of the initial design (Fig. 4a) and the optimal design equals $C^{(init)}/C^{(opt)} = 1.6424$.

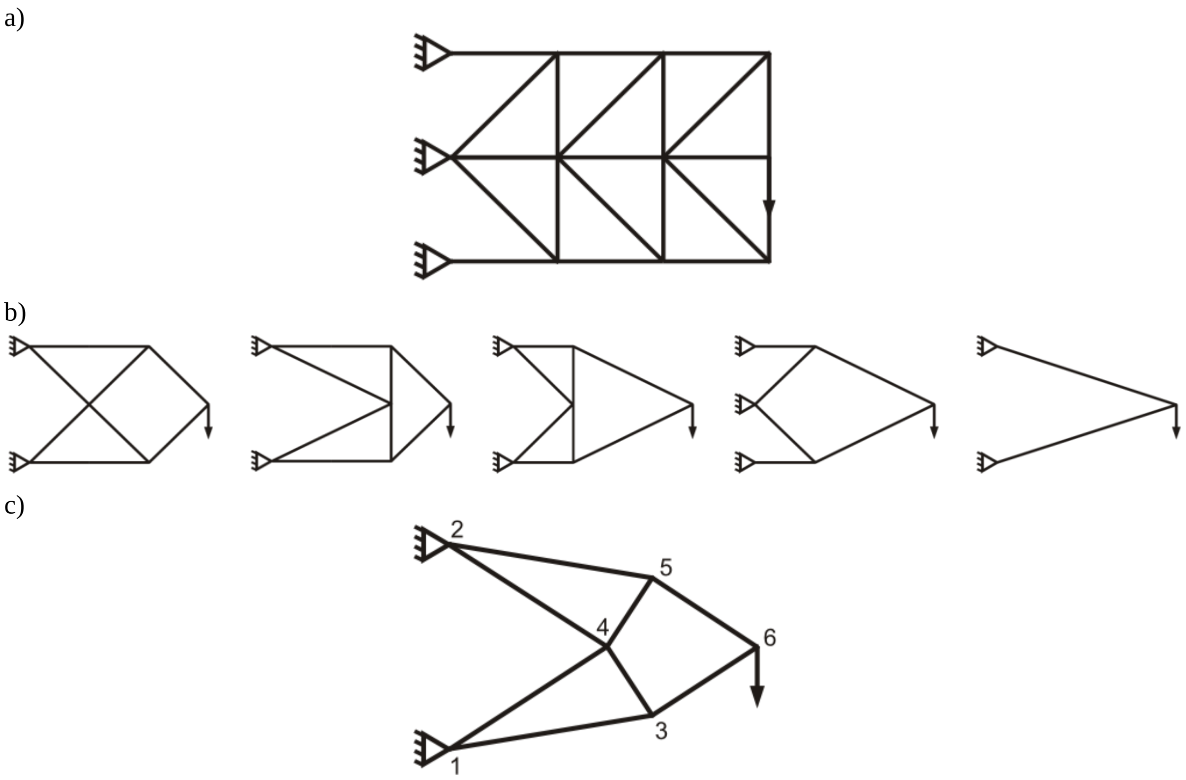


Fig.4. Alternating optimization: a) The initial design, b) The optimal topologies after removal of zero-bar systems, c) The optimal design

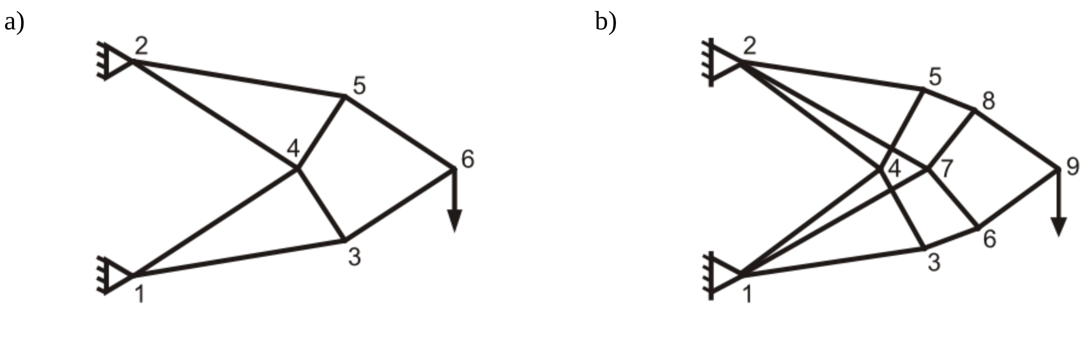


Fig. 5. Simultaneous optimization: a-b) Selected optimal designs

In the second approach the optimization problem (15) is solved simultaneously with respect to topological parameters and configurational parameters. It leads to two different optimal designs presented In Fig. 5a ($C^{(init)}/C^{(opt)} = 1.6424$) and in Fig. 5b ($C^{(init)}/C^{(opt)} = 1.6506$). The optimal designs presented in Fig. 4c and Fig. 5a have the same layout of non-zero bars, however they differ a little because of different systems of zero-bars and some numerical errors.
Moreover, let us consider simultaneous optimization of this truss presented in [1]. Here, the initial design (Fig. 6a)

