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Numerical study of avoiding mechanism issues in structural topology optimization

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1. Abstract

In the work of paper [1], it was found that the optimal topological configurations depended on the value of the volume ratio; especially when the volume ratio was too small, the structure tended to be broken off or disconnected and degenerated into a mechanism. In order to address the issue of yielding mechanisms in structural topology optimization by using the MCVC (minimizing compliance with a volume constraint) model, based on the 99-line code in paper [2], this paper calculates a large number of examples with mechanisms appearance by referring to the solutions from the 120-line code of the MWDC (minimizing weight with a displacement constraint) model in paper [1]. In each example, besides the volume ratio, the maximum value of the ratio of the strain energy of each configuration in iterative process to the strain energy of the initial ground structure with full materials is investigated. It is found that, the mechanism occurs when the maximum strain energy ratio is larger than a threshold, or when the volume ratio is less than a threshold. The two thresholds can vary for different structures and/or loading cases, but for those examples that were tested, the range of the threshold of the maximum strain energy ratio turns out to be between 4.59 and 12.38, and the range of the threshold of the volume ratio is between 0.15 and 0.26. Researchers of structural topology optimization and engineers can use above numerical regularities to prevent mechanisms appearance in the final topological configurations. This work will help them to avoid unadvisable constraint values, which yield the final configuration with a mechanism, and to use advisable constraint values for calculation in structural topology optimization.

2. Keywords: structural topology optimization, volume constraint, displacement constraint, mechanism, structural strain energy ratio

3. Introduction

In the study of paper [1], the 99-line topology optimization code in paper [2] had been fully used to calculate a large number of numerical examples. Based on those examples, a question about constructing a more reasonable model in structural topology optimization was proposed, and a 120-line topology optimization code was studied and developed in the meantime. Whether a more reasonable topology optimization model can be constructed, it depended on mutual conformity and appropriate selection of the objective function and constraints. In previous studies, objective functions could usually be the compliance [2-3], stiffness [4], strain energy [5], weight [6-7] and cost [8] et al.. Meanwhile, the constraints could be structural volume constraint [3], structural weight constraint [9], displacement constraints [10], stress constraints [11], buckling constraints [12] and frequency constraints [13] et al.. In practical engineering problems, the objective function, as the goal pursued by engineering designs, should be chosen demonstrably. However, if a topology optimization problem had only an objective function, and no definition about constraints, no matter what the objective function was, the results of this unconstraint optimization problem would be all 0 or 1, and the structural topology optimization would be meaningless. If an objective function was chosen with unreasonable constraints, no matter what kind of numerical algorithm was used for solving, the structure would be optimized in the direction against expectations in the process of adding and eliminating elements constantly. Obviously, choosing constraints reasonably is a key point to implement structural topology optimization.

The constraints' formulations and bound values are generally determined by the needs of practical engineering designs and the wealthy experience of engineers. In paper [1], it was clearly expounded that an optimization model could be more reasonable by choosing an economic indicator as the objective function and mechanical performance indicators as the constraints. Therefore, in most cases, this kind of model can avoid structures yielding mechanisms and get safe structural designs.

On the contrary, if a structural mechanical performance is taken as the goal of our pursuit, such as minimizing the structural compliance, and in the meantime, the structural volume ratio is taken as the constraint, it has been found in the calculations of paper [1] that, the final configuration might be a mechanism when the pre-setting volume ratio is too small. What's the reason for this issue? The question is easily to be answered, because a quite small pre-setting volume ratio causes a lack of materials, whose distribution is not enough for resisting the external loads. Numerically speaking, when a mechanism occurs in the final topological configuration, the number of effective elements (black and grey elements) is very small, or almost none. Besides, the average value of grey element densities is quite small. According to the illustration of grey images, some parts in the final topological configurations, where mechanisms occur, show suspension, disconnection or articulation.

Due to the inadequate consideration of practical engineering needs and the limitation of subjective experience accumulation, the possibility that structures yield mechanisms in structural topology optimization, still exists. This means that the safety and validity of structural designs can not be guaranteed. In order to avoid the ground structures yielding mechanisms during the searching process for the optimum in topology optimization, a formulation of weight minimization as the pursuit goal and structural performance such as a structural displacement limitation as the constraint was proposed in paper [1]. Because of the limitation of the length of paper [1], it didn't involve another question: can the mechanisms be avoided proactively when the 99-line topology optimization code is used in structural topology optimization? If this question could be answered, two benefits will be obtained: (1) satisfying the needs of the theory development of structural topology optimization; (2) providing effective help for users of the 99-line topology optimization code. Therefore, the present paper continues the research of paper [1]. Situations in structural topology optimization with mechanism appearance are specifically explored in this paper. In these situations, the respective changes of the structural volume ratio and strain energy are investigated to look for the numerical regularities that cause mechanism appearance in structural topology optimization. Here, the reason that the structural strain energy is selected as an objective of the investigation of mechanism appearance, lies in the fact that: the structural strain energy is classified as a structural mechanical performance indicator, and it doesn't vary with the changes of structural shape or material distribution. Therefore, the structural strain energy is not related to constraints selection in the optimization, and its changes can reflect well if a structure yields a mechanism.

This paper still uses the 99-line and 120-line topology optimization codes, which are, respectively, representing the MCVC and MWDC models. Both of the models are briefly described in the 4th section. They are also analyzed deeply and compared on the basis of mechanism appearance. In the 5th section, numerical study is made for avoiding mechanism appearance in the MCVC model, and a method to look for the thresholds of maximum strain energy ratios and volume ratios is proposed and provided with detailed implementation procedures. Based on this method, four numerical examples are tested in the 6th section. The numerical results are summarized and the numerical regularities to avoid mechanism appearance in structural topology optimization are obtained. At the end, the conclusion of this paper is summarized in the 7th section.

