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A new approach for stress-based topology optimization: Internal stress penalization

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Abstract
In this paper, a new method is proposed for stress-based topology optimization. The basic idea is to perform compliance-based topology optimization while concurrently penalizing material in which the stress exceeds the admissible stress. Penalization of overstressed material is realized by considering it as damaged. Since damaged material will contribute less to the overall stiffness of the structure, the optimizer will promote a design with a minimal amount of damaged/overstressed material. The process of penalizing overstressed material can be regarded as analogous to the penalization of intermediate densities to obtain a black and white design, often used in topology optimization. In our problem formulation, the admissible stress criterion is not imposed as a hard constraint. Therefore, using this method, the considered problems do not suffer from singular optima related to vanishing stress constraints. Furthermore, the computational costs are manageable since only one objective and one constraint are considered. The proposed method is verified by applying it on two test problems.

Keywords: topology optimization, stress constraints, stress penalization

1. Introduction

Local material failure criteria are often necessary to consider in lightweight design applications. A powerful design tool in lightweight design is topology optimization. In order to make topology optimization generally applicable for industrial applications, it is critical to properly handle material failure criteria (e.g. an admissible stress criterion).

However, several difficulties arise when considering an admissible stress criterion in topology optimization. For example, the stress is a local state variable which results in a large number of constraints. This property makes it computationally expensive using gradient-based optimization techniques to solve the problem, since it requires sensitivity analysis of every local stress constraint. Another particular problem in density-based topology optimization (e.g. based on the concept of SIMP [1]) is that the stress is non-uniquely defined for intermediate densities. Finally, problems involving stress constraints may suffer from singular optima [2] that cannot be reached by gradient-based optimization techniques. The existence of singular optima is related to the presence of vanishing constraints and was already observed in truss optimization [3]. A comprehensive overview on singular optima and its fundamental characteristics can be found in [4].

Over the years, a variety of solution techniques were proposed to deal with these difficulties. For example, constraint aggregation techniques are used to reduce the order of the optimization problem by transforming the local stresses into a single aggregation function (e.g. the P-norm [5]). Duysinx and Bendsøe [6] proposed a solution for the non-uniquely defined stress for intermediate densities by modeling the stress mimicking the behaviour of microstress in porous layered material. Finally, problems related to singular optima are most often circumvented by applying constraint relaxation techniques (e.g. ϵ-relaxation [7] and qp-relaxation [8]). Unfortunately, these solution techniques may introduce additional difficulties.

For example, when using constraint aggregation techniques, two conflicting requirement exist: 1) a certain level of smoothness of the aggregation function is required to prevent numerical instabilities and 2) the aggregation function should accurately approximate the local stress levels. Large values of the aggregation parameter will give a better approximation of the local stress levels but could lead to numerical instabilities when gradient-based techniques are used. Therefore, in general, one seeks for a trade-off between these two requirements. However, as a result, the maximum local stress of the optimized design will generally not match the admissible stress criterion when following this approach. Furthermore, one does not know *a priori* the best value of the aggregation parameter which may be problem dependent.

Recently, Le *et al.* [9] proposed a solution to this problem which is based on adaptive normalization of the aggregation function during optimization to better approximate the maximum local stress. Another solution technique employed to obtain a better approximation of the maximum local stress, is to subdivide the design domain into subregions and calculate an aggregated function for each specific subregion [9,10]. This approach is a compromise between considering every local stress constraint (computationally expensive) and only considering a single aggregation function (poor approximation of the local stress levels). Using these techniques optimized designs were obtained which satisfy an admissible stress criterion. However, it remains difficult, to determine in advance, how many regions will give a good representation of the local stress levels since this will be problem and mesh-dependent. Furthermore, adaptive scaling is inconsistent and interferes with the optimization procedure. This is even more severe when combined with a regional approach in which the composition of each region changes during optimization (e.g. a subdivision based on the current stress levels) [11].

Other difficulties are related to constraint relaxation techniques used to make singular optima accessible. These techniques are based on making a smooth approximation of the original constraint such that the design space is enlarged and gradient-based techniques can access (local) optima. However, this also results in a highly non-convex design space which makes these problems prone to convergence to local optima. For that reason, constraint relaxation is often applied in a continuation strategy; starting off from a highly relaxed design space and then gradually decreasing the amount of relaxation toward the original problem. However, it was shown by Stolpe and Svanberg [12] on a truss example, that the global trajectory (i.e. path of the global optimum in a continuation strategy), may be discontinuous; the global optimum of the relaxed problem may not converge to the true global optimum. Bruggi showed the same result for the qp-approach [8], also using a truss example. Finally, the same effect was shown by the author for the continuum case [13].