is obtained by modified ground structure approach, where only connections with neighboring nodes are taken into account. The optimal design is presented in Fig. 6b and the ratio of the costs of the initial design from the Fig. 4a and this optimal design equals $C^{(init)}/C^{(opt)} = 1.6434$.
The best solution is presented in Fig. 5b. The node coordinates for all optimal designs are assembled in Table 2.

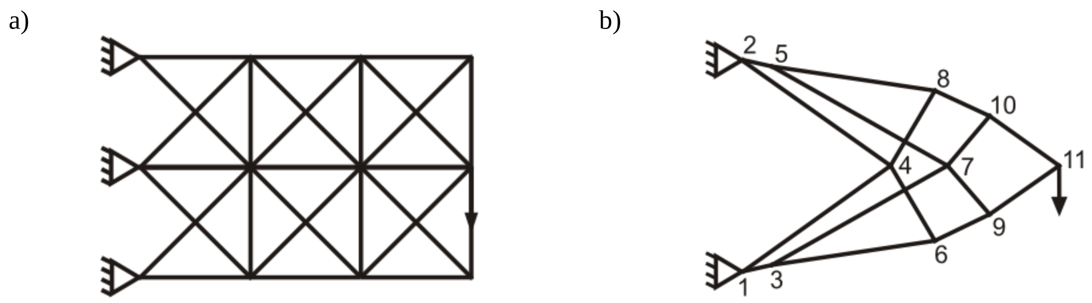


Fig. 6. Simultaneous optimization (solution from the paper [1]): a) The initial design, b) The optimal design

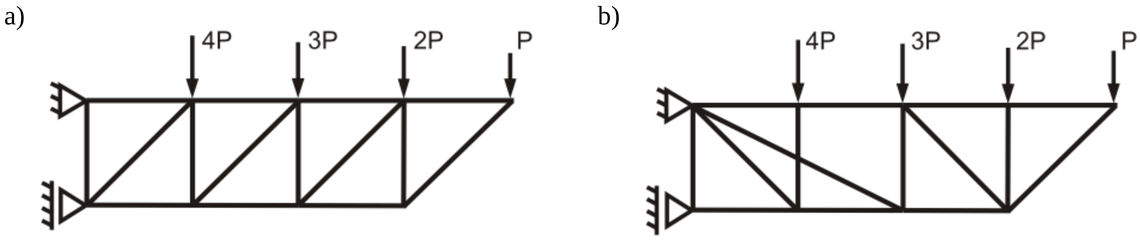
Table 2: Node coordinates for the optimal designs

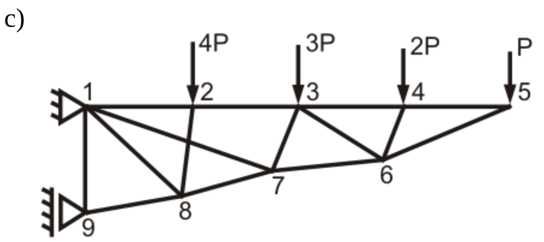
No. of node	Design from Fig.4c		Design from Fig.5a		Design from Fig.5b		Design from Fig. 6b	
	horizontal coordinate	vertical coordinate	horizontal coordinate	vertical coordinate	horizontal coordinate	vertical coordinate	horizontal coordinate	vertical coordinate
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	2.00	0.00	2.00	0.00	2.00	0.00	2.00
3	1.97	0.33	1.97	0.33	1.74	0.26	0.27	0.06
4	1.53	1.00	1.54	1.00	1.32	1.00	1.40	1.00
5	1.97	1.67	1.97	1.67	1.74	1.74	0.27	1.94
6	3.00	1.00	3.00	1.00	2.22	0.45	1.82	0.29
7					1.77	1.00	1.94	1.00
8					2.22	1.55	1.82	1.71
9					3.00	1.00	2.34	0.54
10							2.34	1.47
11							3.00	1.00

