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Structural optimization under uncertainties considering reduced-order modeling

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1. Abstract

This paper focus on the development of a optimization tool in which uncertainties are taken into account to obtain robust and reliable designs. The robustness measures considered here are the expected value and standard deviation of the function involved in the optimization problem. To calculate such quantities, we employ two non-intrusives uncertainty propagation analysis techniques that exploit deterministic computer models: Monte Carlo (MC) method and Probabilistic Collocation Method (PCM). When using these robustness measures combined, the search of optimal design appears as a robust multi-objective optimization (RMO) problem. Reliable design address uncertainties to restrict the probability of failure of structures. These are computed through reliability analysis techniques which are here computed by both MC and FORM methods. The insertion of reliability constraints into the RMO problem formulation turns the formulation for the robust and reliability design optimization (R^2DO) problem. As both, statistics calculations and the reliability analysis could be very costly, especially when using the MC method, approximation techniques based on reduced-order modeling (ROM) approach are also incorporated in our procedure via proper orthogonal decomposition (POD) method. For fast outputs considering structural nonlinear behavior. Optimization studies will be conducted for trusses problems considering different loads level, exploring the material plasticity.

2. Keywords: Robust Optimization, Reliability-Based Design Optimization, Multiobjective Optimization, Reduced-Order Modeling, PCM, POD.

3. Introduction

On the design of most engineering applications, the traditional optimization approach is to consider deterministic models and parameters. However, some degree of uncertainty in characterizing any real engineering system is inevitable. Unfortunately, the deterministic approach generally leads to a final design whose performance may degrade significantly or constraints can be violated because of perturbations arising from uncertainties. In this scenario a better target that provides an optimal design is one that gives a high degree of robustness and low probability of failure. In this work some approaches will be used such that uncertainties are incorporated in an optimization procedure in order to obtain robust and reliable designs. The robust measures are the expected value and standard deviation.

4. Robust and Reliable Optimization

As already mentioned, R^2DO considers problem uncertainties to obtain a reliable design less susceptible to variability.

In this work, two objective controls will be considered: the mean and the standard deviation of a selected output function. Under such consideration, it implies that when the expected value is minimized, a less conservative design is found while when the standard deviation is minimized a design with a much smaller range of variation is obtained [1]. A part from geometric constraints, reliability based constraints are imposed to the problem. The task of R^2DO is therefore to obtain the trade-off between the two above aims, keeping control on the probability of failure or the reliability index. Such compromising solutions are obtained by multiobjective optimization techniques [2, 3].

4.1. Problem Definition

Here, the R^2DO problem is mathematically formulated as

$$\min_{x} F\left(x\right) = \left\{E\left(F(x,U)\right), \sigma\left(F(x,U)\right)\right\} \tag{1}$$

$$\text{subject to :} \quad \begin{array}{l} g_i^r(\mathbf{x}) \leq 0 \\ g_i(\mathbf{x}, U) \leq 0 \;\; i = 1, ...m \\ h_j(\mathbf{x}, U) = 0 \;\; j = 1, ...\ell \\ x_k^\ell \leq x_k \leq x_k^u \;\; k = 1, ...n_{dv} \end{array} \tag{2}$$

where x is the design variable vector, U is the random variable vector, $F(x)$ is the set of objective functions to be minimized, $E(*)$ is the expected value , $\sigma(*)$ is the standard deviation, $F(x, U)$ is the selected output, $g_i^r(x)$ is the reliability analysis based constraint: which could be related to the probability of failure or the reliability index, $g_i(x, U)$ and $h_j(x, U)$ are inequality and equality constraints, respectively, that could (or not) be dependent on U, x_k^l , x_k^u, are, respectively, the lower and upper bounds of a typical design variable, m, l and n_{dv} are the number of inequality constraints, equality constraints and design variables, respectively. The MO problem presented above is solved using the techniques based on the Pareto minima concept that are described in Section 6.

4.2. Statistics Calculation

Assuming U as a random variable, any function $f(U)$ will be random, with its specific probability density function (PDF) $P(U)$. The expected value of $f(U)$, called mean of $f(U)$, can calculated as [4]:

$$E[f(U)] = \bar{f} = \int_{-\infty}^{\infty} f(U)P(U)dU \tag{3}$$

and its variance $\sigma_f^2 = \sigma[f(U)]^2$

$$\sigma_f^2 = E\left[\left(f(U) - \bar{f}\right)^2\right] = \int_{-\infty}^{\infty} \left(f(U) - \bar{f}\right)^2 P(U)dU \tag{4}$$

in which σ_f is the standard deviation.