In this paper, a new method is proposed for stress-based topology optimization. The main idea is to perform compliance-based topology optimization and concurrently penalize overstressed material by considering it as damaged. Since damaged material will contribute less to the overall stiffness of the structure, it will be advantageous to limit the amount of damaged material (i.e. material in which the stress exceeds the admissible stress) in the final design. Penalization of damaged material is achieved by introducing an additional linear elastic model in which the material properties of overstressed material are degraded relative to the corresponding stress in the original undamaged model. A key concept here is that for the final design, it is required that both the compliance of the original undamaged model and the damaged model should be approximately equal. In that case, no damaged/overstressed material is present in the final design. Here, we do not focus on accurately describing damage behaviour, rather our aim is to penalize overstressed material, analogous to the penalization of intermediate densities to obtain a black and white design.

An advantage of this method is that there is no need to use aggregation functions to reduce the computational costs. Furthermore, no hard stress constraints are defined which avoids problems of singular optima related to vanishing stress constraints such as large areas of intermediate densities. We must note here that a relaxed stress definition *is* used for most problems in this paper to ensure that the stress in void material equals zero. However, it will be shown that, *not* using such a relaxed stress definition will not lead to problems in the design process and a similar optimized design is obtained. A disadvantage of this method is that it is a penalization method and therefore the maximum stress in the final design generally does not accurately match the admissible stress. However, as was already discussed, this is also the case in the traditional approach using aggregation functions.

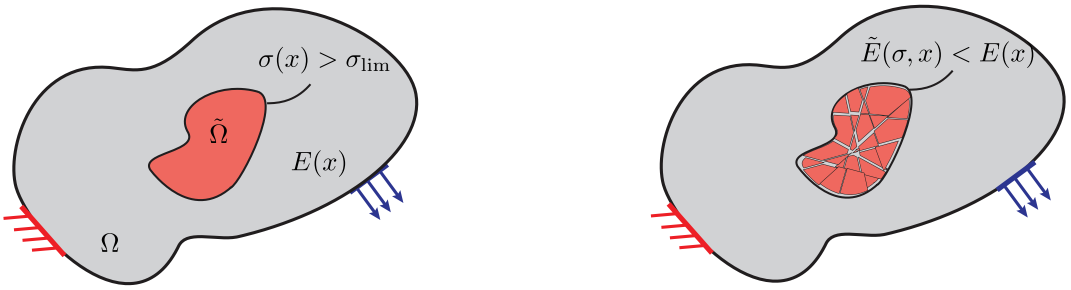
The paper is structured as follows: in Section 2 the method is discussed in the context of density-based topology optimization, in Section 3 the method is applied on two test problems and some preliminary results are presented and in Section 4 the conclusions are presented.

2. Density-based topology optimization with internal stress penalization

In this section, the general approach of the method and its implementation in the context of density-based topology optimization is presented. Furthermore, the sensitivity analysis (necessary for solving the optimization problem using gradient-based topology optimization) and the associated computational costs are discussed.

2.1. General approach

In Figure 1 the main idea of the proposed method is presented schematically. Instead of considering only one model for the structural problem, an additional linear elastic model is introduced for the same



(a) Original model: displacements and stresses are calculated and the red region represents the area where the stresses exceed the admissible stress.

(b) Damaged model: the material in overstressed areas is damaged relative to the stress level in the first analysis.

Figure 1: Schematic representation of the two models: the original undamaged linear elastic model (left) and the damaged model (right), still considering linear elasticity but with degraded material properties in the regions where the stress exceeds the admissible stress.

structural problem (with the same loading and boundary conditions) in which the material properties for overstressed material are considered as damaged. From now on we will refer to the initial undamaged model as the 'original model' and the additional model with damaged material as the 'damaged model'. Furthermore, all quantities belonging to the damaged model are denoted by a tilde over the symbol.
This concept is realized by establishing a relationship between the stiffness in the damaged model and the stress level in the original model. Suppose that a certain stress measure $\sigma(x)$ at every material point x in the red region $\tilde{\Omega}$ in the original model (Figure 1a) exceeds the admissible stress σ_{lim}. Then, the stiffness in the damaged model (Figure 1b) is defined as a function of the stress in the original model $\tilde{E}(x, \sigma)$, as:

$$\begin{cases} \tilde{E}(x, \sigma) < E(x) & \forall x \in \tilde{\Omega} \\ \tilde{E}(x, \sigma) = E(x) & \forall x \in \Omega / \tilde{\Omega}, \end{cases} \tag{1}$$