4. Two topology optimization formulations involved in this paper

The 99-line topology optimization code in paper [2] aimed to solve the structural compliance minimization problem under a volume ratio constraint, in another word, minimizing compliance with a volume constraint, which is the MCVC model. Meanwhile, the 120-line topology optimization code in paper [1] aimed to solve the structural weight minimization problem under a displacement constraint, which is the MWDC model. Their optimization models are, respectively, shown in below Eq. (1) and Eq. (2) as follows:

$$\begin{cases} \text{For} & \boldsymbol{\rho} \\ \min & c(\boldsymbol{\rho}) = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{U} \\ \text{s.t.} & \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \\ & V(\boldsymbol{\rho})/V^{*} = f \\ & 0 < \rho_{\min} \le \rho_e \le 1 \quad (e = 1, \cdots, N) \end{cases} \tag{1}$$

$$\begin{cases} \text{For} & \boldsymbol{\rho} \\ \min & W(\boldsymbol{\rho}) = \sum_{e=1}^{N} \rho_e w_e^0 \\ \text{s.t.} & u(\boldsymbol{\rho}) = \bar{u} \\ & 0 < \rho_{\min} \le \rho_e \le 1 \ (e = 1, \cdots, N) \end{cases} \tag{2}$$

where N is the total number of elements, and $\boldsymbol{\rho}$ is the artificial relative density vector in design domain. The minimum value of the relative density is set to be 0.001 to avoid singularity of the stiffness matrix. In Eq. (1), c represents the structural compliance function. $\boldsymbol{K}$, $\boldsymbol{U}$ and $\boldsymbol{F}$ are structural global stiffness matrix, total displacement vector and total force vector respectively. V and V^* are separately the total volumes of the designed structure and the ground structure, and f is the pre-setting volume ratio. In Eq. (2), w_e^0 is the weight of the e-th element with a relative density of 1. W is the structural total weight. u and $\bar{u}$ are respectively the displacement and the allowable displacement at the point of interest.

Both of the above optimization models are constructed with an objective function and a constraint function related to design variables based on the SIMP (Solid Isotropic Material with Penalization) model [14] of the artificial material method, which puts penalty on the intermediate densities to make them close to 0 or 1. The differences between Eq. (1) and Eq. (2) are, the volume ratio constrain in Eq. (1) can be expressed directly by a linear function of design variables and the optimization model can be easily solved by an optimality criteria method, while the displacement constraint in Eq. (2) is a nonlinear function of design variables, which is always replaced by the analytic expression of its approximate function [16-17] obtained by the unit virtual load method [15], and the optimization model can be solved by the Lagrangian multiplier method.

From the perspective that whether an optimization model can lead its final topological configuration into a mechanism, the MCVC model in Eq. (1) might have mechanism appearance, while the MWDC model in Eq. (2) does not have mechanism appearance. The main reason that Eq. (1) causes mechanism appearance is due to the unreasonable volume ratio constraint, which has been discussed in paper [1] through numerical examples. Actually, the compliance minimization is the stiffness maximization, and in such kind of problem, a mount of materials are used to form a structural with the maximum stiffness. No matter from the perspectives of engineering or research, no one has an idea how much materials are needed for a structure with the maximum stiffness, or how large the material volume should be to get a structure with as much stiffness as we want. Furthermore, different material volumes come out with topological configurations of different shapes and stiffness. This causes instability in the structural design. When the allowable volume ratio value is comparatively large, the optimal topological configuration shows as a “cumbersome” structure [1] (a structure with very little sub-domains deleted from the ground structure); when the allowable volume ratio value gets smaller and smaller, the optimal topological configurations transfer from Michell-truss-like structures [1] to “light” structures [1] (appropriate super-statically determinate structures or statically determinate structures); when the allowable volume ratio value is too small or unreasonable, in order to meet the unreasonable constraint condition, structural elements are eliminated too many, and it leads to the mechanism or no appropriate configuration. Moreover, in order to save materials and cost, it’s expected to use least materials to get the stiffest structure. But, when the structural volume is too small, a mechanism will inevitably appear if the stiffest structure is still sought for hard.

However, such kind of problem does not exist in Eq. (2) with the displacement constraint. It has been pointed out in paper [1] that, as a mechanical performance indicator, the displacement constraint can be satisfied as long as the displacement in the loading process meets the condition of guaranteeing the structure safety. It’s meaningless to have a very large allowable displacement value. In the meantime, the displacement constraint can be set on the basis of engineers’ accumulated experience, so the allowable displacement value is usually not very large, and it doesn’t cause mechanism appearance. Such kind of displacement constraint problem still belongs to the area of small deformation. Since if a very large value is set as the allowable displacement value, the displacement constraint will loose the significance of constraints, and it does not satisfy the engineering practice either. Moreover, a displacement constraint with a large allowable value is not a small deformation problem anymore, but a large deformation problem. Therefore, the MWDC model in Eq. (2) will not produce mechanisms no matter from the perspectives of engineering or theory. Actually, the requirement for reasonable constraints is an implicit constraint for no mechanism appearance in structural topology optimization, because the allowable values themselves in constraints can avoid mechanism appearance in structural topology optimization, and they guarantee not to yield mechanisms in topology optimization.

5. Numerical approaches for avoiding mechanism appearance in the MCVC model

In order to find the numerical regularities to avoid mechanism appearance in the MCVC model, this paper focuses on studying the maximum value of the ratios of the structural strain energies of iterative topological configurations to the structural strain energy of the ground structure with full materials, and the volume ratio of the final optimal configuration to the ground structure. It’s found that the mechanism can be avoided when the maximum strain energy ratio is smaller than a threshold, or when the volume ratio is larger than a threshold. In order to find the values of the two thresholds, a strategy is made as below: according to the solution approach for the MWDC model in paper [1], the corresponding displacement at the point of interest to the minimum volume ratio, which is also the displacement upper bound in the MWDC model, as a bridge between the MCVC and MWDC models, will be narrowed down the range of its minimum value firstly via setting a threshold for grey elements in configurations of the MWDC model to check the possibility of mechanism appearance. Then the bisection method is used to address the minimum allowable displacement value from that range. Meanwhile, the corresponding thresholds of the maximum strain energy ratio and the volume ratio are obtained.