In the present work two methodologies are employed for statistics calculations of several responses. They are Monte Carlo method and Probabilistic collocation method. Both methodologies are described in the following subsections.

4.2.1 Monte Carlo Method

The MC method is the most popular non-intrusive method and can be used for any problem related to uncertainty propagation [5]. Given the joint probability distribution function of the involved random variables, the MC method can be applied for approximated calculations of the statistics response of a particular quantity, including its distribution, with an arbitrary error, as long a sufficient number of samplings points is given. This approach has also been used as a ¨benchmark¨to validate other techniques for statistics calculations. In this method the functions $f(\mathbf{U})$ of interest are calculated in several random points U^k , generated taking into account their probability distribution $P(U)$, then the integrals of Eqs. 3 and 4 are respectively approximated as

$$\begin{array}{c} \bar{f} \approx \bar{f}_{MC} = \frac{1}{m} \sum\limits_{i=1}^{m} f(\mathbf{U}_{(i)}) \\ \sigma[f(\mathbf{U})]^2 = \sigma_f^2 \approx \hat{\sigma}_f^2 = \frac{1}{m-1} \left[\sum\limits_{i=1}^{m} \left(f(\mathbf{U}_{(i)})^2 \right) - m\bar{f}_{MC}^2 \right] \end{array} \tag{5}$$

in which m is the number of sampling points, $\bar{f}_{MC}$ is the MC approximation for the mean values of $f(U)$ and $\hat{\sigma}_f$ is the MC approximation for the standard deviation.

4.2.2 Probabilistic Collocation Method

The basic idea of PCM is to approximate the function $f(U)$ by polynomial functions and to evaluate the integrals of Eqs. 3 and 4 by Gaussian quadrature. Gaussian quadrature is based on the concept of orthonormal polynomials. These concepts are briefly described here.

In the numerical integration by Gaussian quadrature for integrals of the form

$$F = \int f(U)\, P(U)\, dU \tag{6}$$

The function $f(U)$ is approximated by a polynomial of order $2n-1$ as follows [6]

$$f(U) \approx \hat{f}(U) = \left(\sum_{i=0}^{n-1} b_i h_i(U)\right) + h_n(U)\left(\sum_{i=0}^{n-1} c_i h_i(U)\right) \tag{7}$$

for $i = 1 \ldots n$ in which b_i and c_i are the coefficients of the approximation, to be obtained, and $h_i(U)$ are polynomials of order i from a orthonormal basis with respect to the weight function $P(U)$.

The statistics evaluations defined in Eqs. 3 and 4 via PCM is a direct application of Gaussian quadrature in which the PDF is the weighting function. Hence, by orthonormality, the approximated Gaussian quadrature integral Eq. (6) can be expressed as follows

$$F \approx b_0 h_0 \int_F P(U)\, dU = b_0 \tag{8}$$

To find the coefficients b_i and c_i of the Eq. (7) would be necessary to evaluate the function $f(U)$ in 2n points. However, as the integral presented in Eq. (8) does not depend on the coefficients c_i, it is required the calculations of function $f(U)$ only at the n roots (U^*) of $h_n(U)$, in this way canceling the second part of Eq. (7), as $h_n(U^*) = 0$. For more details concerning coefficients evaluations see [7].

The orthonormal polynomials are defined for each PDF and the roots (U^*) of each polynomial $h_i(U)$ are the quadrature points or integration point. Solving the approximation of Eq. (7) to find b_0, it follows that the mean value and, analogously, the standard deviation of an output of interest are approximated by PCM as

$$\begin{aligned} \bar{f}_{PC} &= \sum_{i=1}^{n} P_i f(U^{k*}) \\ \hat{\sigma}^2_{PC} &= \sum_{i=1}^{n} P_i f(U^{k*})^2 - \bar{f}^2_{PC} \end{aligned} \tag{9}$$

in which $P_i, : i = 1 \ldots n$, are the weight coefficients and U^* the integration points, calculated once PDF is given.