As a result the damaged model will always be more (or at least equally) compliant,

$$\tilde{C} \geq C, \tag{2}$$

where $\tilde{C}$ and C denote the overall compliances of the damaged structure and the original structure, respectively.
The next step is to define the optimization problem to be solved. Note that the general objective is to perform weight optimization without violating a certain admissible stress. This last requirement implies that both models should be equal and have the same compliance. Since, it is defined that, if there is no overstressed material, there is no damaged material and as a result both models have the same material properties and performance (compliance). For this purpose the design problem is posed as,

$$\begin{aligned} \min_{\mathbf{s}} \quad & V(\mathbf{s}) \\ \text{s.t.} \quad & \frac{\tilde{C}(\mathbf{s})}{C(\mathbf{s})} - 1 \leq \delta, \end{aligned} \tag{3}$$

where V is the volume objective to be minimized and $\mathbf{s}$ are the design variables which can be, for example, the densities in topology optimization, while δ is a small but positive number introduced to relax the condition that both models should have the same compliance in the final design. Next, an implementation of this concept in density-based topology optimization is discussed.

2.2. Density-based topology optimization and stress definition

First a relationship is established between the stiffness of the discretized versions of the original model and the damaged model. For the original model in Figure 1a the SIMP model [1] is adopted. The design domain Ω is discretized into a fixed finite element mesh and density variable is assigned to each element which can continuously vary between zero and one representing 'void' and 'solid' material, respectively.

This is achieved by scaling the effective Young's modulus for each element as,

$$E = E_{\min} + \rho^p (E_0 - E_{\min}), \qquad \text{where,} \qquad \rho \in [0,1], \qquad \text{and,} \qquad p > 1, \tag{4}$$

E_0 is the Young's modulus associated with solid material ($\rho = 1$) and $E_{\min}$ is a small but positive number (typically 1E-6) and acts as a lower bound on the Young's modulus to avoid singularity of the global stiffness matrix. Finally p is a penalization exponent introduced to penalize intermediate densities and promote a zero-one solution. The linear structural problem over the design domain Ω is then defined as,

$$\mathbf{K}\mathbf{u} = \mathbf{f}, \qquad \text{where,} \qquad \mathbf{K} = \sum_{e\in\Omega} \rho_e^p \mathbf{K}_e^0. \tag{5}$$

$\mathbf{K}$ is the global stiffness matrix, $\mathbf{K}_e^0$ is the element stiffness matrix associated with solid material properties, $\mathbf{u}$ is the vector with nodal displacements and $\mathbf{f}$ is the external load vector. One can now solve this problem for nodal displacements $\mathbf{u}$ and the stress $\boldsymbol{\sigma}$ for each element.
A difficulty when considering a stress criterion in density-based topology optimization is that the stress is non-uniquely defined for intermediate densities. An overview of different stress definitions used can be found in [9]. Here, we adopt the qp-relaxed stress [8] definition for which the stress vector $\boldsymbol{\sigma}$ is defined as,

$$\boldsymbol{\sigma} = \rho^{p-q} \boldsymbol{\sigma}^0, \qquad \text{where,} \qquad 0 \leq q < p. \tag{6}$$

Here, $\boldsymbol{\sigma}^0$ is the stress vector associated with the material properties belonging to 'solid' material, E_0. Furthermore, q is a relaxation exponent which is defined to be smaller than p, to ensure that $\boldsymbol{\sigma} = \mathbf{0}$ for $\rho = 0$. For $q = p$ the stress is unrelaxed and is consistent with the microstress in porous layered material [6]. In the traditional method (involving hard stress constraints) such an unrelaxed stress leads to the presence of singular optima and therefore stress relaxation is applied. In the current problem definition there are no singular optima since there are no vanishing constraints. Here, the main motivation to use this relaxed stress definition, is to have a correct representation of the stress in the void elements ($\boldsymbol{\sigma} = \mathbf{0}$ for $\rho = 0$) and *not* to prevent singular optima.