5.1. Threshold of the volume ratio, $f_{\min}$, and threshold of the maximum strain energy ratio, $r^*_{\max}$

According to the analysis of the last section, it’s known that the MWDC model does not produce mechanisms, while the MCVC model with a quite small allowable value for the volume ratio constrain might lead to mechanisms. But one has no idea that, when a structure starts to become a mechanism, how much small this volume ratio should be. Therefore, in order to avoid mechanism appearance in the MCVC model, it is expected to find $f_{\min}$, the threshold of the volume ratio when the structures begin to become mechanisms. It is the minimum value among all the allowable volume ratio values that can avoid mechanism appearance, and also the dividing line for a structure between being optimized into the optimal configuration and degenerating into a mechanism: when $f \le f_{\min}$, the structure might yield a mechanism; when $f > f_{\min}$, there will be no mechanism appearance, and the structure will be designed into a safety optimal configuration. For practicality and convenience, it's expected that $f_{\min}$ is a common threshold of volume ratio for different problems. However, it is actually a numerical range because of differences of problems.

By studying examples with mechanisms appearance and observing structure changes in each iteration, it is found that, too many elements are eliminated quickly after the optimization begins. In order to meet the quite small volume ratio constraint, the remaining elements are few. In another word, the mechanism-like iterative configurations occur during the iterative process. After the first appearance of the mechanism, elements in the structure self adjust or "add" according to the results in the following optimization process until the topological configurations tend to be stable. However, in order to find the accurate $f_{\min}$, since it is just a point of view to try different allowable volume ratios and observe the changes when mechanisms occur, it is believed that there is another corresponding structural mechanical performance parameter to improve the ability of indicating mechanism appearance.

Since the structural strain energy doesn't vary with the changes of structural shape and material distribution, it doesn't involve in selecting constraints in the optimization process. When a structure yields a mechanism, it has large deformation, and its strain energy correspondingly becomes very large. Therefore, the changes of the structural strain energy can reflect very well whether a mechanism occurs in the optimization process. But in the view of the proportional relationship between the structural compliance and the structural strain energy in the MCVC model, the dramatic changes of the structural strain energy can not be distinguished correctly. Hence, the MWDC model is adopted to help us achieve this perception.

In the MWDC model, the structural strain energy can be expressed as follows:

$$U = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{u} = \frac{1}{2}\sum_{e=1}^{N}(\rho_e)^p \boldsymbol{u}_e^{\mathrm{T}}\boldsymbol{K}_0\boldsymbol{u}_e \tag{3}$$

where ρ_e and $\boldsymbol{u}_e$ are, respectively, the relative density and the displacement vector of the *e*-th element, and $\boldsymbol{K}_0$ represents the element stiffness matrix. Because the initial ground structure in the MWDC model only uses the full material, where the relative density of each element equals to 1, the relative density vector $\boldsymbol{\rho}$ will have different values in each iteration. Thus, the ratio of the structural strain energy $U^{(k)}$ of the *k*-th iteration to the structural strain energy of the initial ground structure U^0 is represented as follows:

$$r^{(k)} = \frac{U^{(k)}}{U^0} \tag{4}$$

Let the maximum value among the strain energy ratios for the topological configurations over all iterations be represented as follows:

$$r_{\max} = \max\{r^{(k)}, k = 1, \cdots, n\} \tag{5}$$

where n is the total iteration number, and the above maximum value of the strain energy ratios of iterative configurations, $r_{\max}$, is simplified to be called as the maximum strain energy ratio.

Different allowable displacement values will come out with different maximum strain energy ratios. Assuming under a certain allowable displacement value $\overline{u}_{\min}$ in the MWDC model, the ratio of the volumes of the final configuration to the initial ground structure is $f_{\min}$, and the corresponding maximum structural strain energy ratio is $r^*_{\max}$. Since the smaller the volume ratio f is, the larger the value of $r_{\max}$ is, and the easier a mechanism occur. Furthermore, $f_{\min}$ is the minimum among all volume ratios that will not cause mechanism appearance, and the maximum among those volume ratios that can cause mechanism appearance. Correspondingly, $r^*_{\max}$ is the maximum among all strain energy ratios with mechanism appearance, and the minimum among those strain energy ratios without mechanism appearance, which means: when $r_{\max} \geq r^*_{\max}$, a mechanism will occur, when $r_{\max} < r^*_{\max}$, a mechanism will not occur. Therefore, two numerical parameters are found to avoid mechanism appearance in the MCVC model: the threshold of the maximum strain energy ratio, $r^*_{\max}$, and the threshold of the volume ratio, $f_{\min}$.

Since the MWDC model doesn’t yield mechanisms, how can we know whether the corresponding volume ratio constraint in the MCVC model to a certain allowable displacement value in the MWDC model will lead the final topological configuration into a mechanism? The most direct way is to observe the connection situations in grey images of the final configurations. So a threshold for grey elements is needed to help us to distinguish mechanisms in grey images.

5.2 The grey element threshold

It is known that the existence of a large number of grey elements in grey images of topological configurations makes the boundaries of the final topological configuration indistinct, and it is difficult to tell whether the elements connect to each other or not. In order to address this issue, the grey element threshold [18-20] is introduced into the MWDC model to get distinct boundaries by deleting elements with small densities.

