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A discussion about choosing an objective function and constraint conditions in structural topology optimization

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1. Abstract
Based on some examples calculated using the 99-line code made by Sigmund [1], the structural topology optimization model construction is discussed to propose a problem that whether the topology optimization model construction could be more reasonable. In order to solve this problem, a topology optimization model according to practical engineering is constructed to search for the minimization of structural weight with a displacement constraint. And a corresponding 120-line code written in Matlab is made by us. Calculation results of the examples show that the model of the 120-line topology optimization code is more reasonable.
2. Keywords: structural topology optimization; optimization model; choosing an objective function and constraint conditions; 99-line topology optimization code; 120-line topology optimization code

3. Introduction
The purpose of structural topology optimization is to seek the optimum layout of structural components or sub-domains within a given design space under a given set of loads and boundary conditions such that the resulting layout meets a prescribed set of performance targets. As an important branch of structural optimization research, topology optimization has been enriched by lots of solution methods. An exhaustively detailed summary for the development of topology optimization was made by Bendsøe and Sigmund [2], and the application and development of numerical modeling methods used in structural topology optimization were also reviewed in details by Rozvany [3].

Looking back the research of truss structure topology optimization led by Michell [4], a great progress has been made since then. The Michell truss theory is not only developed by Rozvany et al. [5-6], but also the research objects of topology optimization have developed from skeleton structures such as truss and frame into continuum structures. Although the paper [7] superficially stated to be the homogenization method, Bendsøe and Kikuchi broaden the research objects of structural topology optimization, because they presented the concept of topological optimization for the continuum structures. The ground structure approach [8], which was only used in topological optimization of skeleton structures, has become the foundation of the homogenization method in topological optimization of continuum structures, and also the foundation of the methods seeking the optimum layout of sub-regions for a given domain, such as the variable thickness method [9], the artificial material method [1,10-12], the evolutionary structural optimization method [13], the independent continuous and mapping method [14-17], the level set method [18-20], etc. Due to limited length of the paper, we can't list all of the research here and can only mention the methodologies above, which are a few of thousands.

Among many studies, the 99-line topology optimization code written in Matlab by Sigmund has been a unique significant research [1]. The code has been published on web (http://www.topopt.dtu.dk/?q=node/2), which could be downloaded. We notice that Rozvany gave a good evaluation for this work. He said it "played an important role in SIMP's general acceptance" [3].

In our opinion, "an important role" has two senses: it is not only extracurricular materials for students to master the knowledge of structural topology optimization, but also an introductory tool for engineers to understand and use a method of structural topology optimization. The 99-line code is short and includes finite element analysis and optimizer subroutines, where the users can change the definition of structural sizes, set different loading and boundary conditions and solve problems. Beginners can get great inspiration and help from it.

The 99-line topology optimization code is based on the SIMP (Solid Isotropic Material with Penalization) model of the artificial material method and its formulation shows as below:

$$E_e = E(\rho_e) = \rho_e^p E_0 \qquad \rho_e \in [0,1] \tag{1}$$

where E_e and E_0 are the Young's modulus of artificial and real material respectively, ρ_e is the elementary artificial relative density, and p is the penalty factor.

On one hand, Eq. (1) involves the artificial relative density variables on a closed interval [0, 1], and in fact the discrete 0/1 topology variables are made to be continuous ones. On the other hand, the penalty function is used to establish the relationship between artificial relative densities and Young's modulus, and the explicit relationship of element stiffness and continuous topology variables is formulated essentially. Actually, the artificial material method has overcome two difficulties of structural topology optimization through Eq. (1). The first difficulty is that the model can not be constructed and the optimization problem can not be solved owing to discrete 0/1 topology design variables. The second difficulty is that the structural performance couldn't be expressed by topology design variables.

The structural topology optimization model in paper [1] is to seek minimum compliance, or maximal stiffness, with a volume constraint. We call it the MCWC (minimum compliance with a weight constraint) formulation. It is presented as below:

$$\begin{cases} \text{For} & \boldsymbol{\rho} \\ \min & c(\boldsymbol{\rho}) = \boldsymbol{U}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{U} = \sum_{e=1}^{N} \boldsymbol{u}_e^{\mathrm{T}} \boldsymbol{k}_e \boldsymbol{u}_e \\ \text{s.t.} & \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \\ & \dfrac{V(\boldsymbol{\rho})}{V^*} = f \\ & 0 < \rho_{\min} \le \rho_e \le 1 \quad (e = 1, \cdots, N) \end{cases} \tag{2}$$

where $\boldsymbol{K}$, $\boldsymbol{U}$ and $\boldsymbol{F}$ are the structural global stiffness matrix, total displacement vector and total force vector, respectively, $\boldsymbol{u}_e$ is the displacement vector of the e-th element, $\boldsymbol{k}_e = \boldsymbol{k}(\rho_e) = E(\rho_e)\boldsymbol{k}_e^0$ is the element stiffness matrix of the e-th element, $\boldsymbol{k}_e^0$ is the element stiffness matrix for unit Young's modulus, $\boldsymbol{\rho}$ is the artificial relative density vector. In order to avoid singularity of the stiffness matrix, the minimum relative density $\rho_{\min}$ is usually taken as 0.001. N is the total number of elements, and the penalty factor p is usually equal to 3. $V(\boldsymbol{\rho})$ and V^* are the total volumes of the designed structure and the ground structure, respectively, f is the pre-setting volume ratio. Eq. (2) has been solved by an optimality criteria method.