4.3 Reliability Measures

The reliability measures needed to compute the g^r constraint are obtained through reliability analysis. The reliability analysis computes the structural probability of failure (Pf) and its reliability index (β). For a set of random variables U with a joint probability distribution function $f_U(U)$ and a failure function $G(U)$, the Pf can be computed as follows:

$$Pf = \int\limits_F f_U(U) dU \tag{10}$$

in which F is the failure region, defined as:

$$F = \{U : G(U) > 0\} \tag{11}$$

The reliability analysis could be performed by several methods. Here the Pf computation will be proceed via Monte Carlo (MC) method and First Order Reliability Method (FORM) [8]. The MC is the simplest method used for reliability analysis. The Pf is obtained after the evaluation of the failure function in a set of samples in the random variable space. These random points are generated considering the joint probability distribution function $f_U(U)$ of the random variables. The Pf is the proportion of points that lie over the S region, in which $G(U) < 0$.

4.3.1 Monte Carlo

In the MC method, to Pf computation, the integral presented in Eq. (10) is numerically approximated as [8]

$$Pf_{MC}(x) = \frac{1}{ns} \sum_{i=1}^{ns} \left[G(\mathbf{U}_{(i)}) < 0\right] \tag{12}$$

in which ns is the number of MC sampling and $\left[G(\mathbf{U}_{(i)}) < 0\right] = 1$ for situations that failure occurs $(G(\mathbf{U}_{(i)}) < 0)$ and zero otherwise.

4.3.2 FORM

FORM approximates the problem around the most probable failure point (MPP) to an equivalent basic problem by transforming the original random variables space (U) into a standard space (V), called Nataf transformation (T). A graphical interpretation of the method is shown in Figure 1. The MPP is the shortest distance point from the limit state $G(U) = 0$ to the origin of the standard space (V). The minimum distance $\beta = |V^*|$ is called the reliability index. The standard space is a space of equivalent standard normal distribution of uncorrelated random variables, i.e. $f_V(V)$ is the standard normal distribution (mean, $\mu = 0$ and standard deviation, $\sigma = 1$). In the standard space the limit state is linearly approximated around the MPP [8].

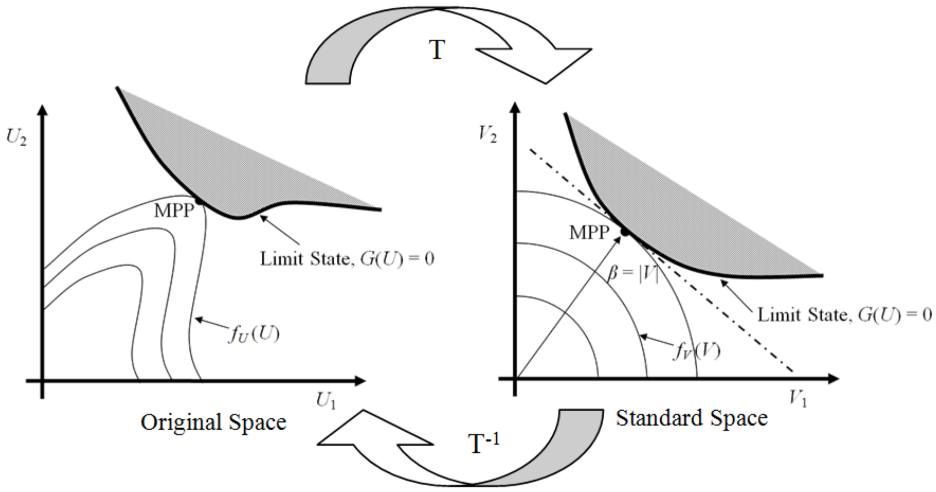

Figure 1: FORM overview.

In the standard space Pf is approximated by $\Phi(-\beta)$, where Φ is the cumulative distribution function of the standard normal distribution. The opposite procedure could be used to approximate the reliability index from the Pf value (obtained via MC, for instance). To find the MPP an iterative algorithm, called HL-RD from Hasofer, Lind, Rackwitz and Fiessler [9, 10], is used in the standard space, from which the following updated rule is employed.

$$V^{K+1} = \frac{\nabla G_V(V^K)}{|\nabla G_V(V^K)|^2}\left[\nabla G_V(V^K)^T V^K - G_V(V^K)\right] \tag{13}$$

in which $G_V(V) = G(U)$ and $U = T^{-1}V$.