The magnitude of the grey element threshold is generally a constant in (0, 1). Elements whose densities are larger than this threshold are solid elements, and elements whose densities are less than this threshold are void elements. For example, if the grey element threshold is set as $T = 0.5$, which means to delete grey elements with densities less than 0.5 and to keep grey elements and their initial relative density values larger than or equal to 0.5. About the function of the grey element threshold during hunting for the numerical regularities of avoiding mechanism appearance in structural topology optimization, it can be summarized as follows:

(1) A small grey element threshold means elements with densities larger than or equal to this threshold will be retained. The smaller this threshold is, the looser the condition of deleting grey elements becomes. For a

large threshold, all elements with densities less than this threshold will be deleted. The larger the threshold is, the stricter the requirement for grey element densities will be.

(2) If a mechanism occurs when the grey element threshold is quite small, a mechanism will definitely occur when the threshold is large. But on the contrary, if a mechanism occurs when the threshold is large, a mechanism does not always occur when the threshold is small. For example, if a mechanism occurs when the threshold equals to 0.4, mechanisms will certainly occur with a threshold of 0.5, 0.6, or larger than 0.6. But on the contrary, if a mechanism occurs when the threshold equals to 0.6, a mechanism may not occur with a threshold of 0.5 or 0.4.

(3) The function of the grey element threshold here is "the postponement of mechanism appearance", and "the transition from a structure to a mechanism". It helps to improve the ability of indicating mechanism appearance.

It should be noted that, after introducing the grey element threshold, the real strain energy, which is the strain energy of the structure without the grey element threshold, is still used for the calculations of the strain energy of an iterative configuration and the strain energy ratio. Here, the grey element threshold is only an approach to help us to improve ability of indicating mechanism appearance. Mechanical properties of the real structure should be abided by for the research on numerical regularities for avoiding mechanism appearance.

5.3. Implementation procedures

In the concrete implementation, the 120-line and 99-line topology optimization codes from paper [1] and paper [2] are adopted to represent the MWDC and MCVC models, respectively. After a large number of example calculations, it is found that, the smaller the allowable volume ratio in the MCVC model is, the larger the displacement at the point of interest will be. The larger the allowable displacement value in the corresponding MWDC model is, the larger the maximum strain energy ratio of the iterative configurations will be, and the more likely a mechanism occurs. On the contrary, the larger the allowable volume ratio in the MCVC model is, the smaller the allowable displacement value in the corresponding MWDC model will be, the smaller the maximum strain energy ratio of the iterative configurations will be, and the less likely a mechanism occurs. But the allowable displacement value is not the larger the better. The minimum allowable displacement value when mechanisms occur will be determined, and the maximum strain energy ratio of the corresponding final configuration to this minimum allowable displacement value will be determined too. Moreover, this particular maximum strain energy ratio is the minimum value of the maximum strain energy ratio for indicating mechanism appearance.

In order to achieve this goal, the possible value for the minimum allowable displacement value will be explored when a structure yields a mechanism after applying the grey element threshold. Once this possible minimum allowable displacement value or its interval is found, the bisection method will be used to determine the exact value for the minimum allowable displacement value in that interval. In the meantime, the minimum maximum strain energy ratio of the iterative configurations, $r_{\max}^{*}$, which can indicate mechanism appearance in structural topology optimization, and the volume ratio, $f_{\min}$, which indicates a mechanism showing up in the final configuration for the first time, will be obtained under this minimum displacement constraint. The detailed procedures are shown as follows:

Step 1: Run the 99-line code with a small allowable volume ratio value in the MCVC model, such as f =0.2 or 0.1, and obtain the displacement at the point of interest, $\overline{u}$;

Step 2: Use $\overline{u}$ obtained from Step 1 as the allowable displacement value in the MWDC model for calculation, and get the corresponding maximum strain energy ratio, $r_{\max}$. Then, apply the grey element threshold on all elements.

Step 3: Choose smaller volume ratios and repeat Step 1 and Step 2. If it is difficult to choose a small volume ratio, a large volume ratio could be selected directly for calculation in Step 2. By observing grey images, the allowable displacement value for the final configuration showing up as a mechanism when the grey element threshold $T = 0.6$ is noted as $\overline{u}^{l}$, and the allowable displacement value for the final configuration showing up as a mechanism when the grey element threshold $T = 0.5$ is noted as $\overline{u}^{u}$, which is called the possible minimum allowable displacement value. Then, the interval of the minimum allowable displacement value to avoid mechanism appearance in the MWDC model when the grey element threshold $T = 0.5$, can be determined as $(\overline{u}^{l}, \overline{u}^{u}]$.

Step 4: Fix the grey element threshold in the MWDC model as $T = 0.5$, and use the bisection method to look for the minimum allowable displacement value $\overline{u}_{\min}$ from the interval $(\overline{u}^{l}, \overline{u}^{u}]$ while its final configuration shows as a mechanism. The corresponding maximum strain energy ratio under this displacement constraint of $\overline{u}_{\min}$, $r^{*}_{\max}$, and the volume ratio without the grey element threshold, $f_{\min}$, will be also determined.

6. Numerical studies based on examples

The same numerical examples used in paper [1] are tested in this paper to study mechanism issues in structural topology optimization. The initial ground structures of four examples are shown in Figure 1. Similarly, the points of interest of each structure are the loading points.

Example 1, shown in Figure 1 (a), is the half of MBB-beam, with a ground structure of 60mm×20mm×1mm, unit force F=1, unit Young's modulus E^0=1.0 and the Poisson's ratio $\nu = 0.3$.

Example 2, shown in Figure 1 (b), is the short cantilever beam, with a ground structure of 32mm×20mm×1mm, unit force F=1, unit Young's modulus E^0=1.0 and the Poisson's ratio $\nu = 0.3$.