2.3. Damage representation

The next step is to degrade material in regions in which the stress exceeds the admissible stress. Here, the same discretization is used for the damaged model as for the original model. The concept of degrading material is implemented by establishing a relationship on an element level between the Young's modulus of the damaged model $\tilde{E}$ and the original model E as,

$$\tilde{E} = E_{\min} + \beta (E - E_{\min}), \qquad \text{where,} \qquad \beta(\sigma) \in [0,1]. \tag{7}$$

where β is some monotonically decreasing interpolation function that represents the degree of damage as a function of the stress σ in the first analysis of the original model. Here, σ is a stress measure per element (e.g. the Von Mises stress evaluated at the centroid).
Since the primary focus is to obtain a design in which damaged material is minimal (i.e. the maximum stress is approximately equal to the admissible stress), we are less interested in accurately modeling the real damage behavior. Therefore, β should be regarded as a penalty function rather than a function that accurately describes the material degradation once the stress exceeds the admissible stress. It is important that β is at least (piecewise) continuous of the first order, since the problem is solved using gradient-based optimization techniques. Furthermore, it should be approximately equal to one when the stress is below the admissible stress, and monotonically decreasing when the stress exceeds the admissible stress. There are infinitely many functions to choose from that fit within these criteria. In this paper, the choice was made to use an approximated reverse Heaviside function as shown in Figure 2, defined as,

$$\beta = 1 / \left[1 + e^{\alpha(\sigma/\sigma_{\lim} - 1)} \right] \tag{8}$$

This function is continuous and the steepness of the curve can be controlled easily by the tuning parameter α.

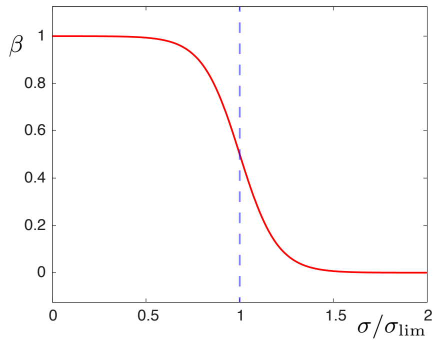


Figure 2: Beta-function: approximate reverse Heaviside function for $\alpha = 10$

Once the degraded Young's modulus is determined for each finite element, the global stiffness matrix is assembled for the damaged model as,

$$\tilde{\mathbf{K}} = \sum_{e \in \Omega} \beta_e \rho_e^p \mathbf{K}_e^0. \tag{9}$$

Finally, the structural problem of the damaged model is defined as,

$$\tilde{\mathbf{K}}\mathbf{v} = \mathbf{f}. \tag{10}$$

Here, $\mathbf{v}$ denotes the vector of nodal displacements of the damaged model. In the discrete density-based setting the design optimization problem is now defined as,

$$\begin{aligned} \min_{\boldsymbol{\rho}} \quad & V_{\text{rel}} = V(\boldsymbol{\rho})/V_\Omega \\ \text{s.t.} \quad & G = \frac{\tilde{C}(\mathbf{v})}{C(\mathbf{u})} - 1 \leq \delta, \\ & \boldsymbol{\rho}_{\min} \leq \boldsymbol{\rho} \leq \mathbf{1}, \end{aligned} \tag{11}$$

where V_{rel} is the relative volume, which is the ratio between the material volume V and the design domain volume V_Ω. The compliances of the original model and the damaged model are given as,

$$C = \mathbf{f}^\intercal \mathbf{u}, \tag{12}$$

and,

$$\tilde{C} = \mathbf{f}^\intercal \mathbf{v}. \tag{13}$$

This problem is solved using gradient-based optimization and therefore we need to determine the gradient of the objective and the constraint with respect to the design variables. The gradient of the volume objective is straightforward and will therefore not be discussed here. In the next section it will be discussed how to derive the total derivative of the constraint function G.

2.4. Sensitivity analysis

From Eq. (11) it can be seen that we deal with a problem of only one objective and one constraint and many design variables. Therefore, the sensitivity analysis is performed using the adjoint method. Here, it will be shown how to compute the total derivative of the constraint function G in Eq. (11). Using the chain rule, the total derivative of the constraint function w.r.t. a design variable is,

$$\frac{\mathrm{d}G}{\mathrm{d}\rho_e} = \frac{1}{C}\frac{\mathrm{d}\tilde{C}}{\mathrm{d}\rho_e} - \frac{\tilde{C}}{C^2}\frac{\mathrm{d}C}{\mathrm{d}\rho_e}. \tag{14}$$

The second term in this equation contains the total derivative of the compliance of the original model. This problem is known to be self-adjoint [14] and is,

$$\frac{\mathrm{d}C}{\mathrm{d}\rho_e} = p\rho_e^{p-1}\mathbf{u}_e^\intercal \mathbf{K}_e^0 \mathbf{u}_e, \tag{15}$$

where $\mathbf{u}_e$ is the vector with the nodal displacements of element e in the original model.