In short the main steps of the FORM algorithm are:

1. Transform current point $V = TU$ (Nataf transformation);
2. Compute $G_V(V)$;
3. Compute $\nabla G_V(V) = T^{-1}\nabla G(U)$
4. Compute the new V, Eq. (13)
5. Transform back $U = T^{-1}V$ (Nataf transformation);
6. Verify convergence, if not go to 1;
7. Compute $\beta = \|V\|$.

generally the initial guess point is $U = \mu$

5. Governing Equations

5.1 High Fidelity Model

The high fidelity response needed to build the surrogate (low fidelity model) will be calculated considering nonlinear static analysis performed by the finite element method (FEM). In this sense, the solution field (displacements) is obtained such that the internal forces equals the external forces

$$\mathbf{F}_{\mathrm{i}}(\mathbf{u}) = \mathbf{F}_{\mathrm{e}} \tag{14}$$

The iterative procedure generally used to obtain solution of plastic analysis is the Newton-Raphson (NR) method. The NR method iteratively approximates the nonlinear equation by a linearization in the current point (solution). In the plastic analysis it can be formulated for the kth iteration as

$$\mathbf{F}_{\mathrm{i}}\left(\mathbf{u}^{k}\right) + \frac{d\mathbf{F}_{\mathrm{i}}}{d\mathbf{u}}\left(\mathbf{u}^{k}\right)\Delta\mathbf{u}^{k} = \mathbf{F}_{\mathrm{e}} \quad \text{or} \quad \mathbf{K}\mathrm{t}\Delta\mathbf{u}^{\mathrm{k}} = \mathbf{R} \tag{15}$$

in which $\mathbf{K}t = \frac{d\mathbf{F}_{\mathrm{i}}}{d\mathbf{u}}\left(\mathbf{u}^{k}\right)$ is the tangent stiffness matrix, and $\mathbf{R} = \mathbf{F}_{\mathrm{e}} - \mathbf{F}_{\mathrm{i}}\left(\mathbf{u}^{k}\right)$ is the residual load vector.

The iterative technique on its own can only provide a single 'point solution'. In practice, we will often prefer to trace the complete load/deflection response (equilibrium path). To this end, it is useful to combine the incremental and iterative solution procedures. The 'tangential incremental solution' can then be used as a 'predictor' which provides the starting solution, for the iterative procedure. A good starting point can significantly improve the convergence of iterative procedures and increases the possible incremental load step. Indeed it can lead to convergence where otherwise divergence would occur [11].

There are various incremental load step methods. A constant increment is considered here, in which the increment is proportional to the pressure at the yield stress. The increment (proportional factor) used in the analysis procedure will be specified in the Application section.

5.2 Low Fidelity Model

The low fidelity model will be used for the multiple function evaluations required in the statistic calculations, reliability analysis and optimization procedure. The POD techniques is employed to construct this model.

POD is a ROM that, basically, project the problem into a subspace formed by a optimum orthonormal basis functions, in the sense that it considers the most significant shape (greatest variance) of the output ($\mathbf{u}$) subspace. A practical way to obtain these vectors is computing a set of $\mathbf{u}$ vectors for various system configurations (times, design variables, parameters, loads steps,) then, a singular value decomposition (SVD) is performed and to compute the eigenvalues of the covariance matrix of the outputs [12, 13]. This procedure is related to the method of snapshots. The method of snapshots was introduced by Lawrence Sirovich in 1987 [14] as a way to reduce the computational requirements of the POD basis. The snapshot matrix $\mathbf{X}$ can be written as:

$$\mathbf{X} = \left[\mathbf{u}^{1}, \mathbf{u}^{2}, \ldots, \mathbf{u}^{m}\right] \tag{16}$$

the number of snapshots m is assumed to be sufficiently large for represent the field of solutions $\mathbf{u}$.

This POD basis can be obtained directly through a SVD of the snapshot matrix $\mathbf{X}$.

$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}} \text{ where } \mathbf{X}\mathbf{V} = \mathbf{U}\mathbf{S} \text{ and } \mathbf{X}^{\mathrm{T}}\mathbf{U} = \mathbf{V}\mathbf{S}^{\mathrm{T}} \tag{17}$$

in above equation $\mathbf{S}$ is related to the eigenvalues such as $diag(\mathbf{S}) = [\Lambda_1, \Lambda_2, \ldots \Lambda_n]$, $\Lambda_i = \sqrt{\lambda_i}$, and $\mathbf{V}$ are eigenvectors of the autocorrelation matrix $\mathbf{X}$.