Example 3, shown in Figure 1 (c), is the short cantilever beam with a fixed hole, with a ground structure of 45mm×30mm×1mm, unit force F=1, unit Young's modulus E^0=1.0 and the Poisson's ratio $\nu = 0.3$. The center of the hole is located at the intersection of 1/3 horizontal length and 1/2 vertical length from left to right, while the radius is equal to 1/3 vertical length.

Example 4, shown in Figure 1 (d), is the cantilever beam, with a ground structure of 80mm×50mm×1mm, loading force F=9kN , Young's modulus E^0=1.0×10^6MPa and the Poisson's ratio $\nu = 0.3$.

Summarizing the analysis of previous two sections, the same example is calculated according to Step 1 and Step 2 in the section 5.3. The maximum strain energy for the iterative configurations and the structural final topological configurations for different grey element thresholds are obtained in the calculation. Taking Example 1(a) as an example, when the final configurations show similarly as configurations in Figure 2, where some parts of the final configurations are shown to disconnect with, break off, or articulate to the other parts, we can determine that mechanisms occur under the condition of a grey element threshold.

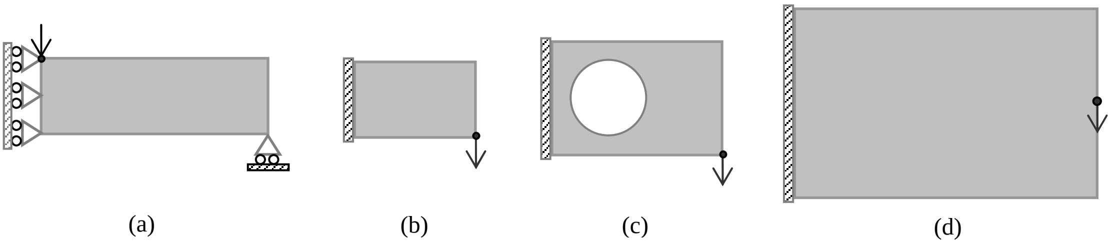


Figure 1: Ground structures and boundary conditions for the four test examples

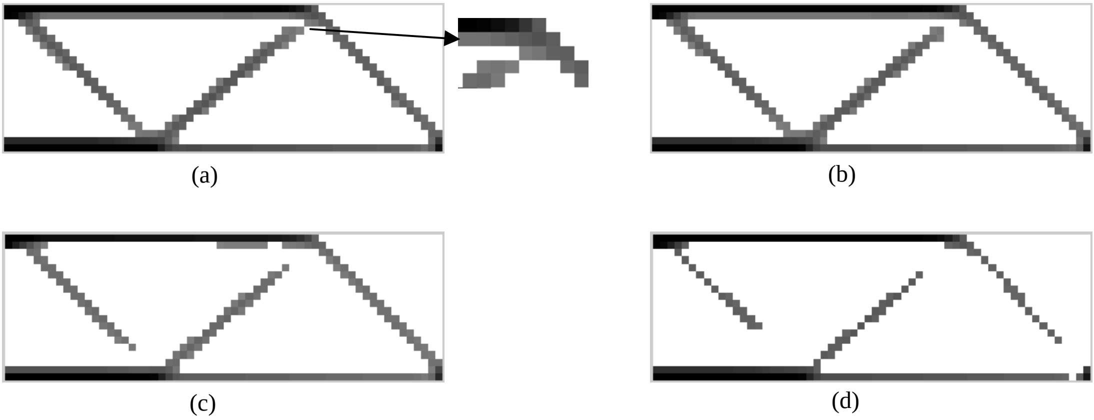


Figure 2: Grey images with mechanism appearance in the final configurations

6.1. The possible minimum allowable displacement value with mechanism appearance

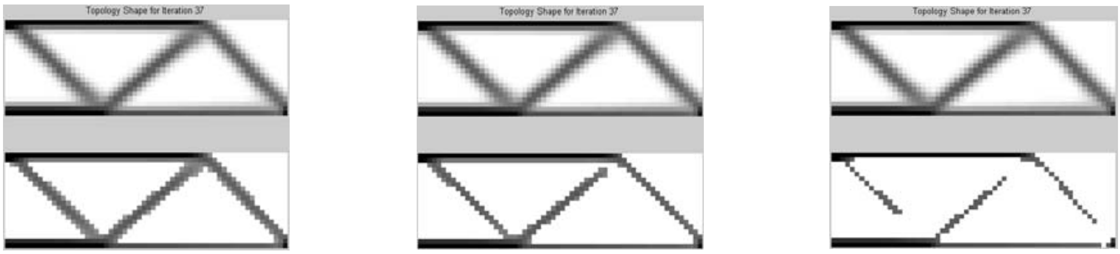
Following the calculation strategy, through the study of volume ratio constraints with too small values causing mechanism appearance, it is revealed that the common unreasonable allowable volume ratio values for four examples with mechanism appearance at the first time, might be f =0.2. This is a first guess and it doesn't need to be accurate. Therefore, the volume ratio values less than 0.2 could be used for calculation in the 99-line code. Meanwhile, grey element thresholds for mechanism appearance are observed in the corresponding 120-line code to explore the possible minimum allowable displacement value and the interval for the minimum allowable displacement value.

Example 1(a): As shown in Table 1, when the allowable volume ratio equals to 0.2, the displacement at the loading point calculated in the 99-line topology optimization code is 728.75, and the final configuration obtained in the 120-line topology optimization code under this displacement constraint becomes a mechanism when the grey element threshold equals to 0.5. Figure 3 shows the corresponding final configurations of different grey element thresholds under the possible minimum displacement constraint $\overline{u}$ =728.75. The words on top of each figure show the convergence iteration, which is the total number of iterations. The above configuration in each figure is the final configuration without grey element threshold, and the below configuration is the final configuration with a grey element threshold. From these figures we can see, when the allowable displacement value is $\overline{u}$ =728.75, the calculation for this example converges after 37 iterations, and the converged final configuration is an integrite structure with lots of grey elements and obscure boundaries. When grey element thresholds T=0.4, 0.5 and 0.6 are applied on the final configuration separately, the final configuration becomes into a zig-zagging structure (T=0.4), or a mechanism (T=0.5 and 0.6). It's obvious that the structure becomes into a mechanism when the grey element threshold T=0.5.