During trying to adopt the 99-line topology optimization code, a number of examples were computed. We found that the pre-setting volume ratio had a decisive influence on the results of structural topology optimization: different volume ratios would produce different topological configurations. This awaken us to think further on the relationship between the pre-setting volume ratios and optimal topological configurations. Based on this, we consider a question: whether structural topology optimization has a more reasonable model?

Following the train of thinking, this paper proposes a structural topology optimization model of minimizing structural weight with a displacement constraint, and develops a corresponding 120-line topology optimization code. Through the example calculations by the 120-line code and then they are compared with the calculation results by the 99-line code. Research results show that constructing a more reasonable structural topology optimization model is a significant mention.

4. Expression of structural topology optimization with a more reasonable model construction

The optimization problems discussed in this paper will be focused on structures that are able to bear external forces, rather than force inverter or compliant mechanisms. Therefore, 4 simple structures are used as examples in this paper, which are shown in Figure 1.

Example 1, shown in Figure 1 (a), is the half of MBB-beam, with a ground structure of 60mm×20mm×1mm, unit force F=1, unit Young's modulus E^0=1.0 and the Poisson's ratio $\nu = 0.3$.

Example 2, shown in Figure 1 (b), is the short cantilever beam, with a ground structure of 32mm×20mm×1mm, unit force F=1, unit Young's modulus E^0=1.0 and the Poisson's ratio $\nu = 0.3$.

Example 3, shown in Figure 1 (c), is the short cantilever beam with a fixed hole, with a ground structure of 45mm×30mm×1mm, unit force F=1, unit Young's modulus E^0=1.0 and the Poisson's ratio $\nu = 0.3$. The center of the hole is located at the intersection of 1/3 horizontal length and 1/2 vertical length from left to right, while the radius is equal to 1/3 vertical length.

Example 4, shown in Figure 1 (d), is the cantilever beam, with a ground structure of 80mm×50mm×1mm, loading force F=9kN , Young's modulus E^0=1.0×10^6MPa and the Poisson's ratio $\nu = 0.3$.

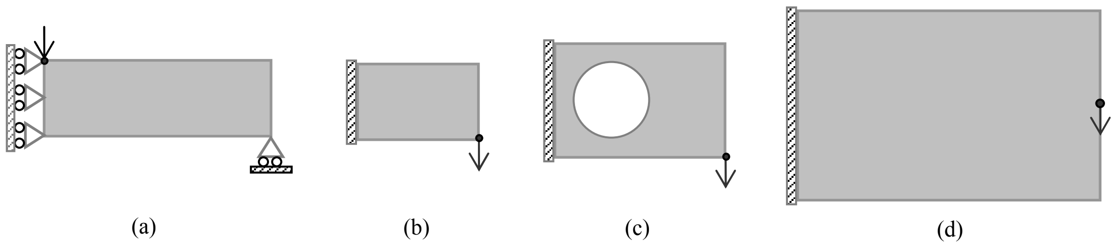


Figure 1: Ground structures and boundary conditions for the four test examples

The first three examples are from paper [1]. There are 9 sub-examples for each example, where the pre-setting volume ratios are 9 equally distributing numbers between 0.1 and 0.9.

A total of 36 sub-examples are all computed by the 99-line topology optimization code and the results are shown in Table 1. Meanwhile, the optimal results of four examples with the pre-setting volume ratio of 0.1 are examined in more detail and shown in Table 2, where in the "Number of elements" column, "Black" presents the number of black elements with a relative density of 1.0, "White" presents the number of white elements with a relative density of 0.001, and "Grey" presents the number of grey elements with various relative densities on the open interval (0.001, 1.0).