6.3. Optimal design of topology and geometry of non-symmetric cantilever truss for minimum cost under buckling and stress constraints
Let us consider now, for the 15-bars truss (Fig. 7a), the cost minimization problem of the form expressed by (19). The truss is loaded by forces P, $2P$, $3P$, $4P$, where $P = 10^3 N$. Moreover, the Young's modulus is $E = 2.1 \cdot 10^{11} N/m^2$, the yield limit equals $R_e = 2.5 \cdot 10^8 N/m^2$, the minimal cross-sectional area is $A_{\min} = 0$ and the specific material costs c_i are the same for all bars. We assume that all cross-sections have circular shape and in this case, the coefficient in (22) equals $\xi = 1/(4\pi)$.
The optimization will be performed using two methods and here 5 virtual bars will be used. At first, the alternating approach is applied. After topology optimization, where the problem (24) is solved directly with respect to topological parameters $\alpha_k, k = 1,2,..., K$, or bars exchanges approach is used, we obtain the structure shown in Fig. 7b. Further cost reduction can be achieved by configuration optimization and finally the optimal design presented in Fig. 7c is determined. The ratio of the costs of the initial design from the Fig. 7a and the optimal design equals $C^{(init)}/C^{(opt)} = 1.6045$. Analogous results for similar problems were also obtained in [7] and [13].
When the simultaneous approach is used, we get the same optimal design as in the previous case. However, here the optimization process is not numerically stable. Moreover, if $A_{\min} > 0$ some difficulties with determination of removed (added) bars may appear.

7. Conclusions
The methods of optimization of trusses topology, configuration and cross-sectional areas using concept of so-called virtual bars are presented in this paper. Application of this approach leads to considerable decrease of

number of design parameters comparing to commonly used ground structure approach and in consequence to significant reduction of computation time required for generation of optimal designs. The considerations are illustrated by some numerical examples. Here, two different methods namely alternating approach and simultaneous approach are applied and the results were compared to each other and with the results available in literature. Generally, the examples confirm usefulness and efficiency of the presented algorithms. However, because of big number of local minima, in each case we can expect different solutions, which differs only a little of value of the objective function.





No. of node	Horizontal coordinate	Vertical coordinate
1	0.00	1.00
2	1.00	1.00
3	2.00	1.00
4	3.00	1.00
5	4.00	1.00
6	2.80	0.50
7	1.75	0.39
8	0.90	1.16
9	0.00	0.00

Fig. 7. Simultaneous optimization: a) The initial design, b) The optimal topology, c) The optimal design

8. References

[1] W. Achtziger, On simultaneous optimization of truss geometry and topology, *Structural and Multidisciplinary Optimization*, 33, 285-304, 2007.

[2] J. Barta, On the minimum weight of certain redundant structures, *Acta Technica Academiae Scientiarum Hungaricae*, 18, 67-76, 1957.

[3] Z.P. Bazant and L. Cedolin, *Stability of structures*, Oxford University Press, 1991.

[4] M.P. Bendsoe and O. Sigmund, *Topology Optimization: Theory, Methods and Applications*, Springer-Verlag, Berlin, 2003.

[5] D. Bojczuk, *Sensitivity analysis and optimization of bar structures*, Publishing House of Kielce University of Technology, Kielce, 1999 [in Polish].

[6] D. Bojczuk, Sensitivity analysis of elastic bar and beam structures, *Engineering Transactions*, 35, 171-204, 1987, [in Polish].

[7] D. Bojczuk and A. Rębosz-Kurdek, Topology optimization of trusses using bars exchange method, *Bulletin of the Polish Academy of Sciences - Technical Sciences*, 60, 185-189, 2012.

[8] W.S. Dorn, R.E. Gomory and H.J. Greenberg, Automatic design of optimal structures, *Journal de Mechanique*, 3, 25-52, 1964.

[9] P. Fleron, Minimum weight of trusses, *Bygningsstatiske Meddelelser*, 35, 81-96, 1964.

[10] U. Kirsch, Optimal topologies of truss structures, *Applied Mechanics Reviews*, 42, 223-239, 1989.

[11] Z. Mróz and D. Bojczuk, Finite topology variations in optimal design of structures, *Structural and Multidisciplinary Optimization*, 25, 153-173, 2003.

[12] Z. Mróz and D. Bojczuk, Shape and topology sensitivity analysis and its application to structural design, *Archive of Applied Mechanics* 82, 1541-1555, 2012.

[13] P. Pedersen, On the minimum mass layout of trusses, *AGARD Conference Proceedings No. 36*, 11.1-11.17, http://www.cso/nato.int, Symposium of Structural Optimization, Istanbul, Turkey, 1969.

[14] P. Pedersen, On the optimal layout of multi-purpose structures, *Computers and Structures*, 2, 695-712, 1972.

[15] G. Rozvany, M. Bendsoe, U. Kirsch, Layout optimization of structures, *Applied Mechanics Reviews*, 48, 41-119, 1995.