The cases of the volume ratios less than 0.2 correspond to those cases of displacement constraint values larger than 728.75. These cases already have mechanism appearance when the grey element threshold equals to 0.3 or 0.4, and thereby they will definitely have mechanism appearance when the grey element threshold equals to 0.5. Thus, the possible minimum allowable displacement value is 728.75, and the exact minimum allowable displacement value is in a left-open and right-close interval (0, 728.75].

Table 1: Studies on grey element thresholds and the allowable displacement values in example 1(a)

Volume ratio in 99-line code	Allowable displacement value in 120-line code	$r_{\max}$ based on 120-line code	Threshold of grey elements when the converged design is a mechanism
0.2	**728.75**	**6.0978**	**0.5**
0.18	900.57	7.5310	0.5
0.15	1287.82	10.7514	0.4
0.10	3056.85	25.5077	0.3
——	5000.00	41.7222	0.3



Threshold=0.4 Threshold=0.5 Threshold=0.6

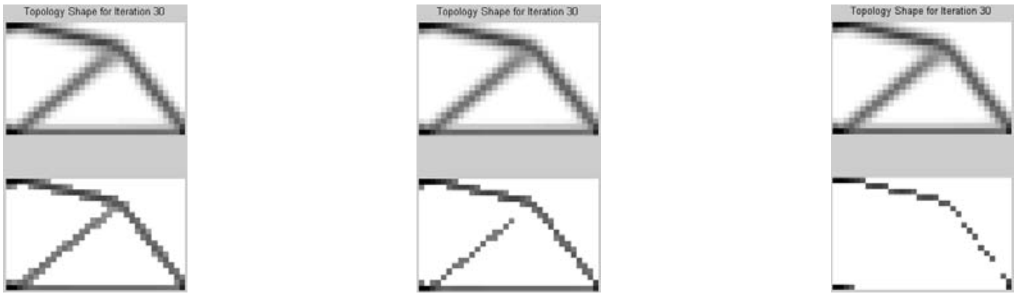
Figure 3: Configurations for different grey element thresholds under the possible minimum displacement constraint of $\overline{u}$ =728.75

Example 1(b): As shown in Table 2, when the volume ratio constraint equals to 0.1, the displacement at the loading point calculated in the 99-line topology optimization code is 382.33, and the final configuration obtained in the 120-line topology optimization code under this displacement constraint becomes a mechanism when the grey element threshold equals to 0.5. Correspondingly, Figure 4 shows the final configurations for different grey element thresholds under the possible minimum displacement constraint of $\overline{u}$ =382.33. From the figures we can see, when the allowable displacement value is $\overline{u}$ =382.33, the calculation for this example converges after 30 iterations. After applying the grey element threshold, it's obvious that the structure becomes into a mechanism when the grey element threshold is T=0.5 or 0.6.

The cases of the volume ratios less than 0.1 correspond to those cases of displacement constraints larger than 382.33. These cases already have mechanism appearance when the grey element threshold is 0.4, and thereby they will definitely have mechanism appearance when the grey element threshold equals to 0.5. While the allowable volume ratio value is 0.15, the corresponding displacement constraint value is 209.38, and its final configuration has no mechanism appearance until the grey element threshold increases to 0.6. Therefore, the possible minimum allowable displacement value is 382.33 in this example, and the exact minimum allowable displacement constraint value is in a left-open and right-close interval (209.38, 382.33].

Table 2: Studies on grey element thresholds and the allowable displacement values in example 1(b)

Volume ratio in 99-line code	Allowable displacement value in 120-line code	$r_{\max}$ based on 120-line code	Threshold of grey elements when the converged design is a mechanism
0.2	139.15	5.3340	>0.6
0.18	178.49	6.8052	0.6
0.15	209.38	7.9310	0.6
0.10	**382.33**	**14.3840**	**0.5**
——	500.00	18.8160	0.4



Threshold=0.4 Threshold=0.5 Threshold=0.6

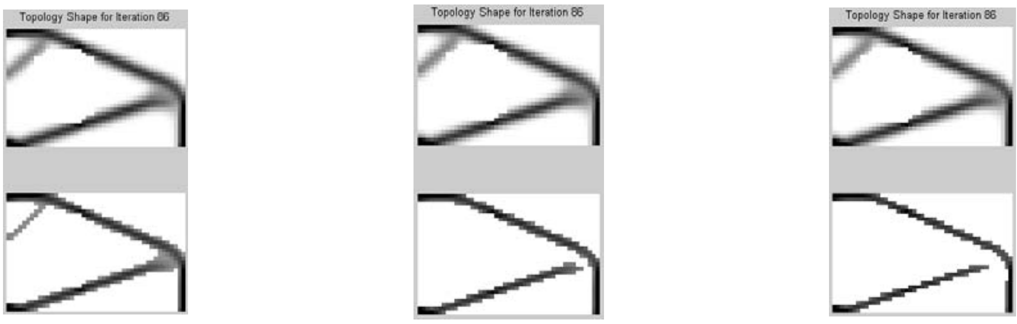
Figure 4: Configurations for different grey element thresholds under the possible minimum displacement constraint of $\bar{u}$ =382.33

Example 1(c): As shown in Table 3, when the volume ratio constraint equals to 0.15, the displacement at the loading point calculated in the 99-line topology optimization code is 260.45, and the final configuration obtained in the 120-line topology optimization code under this displacement constraint becomes a mechanism when the grey element threshold equals to 0.5. Correspondingly, Figure 5 shows the final configurations for different grey element thresholds under the possible minimum displacement constraint of $\bar{u}$ =260.45. From the figures we can see, when the allowable displacement value is $\bar{u}$ =260.45, the calculation for this example converges after 86 iterations. After applying the grey element threshold, it’s obvious that the structure becomes into a mechanism when the grey element threshold is T=0.5 or 0.6.