Table 1: Optimum topological configurations for different pre-setting volume ratios

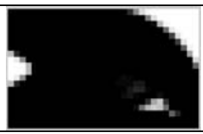
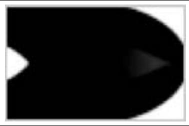
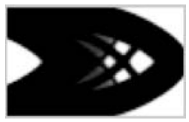
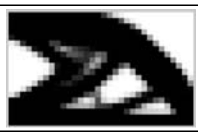
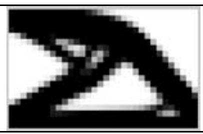
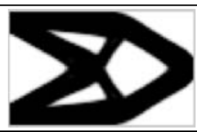
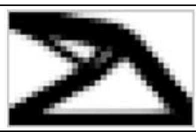
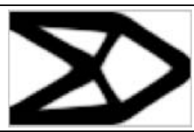
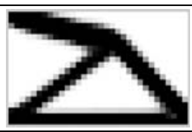
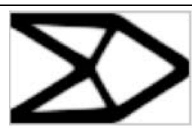
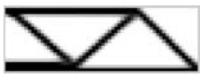
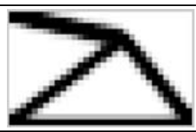
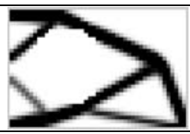
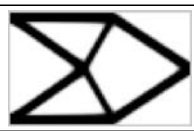
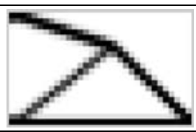
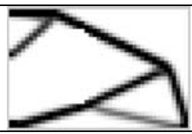
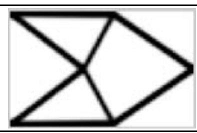
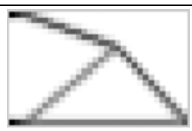
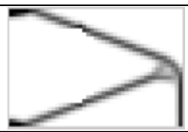
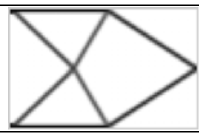
Volume ratio		Example 1	Example 2	Example 3	Example 4
0.9	Iterations	45	44	7	29
	Opt. Config.				
0.8	Iterations	96	80	7	106
	Opt. Config.				
0.7	Iterations	134	56	17	81
	Opt. Config.				
0.6	Iterations	79	50	23	129
	Opt. Config.				
0.5	Iterations	94	69	34	58
	Opt. Config.				
0.4	Iterations	63	71	35	47
	Opt. Config.				
0.3	Iterations	135	42	66	44
	Opt. Config.				
0.2	Iterations	363	222	83	44
	Opt. Config.				
0.1	Iterations	1132	756	740	1971
	Opt. Config.				

Table 2: Results of 4 examples with the pre-setting volume ratio of 0.1

Example	Number of structure analysis	Compliance	Number of elements			Total volume of grey elements	Average density of grey elements
			Black	White	Grey		
Example 1	1132	4785.2511	0	604	596	119.3960	0.2003
Example 2	756	899.0410	0	382	258	63.6180	0.2466
Example 3	740	787.8421	6	904	440	128.0961	0.2911
Example 4	1971	33266.6946	6	2740	1254	391.2600	0.3120

From the results of 36 sub-examples in Table 1, we observe "dependency of the structural optimal topological configurations on the pre-setting volume ratios", and summarize the following three aspects:

(1) 3 or 4 different kinds of typical structural configurations are obtained in all 4 examples.

(2) When the pre-setting volume ratio is too small, the structure will be broken off or disconnected, and it degenerates into a mechanism. According to the data shown in Table 2, when the pre-setting volume ratio of each example is 0.1, the number of the black elements in the structural optimal topological configurations is very small or even zero. At the same time, the average densities of grey elements are 0.2003, 0.2466, 0.2911 and 0.3120, respectively. These show that the structures have already degenerated into mechanisms.

(3) Along with the reduction of pre-setting volume ratios, all 4 examples go through an evolution from "cumbersome" structures, Michell-truss-like structures, "light" structures into degenerating mechanisms at the end.

Since the optimal topological configurations are really dependent on the pre-setting volume ratios, there is a question that how a reasonable pre-setting volume ratio can be selected in the stage of conceptual design, Could the most appropriate volume ratio be found logically? In other words, rather than using a presumed volume ratio, why don't we determine an "optimal volume ratio" at the same time with searching for an optimal topological configuration?

5. Presentation of a topology optimization model best fitting practical engineering problems

Many examples in the last section inspire us to seek the "optimal volume ratio" at the same time with searching for an optimal topological configuration. This means that there will be a new volume ratio objective function besides the compliance objective function when we use the 99-line topology optimization code. If there are a compliance objective function and a volume ratio objective function at the same time, the optimization problem wouldn't be the one with a single objective function and the optimum solutions wouldn't be determined.

Could the volume ratio only be considered as a single objective function? Since the compliance is not taken as the objective function, it should be transformed into a constraint condition. However, what should the compliance constraint value be selected? Therefore, there is a similar difficulty like that one in the 99-line topology optimization code, in which a constraint value of the most appropriate volume ratio is not determinate.

How can the difficulty of selecting a compliance constraint value be overcome? In fact, if we move our focus from compliance into another one, there is a solution. Actually, rather than the compliance constraint, the strength or stiffness constraint is considered for practical engineering problems. That is, we can use the strength or stiffness constraint instead of the compliance constraint.