To compute the POD basis, the first w eigenvalues such that

$$1 - \sum_{i=1}^{w} \bar{\lambda}_i < tol_{\lambda}, \text{ in which } \bar{\lambda}_i = \frac{\lambda_i}{\sum_k \lambda_k} \tag{18}$$

must be found. In the equation 18, the tol_λ parameter is the tolerance related to the energy error in the POD approximation. Note that $[w < m < n]$. After the number w of significant singular components be determined, the POD basis is computed

$$\mathbf{Z} = \mathbf{U}^w \tag{19}$$

where the upper w index, indicate the first w vectors (column) of the matrix.

To proceed the POD in the solution of the nonlinear structural analysis, the standard iterative equation (15) has to be changed, so that the displacement vector $\mathbf{u}$ is the unknown, rather than $\Delta\mathbf{u}$, following to

$$\mathbf{K}_t \left(\mathbf{u}_k - \mathbf{u}_{k-1}\right) = \mathbf{R}_k \text{ or } \mathbf{K}_t\mathbf{u}_k = \mathbf{R}_k + \mathbf{K}_t\mathbf{u}_{k-1} \tag{20}$$

This is due to the fact that the displacement vector $\mathbf{u}$ (correlated) is easier to approximate than the vector $\Delta\mathbf{u}$ (uncorrelated).

As can be seen, the POD has equivalence to the principal component analysis (PCA), the singular value decomposition (SVD) and Karhunen-Love decomposition, and has been widely and successfully applied in various disciplines, including fluid mechanics, static and dynamic structural mechanics, oceanography, statistics, economics, image processing, etc [15].

6. Multiobjective Optimization

Pareto optimality concept [16] is used here to obtain MO solutions. The Pareto minima, are points x_p which for no other point x exist such that:

a) $f_k(x) \geq f_k(x_p)$ for $k = 1, \ldots, nobj$

b) $f_j(x) < f_j(x_p)$

for one objective function (f_j) at least. The discussions about this concept can be found in detail elsewhere [17, 18, 19].

Using the Pareto concept, the designer has to identify as many Pareto points as possible. These points can be used to construct a point-wise approximation to the Pareto front.

There are several techniques to obtain the set of Pareto minima. In this work we will consider the so-called objective weighting sum (WS) method, Min-Max method, the normal boundary intersection (NBI) method [18], and the normalized normal-constrain (NNC) method [20]. Currently, in literature, the later two strategies are pointed to have more success to obtain the Pareto curves. Such techniques are discussed in detail elsewhere [19].

7. Example

The tool developed in this work is used in this section to obtain a reliable optimum design with reduction in the structural performance variability. A plane truss will be optimization here, considering material nonlinearity under static load conditions. The geometric configuration and boundary conditions are presented in Figure 2, where the number of degrees of freedom is 1210 [21].

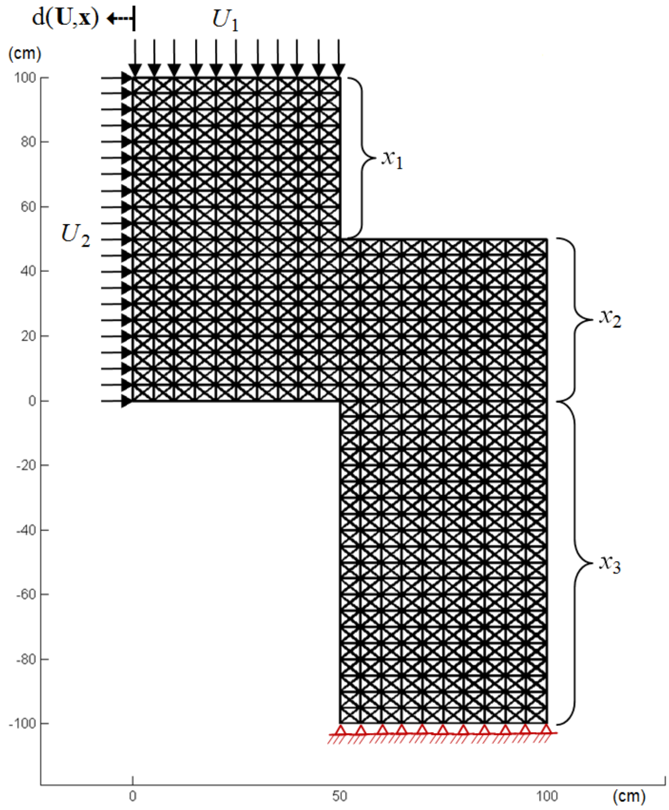

Figure 2: Structure and problem definition.