The cases of the volume ratios less than 0.15 correspond to those cases of displacement constraint values larger than 260.45. These cases already have mechanism appearance when the grey element threshold is 0.3 or 0.4, and thereby they will definitely have mechanism appearance when the grey element threshold equals to 0.5. While the allowable volume ratio value is 0.18, the corresponding allowable displacement value is 186.19, and its final configuration has no mechanism appearance until the grey element threshold increases to 0.6. Therefore, the possible minimum allowable displacement value is 260.45 in this example, and the exact minimum allowable displacement value is in a left-open and right-close interval (186.19, 260.45].

Table 3: Studies on grey element thresholds and the allowable displacement values in example 1(c)

Volume ratio in 99-line code	Allowable displacement value in 120-line code	$r_{\max}$ based on 120-line code	Threshold of grey elements when the converged design is a mechanism
0.2	149.80	3.5633	>0.6
0.18	186.19	4.4303	0.6
0.15	**260.45**	**6.1191**	**0.5**
0.10	628.85	14.7913	0.4
——	1000.00	23.5164	0.3



Threshold=0.4 Threshold=0.5 Threshold=0.6

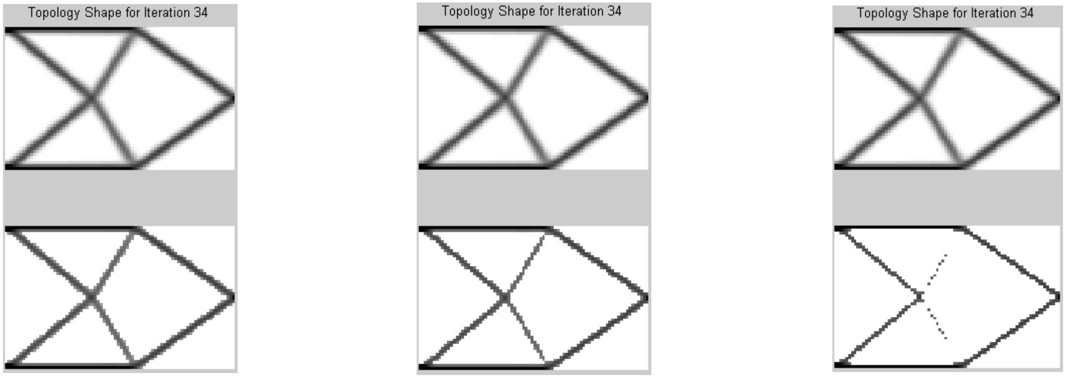
Figure 5: Configurations for different grey element thresholds under the possible minimum displacement constraint of $\bar{u}$ =260.45

Example 1(d): As shown in Table 4, when the displacement at the loading point is $\bar{u}$ =2.50, the final configuration obtained in the 120-line topology optimization code under this displacement constraint becomes a mechanism when the grey element threshold equals to 0.5. Correspondingly, Figure 6 shows the final configurations for different grey element thresholds under the possible minimum displacement constraint of $\bar{u}$ =2.50. From the figures we can see, when the allowable displacement value is $\bar{u}$ =2.50, the calculation for this example converges after 34 iterations. After applying the grey element threshold, it’s obvious that the structure becomes into a mechanism when the grey element threshold is T=0.5 or 0.6.

Table 4: Studies on grey element thresholds and the allowable displacement values in example 1(d)

Volume ratio in 99-line code	Allowable displacement value in 120-line code	$r_{\max}$ based on 120-line code	Threshold of grey elements when the converged design is a mechanism
0.2	0.99	5.0087	>0.6
0.15	1.52	7.7031	0.6
——	2.00	10.0912	0.6
——	**2.50**	**12.5451**	**0.5**
——	5.00	24.9108	0.4

From Table 4 we can also see that, the cases of the displacement constraint values larger than 2.50 already have mechanism appearance when the grey element threshold is 0.4, and thereby they will definitely have mechanism appearance when the grey element threshold equals to 0.5. While the displacement constraint value is less than or equal to 2.00, its final configuration has no mechanism appearance until the grey element threshold increases to 0.6. Therefore, the possible minimum allowable displacement value is 2.50 in this example, and the exact minimum allowable displacement value is in a left-open and right-close interval (2.00, 2.50].



Threshold=0.4 Threshold=0.5 Threshold=0.6

Figure 6: Configurations for different grey element thresholds under the possible minimum displacement constraint of $\overline{u}$ =2.50

6.2. Searching for $r^*_{\max}$ and $f_{\min}$, thresholds of the maximum strain energy ratio and the volume ratio to avoid mechanism appearance by using the bisection method

In the previous section, the intervals for the minimum allowable displacement values for 4 examples with mechanism appearance under the grey element threshold of 0.5 have been found. Four intervals, (0, 728.75], (209.38,382.33], (186.19, 260.45] and (2.00, 2.50] are, respectively, the intervals of the minimum allowable displacement values located when mechanisms occur in the corresponding examples 1(a), 1(b), 1(c) and 1(d). According to analysis and data comparison in Table 2~5, it's found that, the smaller the volume ratio is, the larger the displacement at the loading point will be. As the allowable displacement value increases, the maximum strain energy ratio, $r_{\max}$, increases correspondingly.