The total derivative of the compliance of the damaged model $\tilde{C}$ in the first term of Eq. (14) is less straightforward to derive. Here we switch to index notation with summation convention and the indices run over the number of degrees of freedom. The sensitivities are calculated using the adjoint method. The first step is to add the equilibrium equations multiplied by an adjoint vector to the compliance response in Eq. (13),

$$\tilde{C} = f_i v_i + \lambda_i \left(K_{ij} u_j - f_i\right) + \mu_i \left(\tilde{K}_{ij} v_j - f_i\right). \tag{16}$$

Now let us taking the derivative with respect to a density design variable ρ (now omitting e for reasons of clarity) and collecting the terms containing the displacement sensitivities, gives,

$$\begin{aligned}\frac{\mathrm{d}\tilde{C}}{\mathrm{d}\rho} =& \lambda_i \frac{\partial K_{ij}}{\partial \rho} u_j + \mu_i \left(\frac{\partial \tilde{K}_{ij}}{\partial \rho} + \frac{\partial \tilde{K}_{ij}}{\partial \sigma_l} \frac{\partial \sigma_l}{\partial \rho} \right) v_j + ... \\ &+ \left(\lambda_i K_{ik} + \mu_i \frac{\partial \tilde{K}_{ij}}{\partial \sigma_l} \frac{\partial \sigma_l}{\partial u_k} v_j \right) \frac{\mathrm{d}u_k}{\mathrm{d}\rho} + \left(f_j + \mu_i \tilde{K}_{ij} \right) \frac{\mathrm{d}v_j}{\mathrm{d}\rho}.\end{aligned} \tag{17}$$

The next step is to choose the adjoints such that this leads to elimination of the displacement sensitivities, i.e. the last two terms in Eq. (17) should become zero, which yields,

$$\frac{\mathrm{d}\tilde{C}}{\mathrm{d}\rho} = \lambda_i \frac{\partial K_{ij}}{\partial \rho} u_j - v_i \left(\frac{\partial \tilde{K}_{ij}}{\partial \rho} + \frac{\partial \tilde{K}_{ij}}{\partial \sigma_l} \frac{\partial \sigma_l}{\partial \rho} \right) v_j, \tag{18}$$

where in the first term we made use of the fact that the first adjoint problem in Eq. (17) is self-adjoint, i.e. $\boldsymbol{\mu} = -\mathbf{v}$. The second adjoint, $\boldsymbol{\lambda}$, is obtained by solving the second adjoint problem. Since the global stiffness matrix is symmetric, this can be written as,

$$K_{ki} \lambda_i = z_k. \tag{19}$$

Here, $\mathbf{z} = [z_k]$ is the pseudo-load vector which can be constructed as a summation over the elemental contributions,

$$z_k = -v^i \frac{\partial \tilde{K}_{ij}}{\partial \sigma^l} \frac{\partial \sigma^l}{\partial \rho} v^j = \sum_{e \in \Omega} \frac{\partial \beta_e}{\partial \sigma_e} \frac{\partial \sigma_e}{\partial v_k} \rho_e^p \mathbf{v}_e \mathbf{K}_e^0 \mathbf{v}_e. \tag{20}$$

Here, $\mathbf{v}_e$ are the element nodal displacements of the damaged model and σ_e is the stress measure corresponding to original undamaged model using SIMP.
Summarizing, the computation of the total derivative in Eq. (14) only requires the solution of the adjoint problem in Eq. (19). The two other adjoint problems are self-adjoint. However, note that the method also requires solving the system of linear equations associated with the damaged model in Eq. (10). The total computational costs are dominated by solving three linear systems of equations of the same size. These computational costs are comparable to the costs in the traditional approach for two (aggregated) stress constraints assuming that no information is re-used of the inverse matrices.