In order to try to put this idea into practice, we present a topology optimization model of minimizing structural weight with satisfying a displacement constraint at a certain point of interest. Why is the volume ratio objective function changed into the structural weight objective function? This is because minimizing structural weight is equivalent to minimizing structural volume with the same material, while the minimum structural volume is equivalent to the minimum structural volume ratio with the same ground structure. So minimizing structural weight should be equivalent to minimizing the volume ratio. The advantage of using the structural weight objective function is that the formulation of topology optimization is in line with formulations of the section optimization and the shape optimization.

A minimum weight formulation for topology optimization with a displacement constraint can be presented as follows:

$$\begin{cases} \text{For} \ \ \boldsymbol{\rho} \\ \min \ W(\boldsymbol{\rho}) = \sum_{e=1}^{N} \rho_e w_e^0 \\ \ \text{s.t.} \ \ u(\boldsymbol{\rho}) = \overline{u} \\ \qquad 0 < \rho_{\min} \le \rho_e \le 1 \ \ (e = 1, \cdots, N) \end{cases} \tag{3}$$

where w_e^0 is the element weight when the density of the *e*-th element equals to 1, $W(\boldsymbol{\rho})$ and $u(\boldsymbol{\rho})$ are the structural total weight and the displacement at a point of interest respectively, $\overline{u}$ is the displacement allowable value. Similarly with the 99-line code, the minimum density is set as $\rho_{\min} = 0.001$ for an avoidance of singular stiffness matrix in seeking optimum solutions.

5.1. Derivation for the explicit displacement function at the point of interest

Using the unit virtual load method [21], the displacement at a point of interest can be expressed by calculating the virtual work as follows:

$$u = \sum_{e=1}^{N} u_e = \sum_{e=1}^{N} \int_e \boldsymbol{\sigma}_R^{\mathrm{T}} \boldsymbol{\varepsilon}_v \mathrm{d}v \tag{4}$$

where u_e is the *e*-th element's contribution to the displacement value at the point of interest, $\boldsymbol{\sigma}_R$ and $\boldsymbol{\varepsilon}_v$ denote the element stress vectors associated to the real load and the element strain vectors associated to the virtual load, respectively.

According to the work energy theorem, the virtual work in Eq. (4) is equal to the virtual work obtained from nodal forces and nodal displacements:

$$u = \sum_{e=1}^{N} (\boldsymbol{P}_e^R)^{\mathrm{T}} \boldsymbol{u}_e^v \tag{5}$$

where $\boldsymbol{P}_e^R$ and $\boldsymbol{u}_e^v$ are the element's nodal force vector associated to the real load and the corresponding element's nodal displacement vector associated to the virtual load in the *e*-th element, respectively, and $u_e = (\boldsymbol{P}_e^R)^{\mathrm{T}} \boldsymbol{u}_e^v$.

Due to the element stiffness equation, we have:

$$\boldsymbol{K}_e \boldsymbol{u}_e^v = \boldsymbol{P}_e^v \tag{6}$$

where $\boldsymbol{K}_e$ and $\boldsymbol{P}_e^v$ are the element's stiffness matrix and the element's nodal force vectors associated to the virtual load in the e-th element, respectively.

Substituting Eq. (6) into Eq. (5), we can get:

$$u=\sum_{e=1}^{N}(\boldsymbol{P}_e^R)^{\mathrm{T}}(\boldsymbol{K}_e)^{-1}\boldsymbol{P}_e^v \tag{7}$$

Since the element stiffness is proportional to the Young's modulus, the element's stiffness can be written as:

$$\boldsymbol{K}_e=\boldsymbol{k}_e^0E_e=\boldsymbol{k}_e^0E_0\rho_e^p=\boldsymbol{K}_e^0\rho_e^p \tag{8}$$

where $\boldsymbol{K}_e$ is the element stiffness matrix; $\boldsymbol{k}_e^0$ is the element stiffness matrix for unit Young's modulus; E_e is the element Young's modulus for linear isotropic material (artificial material); E_0 is the Young's modulus of solid material (real material, $\rho_e=1$); $\boldsymbol{K}_e^0$ is the element stiffness matrix when $\rho_e=1$ with the Young's modulus E_0.