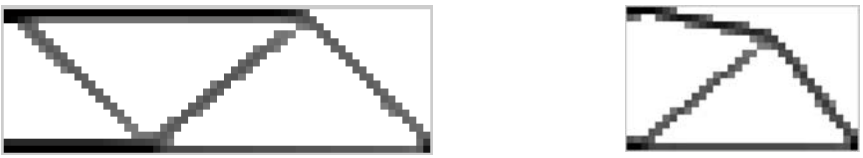
The bisection method is used by getting the middle point in an interval, and this middle point value is taken as the allowable displacement value for calculation in 120-line code. It should be distinguished that whether a mechanism occurs with the grey element threshold of 0.5, and thereby the interval without mechanism appearance is eliminated and the interval with mechanism appearance is retained. Then the next bisection searching is continued to be carried out until the difference between two middle points in neighborhood without and with mechanism appearance is very little. Following this approach, the minimum allowable displacement value, whose final configuration with the grey element threshold of 0.5 showing as a mechanism in each example, is obtained. The corresponding maximum strain energy ratio $r_{\max}$ and volume ratio f are, respectively, taken as the threshold of the strain energy ratio, $r^*_{\max}$, and the threshold of the volume ratio, $f_{\min}$, which are just what we are hunting for.

Example 1(a), the minimum allowable displacement value in the interval (0, 728.75] is $\bar{u}_{\min}$ =701.7065, and the corresponding maximum strain energy ratio is $r^{*}_{\max}$ =5.8719. At this point, the structural volume ratio is $f_{\min}$ =0.26.

Example 1(b), the minimum allowable displacement value in the interval (209.38, 382.33] is $\bar{u}_{\min}$ =228.6319, and the corresponding maximum strain energy ratio is $r^{*}_{\max}$ =8.6352. At this point, the structural volume ratio is $f_{\min}$ =0.21.

Example 1(c), the minimum allowable displacement value in the interval (186.19, 260.45] is $\bar{u}_{\min}$ =193.1882, and the corresponding maximum strain energy ratio is $r^{*}_{\max}$ =4.5886. At this point, the structural volume ratio is $f_{\min}$ =0.21.

Example 1(d), the minimum allowable displacement value in the interval (2.0, 2.50] is $\bar{u}_{\min}$ =2.4659, and the corresponding maximum strain energy ratio is $r^{*}_{\max}$ =12.3791. At this point, the structural volume ratio is $f_{\min}$ =0.15.



(a) 120-line(60, 20, 701.7065, 3.0,1.5)

(b) 120-line(32, 20, 228.6319, 3.0,1.2)

(c) 120-line(45, 30, 193.1882, 3.0,1.5)

(d) 120-line(80, 50, 2.4659, 3.0,1.5)

Figure 7: Final topological configurations of 4 examples with the grey element threshold of 0.5 under the minimum allowable displacement values

Figure 7 shows the final configurations under the minimum allowable displacement values when the grey element threshold is 0.5. According to the results, among the maximum strain energy ratios of 4 examples, the minimum is 4.59, and the maximum is 12.38. Likewise, for different numerical examples, there are different maximum strain energy ratios. So the threshold of the maximum strain energy ratio, which represents mechanism appearance, is not a concrete constant, but in a range. For the examples testes in this paper, the range for the threshold of the maximum strain energy ratio is $r_{\max}^{*} \in [4.59, 12.38]$. Strictly speaking, when the maximum strain energy ratio is $r_{\max} < 4.59$, mechanisms will not occur, and safety structures will be obtained. If this condition is relaxed, and when the maximum strain energy ratio is $r_{\max} < 12.38$, relative safety structures will be obtained, which means that some structures will yield mechanisms and the others will not. Thereby, this numerical regularity about the maximum strain energy ratio is the first criterion that helps us to indicate mechanism appearance.

The minimum volume ratios for structures in 4 examples not to have mechanism appearance are, respectively, 0.26, 0.21, 0.21, and 0.15, which can be called the safety values to avoid the corresponding structures yielding mechanisms. At the same time, these data explain that the threshold of the volume ratio for avoiding mechanism appearance is not a fixed constant either. Since different problems will come out with different topological configurations, and correspond to different safety values that can avoid structures becoming mechanisms, it's hardly to find a common minimum volume ratio. For 4 examples in this paper, from the perspective of structural design, the range for the threshold of volume ratio is $f_{\min} \in [0.15, 0.26]$. Strictly speaking, when the volume ratio is $f > 0.26$, the final topological configuration will not become a mechanism, and the structure is absolutely safe. If this condition is relaxed, and when the volume ratio is $f > 0.15$, the structure will be relatively safe, which means that some structures will yield mechanisms and the others will not. This numerical regularity could be the second criterion to indicate mechanism appearance.

7. Conclusions

According to the study on examples of mechanisms appearance in the final configurations of structural topology optimization, this paper integrates the MCVC and MWDC models. The changes of structural strain energy and the grey images of the final configurations are observed to study the numerical regularities for the MCVC model not to yield mechanisms. It is found out that, the mechanism will occur when the maximum strain energy ratio is larger than a threshold, or when the volume ratio is less than a threshold. Both of the two thresholds can vary for different structures and/or loading cases, but for those examples that we tested, the range of the threshold of the maximum strain energy ratio turns out to be between 4.59 and 12.38, and the range of the threshold of the volume ratio turns out to be between 0.15 and 0.26.

Since different practical problems will have different maximum strain energy ratios and volume ratios, the data obtained here have to be taken as reference values to provide references for researches and designs. Researchers of structural topology optimization and engineers can use the numerical regularities in this paper to predict whether the final configuration of structural topology optimization will become a mechanism, or to prognosis the possibility that a structure with a certain constraint in structural topology optimization yields a mechanism. And unreasonable constraint upper and lower bounds, which make structures yield mechanisms, can also be predicted. This will help them to avoid those unadvisable constraints, and to construct the reasonable constraints and use the advisable constraint values.

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