3. Results and discussion

In this section preliminary results will be discussed which were obtained by testing the method on two numerical problems. First a beam in tension is considered, where a comparison is made between the stress-based design obtained by the method presented in this paper and a purely compliance-based design. Next, the method is applied on the L-bracket benchmark for different definitions of the stress.
For all problems the domain is discretized into a fixed finite element mesh using bilinear quadrilateral elements. Furthermore, plane stress linear elasticity is considered and the following material properties are used: a Young's modulus of $E_0 = 1$ and a Poisson's ratio of $\nu = 0.3$. All results are in SI units. The penalization exponent in SIMP is chosen as $p = 3$. As a stress measure the Von Mises stress is considered. Unless stated otherwise, the Von Mises stress is relaxed using qp-relaxation following Eq. (6) and is defined as $\sigma = \rho^\epsilon \sigma^0$ where the relaxation exponent is set to $\epsilon = p - q = 0.5$. Here, σ^0 denotes the Von Mises stress considering the material properties associated with solid material E_0. Finally, density-based topology optimization is used with a linear 'hat' density filter [15] with a filter radius of 1.5 times the element size. For all problems the initial density distribution consists of only densities equal to one.

3.1. Beam in tension

We consider the test problem shown in Figure 3 which is discretized into 50-by-50 elements. The light grey area is the design domain and the darker grey area is non-design material which has material properties belonging to solid material ($\rho = 1$). A distributed load of total magnitude $P = 10$ is applied on the right-hand-side.

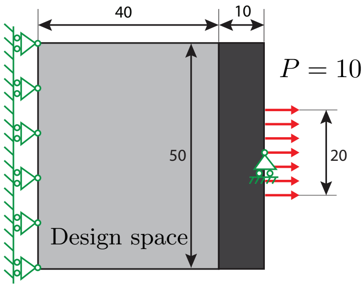


Figure 3: Beam in tension

First, a purely compliance-based design is considered without considering an admissible stress criterion. The optimization problem is defined as minimizing volume subject to a compliance constraint,

$$\begin{aligned} \min_{\boldsymbol{\rho}} \quad & V_{\text{rel}} \\ & \frac{C}{C_{\max}} - 1 \leq 0 \\ & \boldsymbol{\rho}_{\min} \leq \boldsymbol{\rho} \leq \mathbf{1}. \end{aligned} \tag{21}$$

Here, $C_{\max}$ denotes the admissible compliance and is chosen as $C_{\max} = 500$.
Solving the optimization problem in Eq. (21), an optimized design with a relative volume of $V_{\text{rel}} = 22.20\%$ and a maximum Von Mises stress of $\sigma_{max} = 1.33$, was obtained. The optimized design and the corresponding Von Mises stress distribution are shown in Figure 4.

(a) Density distribution (b) Von Mises stress distribution

Figure 4: Compliance-based design (minimizing volume subject to a compliance constraint) with a relative volume of $V_{\text{rel}} = 22.20\%$, a compliance of $C = 500$ and a maximum relaxed stress of $\sigma_{\max} = 1.33$.

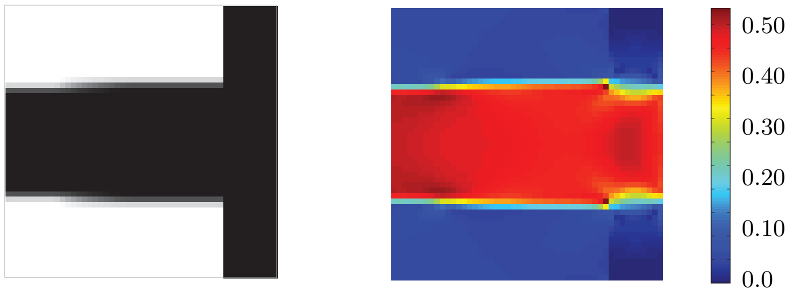
Now, let us introduce an admissible stress criterion for this design problem: $\sigma_{\text{lim}} = 0.6$. In that case, the compliance-based design shown in Figure 4 is not suitable anymore since the stress exceeds the admissible stress. Next, the problem is solved again, but this time taking into account the admissible stress criterion using the method proposed in this paper. For the sake of clarity, we repeat the concept here. The problem formulation is defined as,

$$\begin{aligned} \min_{\boldsymbol{\rho}} \quad & V_{\text{rel}} \\ \text{s.t.} \quad & G = \frac{\tilde{C}(\mathbf{v})}{C(\mathbf{u})} - 1 \leq \delta, \\ & \boldsymbol{\rho}_{\min} \leq \boldsymbol{\rho} \leq \mathbf{1}, \end{aligned} \tag{22}$$

and material is degraded by scaling down the Young's moduli relative to the stress level in the original model. Here, an approximated reverse Heaviside function is used. The Young's moduli in the damaged model are defined as,

$$\tilde{E} = E_{\min} + \beta \left(E - E_{\min} \right), \qquad \text{where,} \qquad \beta = 1/\left[1 + e^{\alpha(\sigma/\sigma_{\lim}-1)}\right]. \tag{23}$$

The positive relaxation parameter is chosen as $\delta = 0.03$ and the parameter which controls the steepness of the Heaviside function is chosen as $\alpha = 5$. The result is shown in Figure 5.