The points of the stress-strain curve considered are illustrated in Table 1.

Table 1: Stress-Strain Curve.

Stress	Strain (KN/cm^2)
0.0025	51.750
0.0037	62.100
0.0050	72.450
0.0100	82.800
0.0175	93.150
0.0350	103.500
0.0750	113.850

Figure 2 shows the random variables (**U**) and the design variables (**x**). Thus, two random variables are considered: the vertical load on the top of the structure (U_1) and the horizontal load on the top-left side of the structure (U_2). The first one (U_1) has a log-normal distribution with mean $\mu_1 = 4$KN/cm and standard deviation $\sigma_1 = 2$KN/cm, the second random variable (U_2) has a normal distribution with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 1$KN/cm. Three designs variables are considered, which are the cross section area of the bars of three regions, as shown Figure 2. The initial cross section areas (designs variables) are equal to two cm^2 and the design variables are bounded by $0.1 \leq x \leq 10$.

The R^{2}DO problem can be formulated as:

$$\begin{array}{l} \min\limits_{\mathbf{x}\in\mathbf{R}^3} \quad \left[E\left(\sigma(\mathbf{x},\mathbf{U})\right),\ SD\left(\sigma(\mathbf{x},\mathbf{U})\right)\right], \\ \\ \text{subject to :} \\ V\left(\mathbf{x}\right) \leq V_0 \\ \beta\left(\mathbf{U},\mathbf{x}\right) \geq 3.3 \\ 0.1 \leq x_i \leq 10,\ i = 1, 2, 3 \end{array} \tag{21}$$

in which $E(\sigma(\mathbf{x},\mathbf{U}))$ and $SD(\sigma(\mathbf{x},\mathbf{U}))$ are the statistical moments of the maximum von misses stresses to be minimized. The current volume should be less or equal to the initial volume ($V_0 = 24741\text{cm}^3$), the reliability index (β) is computed for the following failure function

$$g(\mathbf{x},\mathbf{U}) = d(\mathbf{x},\mathbf{U}) - d_{\max} \tag{22}$$

where $d_{\max} = 1.0$ cm is the maximum horizontal displacement allowed in the top left corner of the structure, see Figure 2.

The analysis parameter for the standard FEM analysis and for the POD reduced model (calibrate process), were obtained for the convergence of the both nonlinear iterative method of analysis. The load increment used was Py /10, in which Py is the load level that lead to the yield stress.

A 1210x246 snapshot matrix ($\mathbf{X}$) was obtained through the analysis via FEM of 30 different cases (considering different values for design variables and random variables). For a required tolerance of $tol_\lambda = 10^{-5}$, the size of POD basis generated was $w = 50$.

The R^{2}DO problem was solved considering the FORM method to evaluate the reliability index and PCM to evaluate the statistics of the structure for each design variable set. The nonlinear analysis responses were obtained using POD reduced order model (for $w = 50$). The Pareto points obtained via the various MO methods cited here, are shown in Fig. 3. As expected, the results via NBI and NNC agree closely. The better Pareto points distribution was obtained by these two methods.

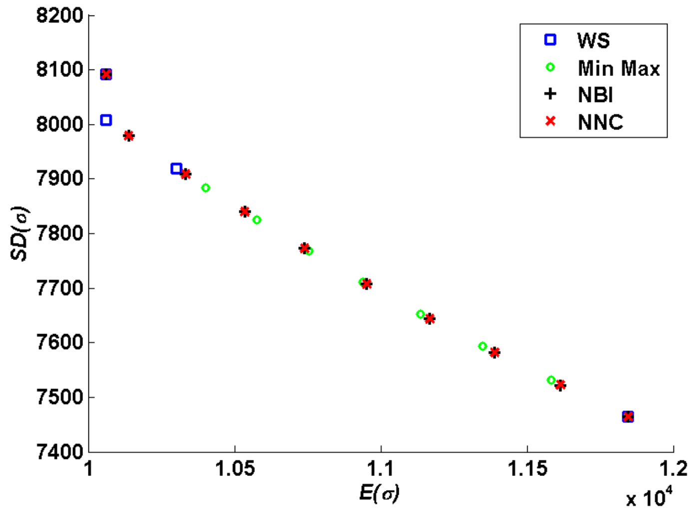

Figure 3: Pareto solutions via different MO methods.