Substituting Eq. (8) into Eq. (7), we obtain:

$$u=\sum_{e=1}^{N}u_e=\sum_{e=1}^{N}(\boldsymbol{P}_e^R)^{\mathrm{T}}(\boldsymbol{K}_e^0)^{-1}\boldsymbol{P}_e^v\rho_e^{-p}\approx\sum_{e=1}^{N}\frac{D_e^0}{\rho_e^p} \tag{9}$$

where D_e^0 is the constant coefficient of the displacement contribution component of the e-th element to the point of interest. In Eq. (9), the reason of $(\boldsymbol{P}_e^R)^{\mathrm{T}}(\boldsymbol{K}_e^0)^{-1}\boldsymbol{P}_e^v\approx D_e^0$ rests with introducing an assumption of the statically determinate structure. Because the displacements of the statically determinate structure have no relationship with the nodal force vectors and our structure is regarded as a statically determinate one in each structural optimization iteration, D_e^0 is a constant which is not dependent on $\boldsymbol{P}_e^R$ and $\boldsymbol{P}_e^v$ for each iteration.

5.2. Solutions of the optimization model

Substituting Eq. (9) into Eq. (3), we obtain an explicit topology optimization model as follows:

$$\begin{cases}\text{For } \boldsymbol{\rho}\\ \min\ W(\boldsymbol{\rho})=\sum_{e=1}^{N}\rho_e w_e^0\\ \text{s.t. } \sum_{e=1}^{N}\dfrac{D_e^0}{\rho_e^p}=\overline{u}\\ \quad 0<\rho_{\min}\le\rho_e\le 1\ (e=1,\cdots,N)\end{cases} \tag{10}$$

According to Lagrangian multiplier method, we can get:

$$L(\boldsymbol{\rho},\lambda)=\sum_{e=1}^{N}\rho_e w_e^0+\lambda\left(\sum_{e=1}^{N}\frac{D_e^0}{\rho_e^p}-\overline{u}\right) \tag{11}$$

The first order derivatives of the Lagrangian function are found as:

$$\frac{\partial L}{\partial \rho_e} = w_e^0 - \lambda p \frac{D_e^0}{\rho_e^{p+1}} = 0 \tag{12}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{e=1}^{N} \frac{D_e^0}{\rho_e^p} - \bar{u} = 0 \tag{13}$$

The design variables can be obtained from Eq. (12) as follows:

$$\rho_e = \left(\frac{\lambda p D_e^0}{w_e^0}\right)^{\frac{1}{p+1}} \tag{14}$$

Substituting Eq. (14) into Eq. (13), we can get

$$(\lambda p)^{\frac{1}{p+1}} = \left(\frac{\sum_{e=1}^{N} (D_e^0)^{\frac{1}{p+1}} \cdot (w_e^0)^{\frac{p}{p+1}}}{\bar{u}}\right)^{\frac{1}{p}} \tag{15}$$

Then, the Lagrangian multiplier λ is eliminated by substituting Eq. (15) into Eq. (14), and the solutions are got as follows:

$$\rho_e = \left(\frac{\sum_{k=1}^{N} (D_k^0)^{\frac{1}{p+1}} (w_k^0)^{\frac{p}{p+1}}}{\bar{u}}\right)^{\frac{1}{p}} \cdot \left(\frac{D_e^0}{w_e^0}\right)^{\frac{1}{p+1}} \tag{16}$$

In order to think of contribution of element densities to the displacement at the point of interest, the partial derivative of Eq. (9) with respect to densities are obtained as follows:

$$\frac{\partial u}{\partial \rho_e} = -\frac{p D_e^0}{\rho_e^{p+1}} \tag{17}$$

When $-D_e^0 > 0$, we have $\frac{\partial u}{\partial \rho_e} > 0$. That's to say, if the element density ρ_e is increased, the displacement u at the point of interest will be increased; if the element density ρ_e is decreased, the displacement at the point of interest will be decreased. So increasing element density ρ_e is not economical in terms of design. In this case, the element density values have to be decreased to reach the minimum value, therefore the displacement contribution and stiffness contribution of the corresponding element can reach minimum and maximum values, respectively. We don't need to design the density of this element in this case, so it is called as passive design variable.

When $-D_e^0 \le 0$, we have $\frac{\partial u}{\partial \rho_e} < 0$. That's to say, if the element density ρ_e is increased, the displacement u at the point of interest will be decreased; if the element density ρ_e is decreased, the displacement at the point of interest will be increased. In other words, the element density ρ_e is increased, displacement contribution and stiffness contribution of the corresponding element will be decreased and increased, respectively. This is a reasonable return on economic investment in increasing the element density ρ_e. In this case, the density should be designed, and it is called as the active design variable.

The discussion above is summarized as following: the elements satisfied with $D_e^0 \geq 0$ are designable elements, and the corresponding density variables are the active variables included in set $I_a = \left\{e \middle| D_e^0 \geq 0, e = 1, \cdots, m\right\}$; the elements satisfied with $D_e^0 < 0$ are not designable and the corresponding density variables are in the passive design variable set.