(a) Density distribution (b) Von Mises stress distribution

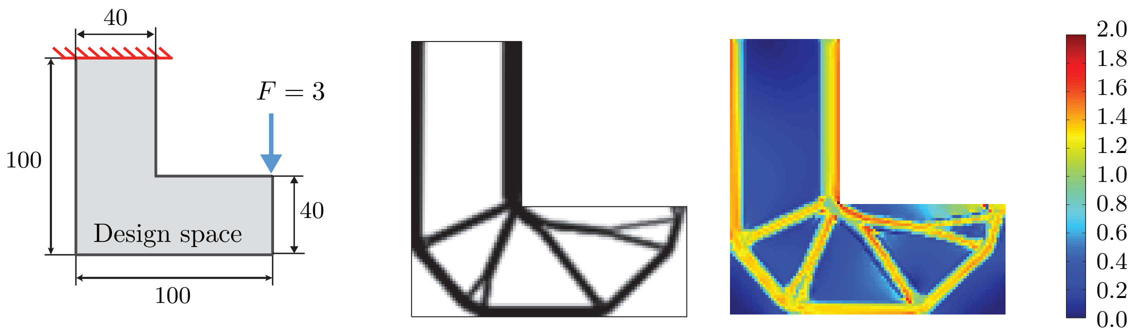
Figure 5: Optimized design obtained for $\alpha = 10$ and $\delta = 0.03$: a relative volume of $V_{\text{rel}} = 43.15\%$, a compliance of $C = 243.30$ and a maximum relaxed stress of $\sigma_{\max} = 0.55$.

Again, a horizontal member was obtained, but now the thickness increased to satisfy the stress criterion. Here, the optimal relative volume is $V_{\text{rel}} = 41.15\%$ and the maximum stress is equal to $\sigma_{\max} = 0.55$ which is close to the admissible stress of $\sigma_{\lim} = 0.6$. The reason that the maximum stress does not accurately match the admissible stress can be explained by the fact that this is a penalization method and the admissible stress criterion is not imposed as a hard constraint. However, also in the traditional approach when using aggregation functions to reduce the computational costs, the maximum stress generally does not match the admissible stress.

3.2. L-bracket

Next, the method is tested on the L-bracket shown in Figure 6a. The L-bracket is a well-known benchmark to study stress-based topology optimization methods due to the presence of a stress singularity in the reentrant corner. The design domain is discretized into a mesh of 6400 quadrilateral elements (100 in vertical and horizontal direction). Here, the admissible stress criterion is set to $\sigma_{\lim} = 2$, the tuning parameter that controls the steepness of the β-interpolation function is chosen as $\alpha = 10$ and the relaxation parameter set to $\delta = 0.04$.

For these settings an optimized design was obtained with a relative volume of $V_{\lim} = 30.00$ and a maximum local stress of $\sigma_{\max} = 2.00$. The optimized design and its corresponding stress distribution are shown in Figure 6. It can be seen clearly, that the optimized design has a rounded shape near the reentrant corner which prevents a stress singularity to occur.



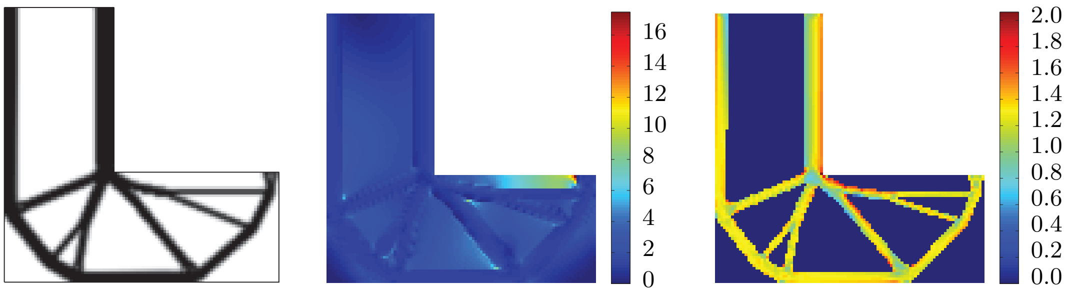
(a) L-bracket (b) Density distribution (c) Von Mises stress distribution

Figure 6: Optimized L-bracket design using internal stress penalization for $\alpha = 10$ and $\delta = 0.04$: a relative volume of $V_{\text{rel}} = 30.00\%$ and a maximum stress of $\sigma_{\max} = 2.00$.