The four multi-objective optimization techniques performance, are shown in Table 2. In that table, the number of function evaluations (F. Count) is the total number of statistic analysis evaluated to obtain the Pareto points. The uniformity distribution of the Pareto points parameter (Evness), that appears in Table 2, indicates the quality of the distribution of the points, the closer to zero the better [20, 19]. The most efficient method in this example was the NNC method.

Table 2: Optimization performance considering PCM with POD methods.

MO Method	Time (min)	F Count	Evness
WS	1.8985	93	1.9695
MinMax	32.8887	225	0.4361
NBI	3.5222	153	0.0517
NNC	2.9911	111	0.0517

Figure 4 shows the MC results considering 10^5 sampling points for the design related to the upper-left Pareto solution in Figure 3 (the one that minimize the expected value of the von Mises stress). This sample size was determined based on convergence test of the Pf value. In that figure, the circles in color scale are the failure function value at each MC integration point, the red cross ('+') is the mean point and the 'x' points represent the failure points.

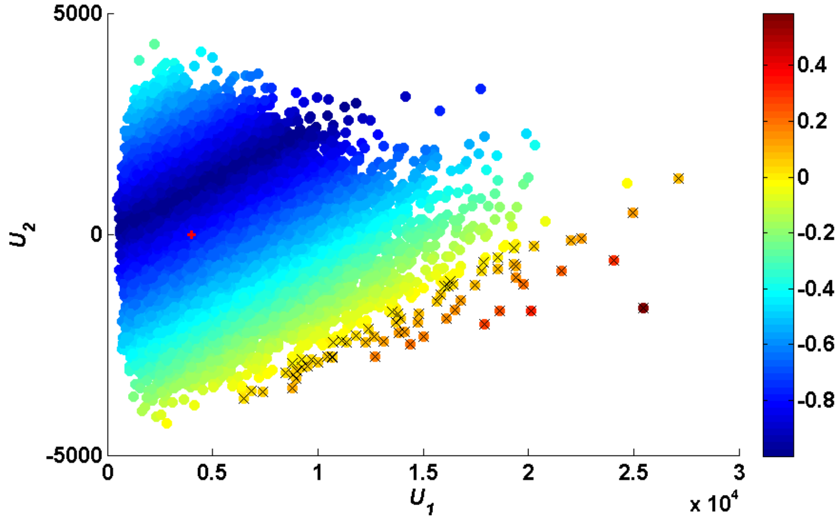

Figure 4: MC results.

For this optimum design, the probability of failure obtained via FORM (through the β index) and via MC was respectively, 0.048% and 0.066%. The β index obtained via FORM and via MC (through the Pf value) was respectively 3.30 and 3.21. According to the MC result this design is unfeasible. However, the MC method, unlike the FORM, is able to obtain the Pf value with a controlled error, but the value of the β index (obteined from the Pf value) has an inerent error. The summary of the FORM and MC results are shown in Table 3.

Table 3: Optimization Results.

Results	MC	FORM
Pf $(\mathbf{x}^*)$	0.066%	0.048%
β $(\mathbf{x}^*)$	3.21	3.30
F Count	10^5	8

8. Conclusions

In this paper a R^{2}MO problem was solved using PCM to evaluate the statistics (1st and 2nd statistical moments) of the response and FORM to evaluate the reliability index of a truss under nonlinear condition. Several multi-objective optimization techniques (Ws, Min-Max, NBI and NC methods) were used to obtain Pareto solutions. A POD algorithm was implemented to approximate nonlinear FEM analysis, considering the material nonlinearity. For an error tolerance of 10^{-5} a basis of just 50 components is used to approximate an output of 1210 components.

The results of the reliability analysis via FORM approximation were confronted to the results via MC for 100.000 integration points. Although some relative differences, on the probability of failure (Pf) and the β index value, computed from both methods the results from FORM are suitable for practical cases. For a better approximation of the Pf value through the β index the SORM (Second Order Reliability Method) could produce a better approximation.

In summary:

- The statistics computation via PCM require about 10^3 times less integration points than the MC method, for the same relative error;
- The structural reliability analysis via FORM require about 10^4 times less structural analyses than the MC method, for a acceptable relative error;
- For the bi-objective example, the most efficient MO methods were the NBI and NNC. Both obtained evenly distribution of the Pareto points with small computational effort.

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