The values of the density variables in the passive design variable set are to remain constant during each iteration of seeking the optimum solutions. Therefore, the topology optimization model in Eq. (9) can be rewritten as follows:

$$\begin{cases} \text{For} \ \ \boldsymbol{\rho} \\ \min \ \ W(\boldsymbol{\rho}) = \sum_{e=1}^{m} \rho_e w_e^0 + W_0 \\ \ \text{s.t.} \ \ \sum_{e=1}^{m} \frac{D_e^0}{\rho_e^p} + u_0 = \bar{u} \\ \qquad 0 < \rho_{\min} \leq \rho_e \leq 1 \ \ (e = 1, \cdots, N) \end{cases} \tag{18}$$

where $W_0 = \sum_{e \notin I_a} \rho_e w_e^0$, and $u_0 = \sum_{e \notin I_a} \frac{D_e^0}{\rho_e^p}$. W_0 and u_0 are constants. So the constant terms, which have nothing to do with seeking optimization, can be eliminated from the objective function. Similar to the previous deduction, we can obtain the following equation to replace Eq. (16) as:

$$\hat{\rho}_e = \begin{cases} \left(\dfrac{\sum_{k=1}^{m} (D_k^0)^{\frac{1}{p+1}} (w_k^0)^{\frac{p}{p+1}}}{\bar{u} - u_0}\right)^{\frac{1}{p}} \cdot \left(\dfrac{D_e^0}{w_e^0}\right)^{\frac{1}{p+1}} & (e \in I_a) \\ \rho_{\min} \ or \ 1 \quad (e \notin I_a) \end{cases} \tag{19}$$

The active design variables calculated from the equation above are not ensured to have a smaller value than 1 or a larger value than the minimum density $\rho_{\min}$. Then an adjustment should be made as below:

$$\rho_e^* = \begin{cases} 1 & \text{where} \ e \notin I_a \ \ (if \ \ \hat{\rho}_e \geq 1) \\ \hat{\rho}_e & \text{where} \ e \in I_a \ \ (if \ \ \rho_{\min} < \hat{\rho}_e < 1) \\ \rho_{\min} & \text{where} \ e \notin I_a \ \ (if \ \ \hat{\rho}_e \leq \rho_{\min}) \end{cases} \tag{20}$$

Now that the active/passive design variable set is adjusted, the calculation according to Eq. (19) has to be re-performed, which is called as a small optimization iteration. The active/passive design variable set classified by the sign of D_e^0 is a global pre-judgment after one structural analysis before the topology optimization model construction for iterative solutions, but Eq. (20) is a local adjustment of reclassifying the active/passive design variable set after a small optimization iteration. This kind of small optimization iteration should be computed continuously according to Eq. (19) until the active/passive design variable sets are no longer changed，and the process of small optimization iterations stops. Thus, the optimal solutions of the model Eq. (18) for the corresponding structural analysis are obtained. Here, we say that a big optimization iteration is finished. Before a new big optimization iteration, a new structural analysis should be made. However, the whole optimization process needs sequential reconstructions of optimization models corresponding to sequential big optimization iterations.

To ensuring mesh-independency, a filter function expressed in Eq. (21) [17] is used in this paper. The displacement contribution component of each element to the point of interest in Eq. (9) is filtered. Thus the new coefficients for the filtered displacement components are obtained in Eq. (22) as follows:

$$\tilde{u}_e = \frac{\sum_{j \in N_e} w(\boldsymbol{x}_j)\rho_j u_j}{\sum_{j \in N_e} w(\boldsymbol{x}_j)} \tag{21}$$

$$D_e^0 = \tilde{u}_e \rho_e^p \tag{22}$$

where $\boldsymbol{x}_j$ is the spatial location of geometric center for the *j*-th element; $N_e = \left\{ j \middle| \left\| \boldsymbol{x}_j - \boldsymbol{x}_e \right\| \le r_{\min} \right\}$ denotes the neighborhood of the *e*-th element within a given filter radius $r_{\min}$ of the center of the *e*-th element; $w(\boldsymbol{x}_j) = \max(0, r_{\min} - \left\| \boldsymbol{x}_j - \boldsymbol{x}_e \right\|)$ is a weight function, also a linearly decaying weighting function, which is linearly reduced along the distances of the neighbor elements to the center element.

In order to keep the comparability of our topology optimization, the same convergence criterion as that in the 99-line topology optimization code is used in this paper. The optimal solutions $\boldsymbol{\rho}^{(k+1)}$ for the (k+1)-th iteration and $\boldsymbol{\rho}^{(k)}$ for the *k*-th iteration should satisfy a relationship as follows:

$$\max(\max(\left| \boldsymbol{\rho}^{(k+1)} - \boldsymbol{\rho}^{(k)} \right|)) \le 0.01 \tag{23}$$

6. Results comparison of two models

The solution process for the optimization model Eq. (3) is programmed into a 120-line topology optimization code, which can be downloaded from website https://sites.google.com/site/yiguilian/link. In order to compare with the 99-line code [1], 4 examples in Figure 1 are solved by the two codes. The point of interest is taken as the loading point.