So far, we have been considering a qp-relaxed Von Mises stress since it ensures that the stress is zero in void elements. Next, the same problem is solved but now considering an unrelaxed stress; i.e. $\epsilon = p - q = 0$ for which the stress is defined as $\sigma = \sigma^0$ following Eq. (6). This means that the stress measure is the Von Mises stress associated with solid material properties, E_0.

For this problem, an optimized design was obtained with a relative volume of $V_{\text{rel}} = 31.02\%$ which is nearly the same as in the previous result for the relaxed stress ($V_{\text{rel}} = 30.00\%$). Comparing the density distribution in Figure 7 with the previous result shown in Figure 6, it can be seen that both designs are similar. On the other hand the stress distributions on which both design are based (relaxed stress vs. unrelaxed stress) are different, as can be seen from comparing Figure 7b with Figure 6c. In this case, the maximum unrelaxed stress is $\sigma_{\max} = 17.56$ and exceeds the admissible stress of $\sigma_{\lim} = 2$. However, it should be noted that the largest stress values are found in void regions, which results from the unrelaxed stress definition in which the stress in the voids remains finite. Since void elements are weak elements, the strains in these elements are relatively large and therefore the stress values are large and dominating. If we would neglect this effect by setting the stress for lower density elements ($\rho < 1/2$) to zero, the maximum stress value is $\sigma_{\max} = 2.06$ and close to the previous result for the unrelaxed stress. In Figure 7c the stress distribution is shown omitting the lower density elements.

The most important observation is that using an unrelaxed stress definition in our method does not lead to the general problems associated with the presence of singular optima. In the traditional methods in which the admissible stress criterion is imposed as a hard constraint, not applying stress relaxation will result in a final design which contains substantial regions of intermediate material. In these 'grey' regions the stress constraints in the elements are active and therefore will not be removed from the design by the optimizer. In this method, large stresses will lead to more penalization and therefore intermediate densities will be reduced to zero. The main reason to use a relaxed stress criterion here is for post-processing reasons to suppress large and dominating stress values from the void elements.



(a) Density distribution (b) Unrelaxed stress distribution (c) Stress distribution omitting elements with density $\rho < 1/2$.

Figure 7: Optimized L-bracket design using internal stress penalization for $\alpha = 10$ and $\delta = 0.04$: a relative volume of $V_{\text{rel}} = 31.02\%$ and a maximum unrelaxed stress of $\sigma_{\max} = 17.56$ if consider all element. Omitting the lower density elements $\rho < 1/2$ gives the density distribution in c) for which the maximum stress is $\sigma_{\max} = 2.06$

4. Conclusions

In this paper, a new method for stress-based topology optimization was presented. Analogous to penalization of intermediate densities, penalization of overstressed material is performed. The method was tested on two design problems and it was shown that designs were obtained for which the maximum stress is close to the admissible stress.

An advantage of this method is that there is no need to transform the local stresses into global/regional stress measures using aggregation functions in order to reduce computational costs related to the sensitivity analysis. Therefore, there is no need to apply additional measures (normalization/regional constraints) to deal with the difficulties associated with the use of aggregation functions. Another advantage is that the admissible stress criterion is not imposed as a hard stress constraint and therefore there are no vanishing stress constraints in this problem formulation. This eliminates the presence of singular optima and its associated problems. It was shown that using an unrelaxed stress definition also gives a meaningful stress-based design. The main reason to use a relaxed stress, as we did in most problems, is to get meaningful stress values in the voids; an unrelaxed stress criterion will give finite stress values in the voids. Finally, what can be considered as a disadvantage of this method, is that it is a penalization

approach which means that it is hard to satisfy the admissible stress criterion accurately. However, the same holds for the traditional method when using aggregation functions. Another disadvantage, may be the computational costs which require the solution of three linear system of equations of the same size. However, here the number of constraints does not increase as the size of the problem increases. Finally, it is not clear yet how to generalize this method, which makes it difficult to apply on different design optimization problems; e.g., minimize the stress subject to volume constraint.
Currently the fundamental characteristics of this method are being studied on some elementary truss optimization examples. Furthermore, it is being tested against existing stress-based topology optimization techniques based on using aggregated stress constraints.

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