Since a small-volume structure has little material and small stiffness, it causes a large displacement at the point of interest. On the contrary, a large-volume structure has a large stiffness and a small displacement at the point of interest. Therefore, whole calculation for each example has two stages. In the first stage, the model in Eq. (2) under a pre-setting volume ratio constraint is solved using the 99-line topology optimization code to obtain the displacement value at the loading point and the optimum topological configuration. This displacement value is taken as a displacement constraint value in the second stage. Then, the model in Eq. (3) under the displacement constraint is solved using the 120-line topology optimization code. All the data and optimal topological configurations are shown in Table 3 and Figure 2, respectively. In the 120-line code, there is a "measure of discreteness" [22] which is used to distinguish whether an optimum solution converges to a discrete one. This measure of discreteness is defined as:

$$M_{nd} = \frac{\sum_{e=1}^{N} 4\rho_e^*(1-\rho_e^*)}{N} \times 100\% \tag{24}$$

where ρ_e^* is the optimum relative density of the *e*-th element. Once the design is converged, M_{nd} with a value of 0% denotes all element relative densities are equal to 0 or 1, and M_{nd} with a value of 100% denotes all element relative densities are equal to 0.5. Therefore, a smaller value of M_{nd} means less grey elements and is expected to give a better design.

In Table 3, the meanings of Black, White and Gray in the "Number of elements" column are the same as those in Table 2. "Total weight" denotes the structural weight without the white elements with a relative density of 0.001, including the weight of all black and grey elements. "Weight of grey elements" denotes the total weight of all grey elements, and "Ratio of grey elements" is the percentage ratio of the number of grey elements in the number of all elements.

Table 3: Results for 4 examples

Example		Number of structural analysis	Total weight	Displacement at the point of interest	Number of elements			Weight of grey elements	Ratio of grey elements (%)	M_{nd} (%)
					Black	White	Grey			
Example 1	99-line code	94	599.67	203.2980	447	321	432	152.67	36.00	17.55
	120-line code	121	599.15	203.2961	467	370	363	132.15	30.25	16.35
Example 2	99-line code	71	255.74	57.3525	197	257	186	58.74	29.06	13.96
	120-line code	70	255.81	57.3543	208	273	159	47.81	24.84	12.75
Example 3	99-line code	34	674.60	52.0993	544	467	339	130.60	25.11	12.91
	120-line code	26	673.29	52.0988	565	491	294	108.29	21.78	11.67
Example 4	99-line code	58	1998.54	0.3488	1718	1458	824	280.54	20.60	9.56
	120-line code	68	1990.17	0.3488	1764	1532	704	226.17	17.60	8.70

In Figure 2, five parameters between parentheses in each of eight captions for configurations all have their own meaning on the left side and right side, respectively. For example on the left side, in Figure 2 a1) there is a caption "99-line code (60,20,0.5,3.0,1.5)", which represents concrete computing data of an example for using the

99-line code, where 60 elements grids in x-direction, 20 elements grids in y-direction, volume ratio constraint value 0.5, penalty factor 3.0 and filter radius 1.5. For example on the right side, in Figure 2 a2), there is a caption “120-line code (60,20,203.2980,3.0,1.5)”, which represents concrete computing data of an example for using 120-line code, where 60 elements grids in x-direction, 20 elements grids in y-direction, displacement constraint value 203.2980, penalty factor 3.0 and filter radius1.5.

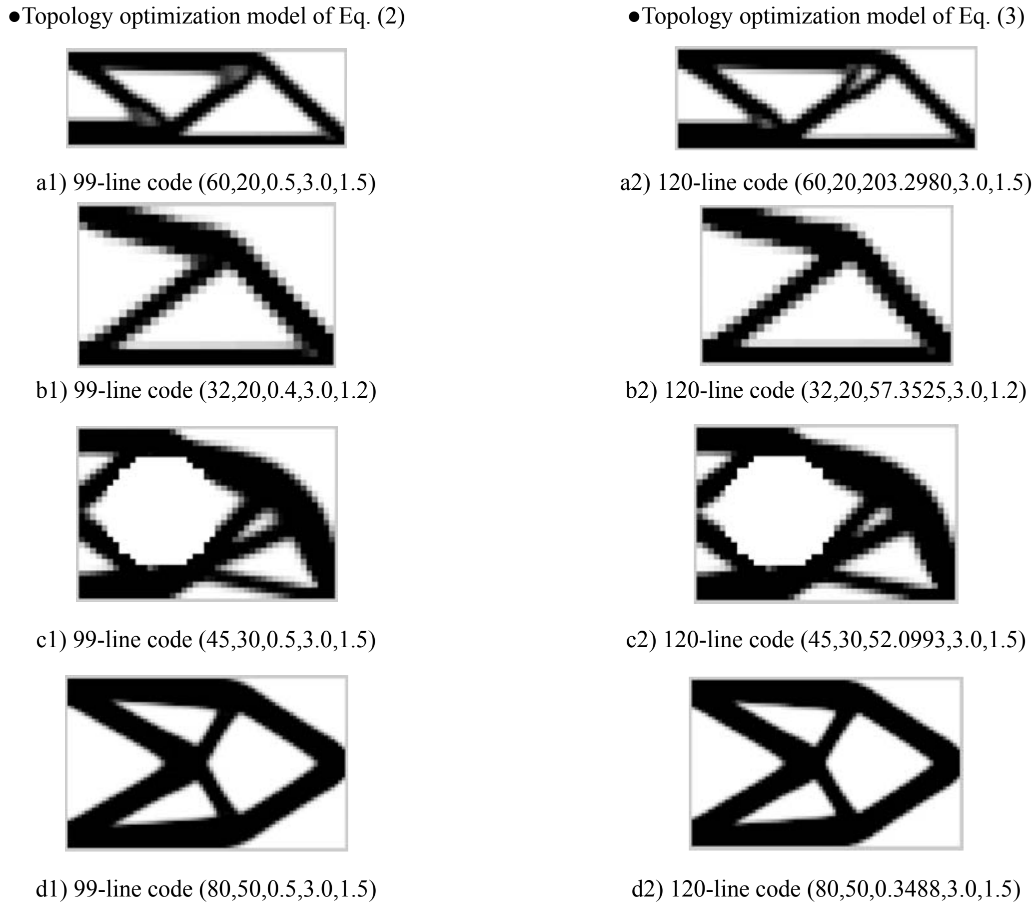


Figure 2: Optimum topological configurations for 4 examples

The computational results show that, the topology optimization model proposed in this paper can basically satisfy the displacement constraint, which is given by the 99-line code at the loading point. And the 120-line code obtains smaller structural weight than the 99-line code. Most of all, the 120-line code can get smaller values of measure of discreteness and less grey elements than the 99-line code. The method of this paper improves the degree of discreteness of design variables in the topology optimization. In computing speed, the convergence rates of both codes are of about the same. They indicate the topology optimization model presented in this paper is more reasonable in the model construction and results.

7. Conclusions

In the introduction of this paper, we pointed out: the 99-line topology optimization code played “an important role” in helping students and engineers to learn topology optimization. Our work here indicates that the 99-line topology optimization code also plays “an important role” for researchers in structural topology optimization. The reason is that the 99-line code inspires us to put forward a more reasonable model construction for topology optimization through the examples calculated by itself.

In structural topology optimization, although there are many researches on optimization model with objective function of structural weight subjected to displacement constraints, we have not seen a detailed comparison on results about above model and a model with objective function of structural compliance under a structural volume ratio constraint. Admitting that the method of structural topology optimization can be used for references to design useful compliant mechanisms such as the force inverter, but we limited ourselves here to the study of topological changes of structures.

(1) The essence of the structural weight or volume ratio is an economic indicator, and the compliance, displacement, stress, buckling, frequency, fatigue etc. are mechanical performance indicators.

(2) The purpose of structural optimization is to find the optimum points representing a compromise in the mutual restraint and mutual game between economic indicators and mechanical performance indicators. Finally, a contradiction between construction cost and structural safety will be solved.

(3) The practical engineering problems usually seek the minimum economic indicator, which is as the objective function, with guarantee of the constraints of mechanical performance indicators. Why can not we reverse them? Because it's enough to guarantee structural safety represented by the mechanical performance indicators mechanical performance has no significance. Since the economic indicator belongs to fabricating cost in essence, its beforehand setting is impossible. Thereby it is a behavior of apriorism. Conversely, seeking the minimum economic indicator should be done along with satisfying the mechanical performance indicators, and the reasonable cost can be obtained in accordance with its natural tendency.

(4) The mechanical performance indicators can be classified into "engineering" and "academic". The engineering performance indicators include the allowable displacements, the allowable stresses and so on. But the compliance belongs to the academic performance indicators. The engineering performance indicators are the main points that should be satisfied directly in design and they have been refined into perfection day by day by the accumulated experience of engineers. But the academic performance indicators are sometimes the information concerned by researchers. If the academic performance is supplied to reference in design process, the reference is only indirect in consideration, but without direct interest.

(5) The mechanical performance indicators are related to the loading cases. If a mechanical performance is taken as the objective function, in multi-loading cases, the optimization problem will become a multi-objective optimization problem, which is difficult to be solved. Since the economic indicators are not the function of loading cases, it is taken as the objective function to avoid producing the multi-objective optimization problem, and we always obtain an optimization problem with the single objective function.

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