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On the adaptive ground structure approach for multi-load truss topology optimization

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1. Abstract
This paper presents new numerical solutions of generalized Michell structures subjected to multiple load conditions. These results were obtained using a new method to determine optimal topologies of multi-load trusses, recently developed by the first author of the paper. The method is based on the adaptive ground structure approach and stress-based formulation of the optimization problem. It leads to a huge but linear programming problem. Thus, the proposed method assures finding the global optimum for the given discretization of the design domain, represented here by the fully connected ground structure. The method makes use of both *active set* and *interior point* methods and allows solving a large-scale optimization problem even with more than billion design variables. The efficiency and robustness of the proposed method have been confirmed in several benchmark tests. An important result of this research is finding a new exact-analytical solution of a specific symmetrical two-load case problem. This solution was predicted numerically using the method described above, and then adjusted to obtain the exact formulae describing the layout of the optimal structure using the concept of *component loads*.
2. Keywords: topology optimization, optimal trusses, multi-load case, ground structure, linear programming.

3. Introduction
Michell trusses are the most efficient structures capable of transmitting the given load to the given supports using the smallest amount of structural material. In fact they are not real trusses because they are composed of an infinite number of infinitesimal bars. Nevertheless, they define the shape and topology of the optimal structure, thus they play a significant role in structural topology optimization. The exact analytical solutions of Michell trusses are very hard to obtain but can be accurately approximated numerically by trusses composed of large but finite number of bars. It requires, however, solving large-scale numerical optimization problems which can be handled only by appropriate optimization methods. This problem is more complex for the case of multiple loads since then we have to deal with independent member forces (design variables) for each load condition. Hence the size of the optimization problem increases several times.
The *classical ground structure approach* was introduced by Dorn, Gomory and Greenberg [4], who showed that truss topology optimization problem can be formulated on the basis of plastic design principles and solved using the linear programming methods. The above paper received little attention in recent decades. This approach leads to a large optimization problem with many design variables and corresponding constraints, and therefore might be regarded as the "brutal force" or inefficient method. But contrary to many other approaches, it assures finding a possibly lightest truss in a global sense. The solution obtained by this method determines at once the optimal topology, shape and cross-section areas. Moreover, the main difficulty of the classical approach has recently been overcome by new linear programming methods based on the *adaptive ground structure approach*, introduced by Gilbert and Tyas [5] and then modified by Sokół [18,19]. The latter method has been tested using ground structures with more than a billion potential bars giving the excellent agreement with known exact solutions of Michell problems (see [21, 22, 23]). The main goal of the present paper is to show that this method can be enhanced and applied to multi-load case problems.
The problem of multiple load cases is not new and was investigated in many papers, see [1-3, 6, 8, 10, 11, 13-17], and some basic properties of the optimal multi-load trusses are known. For instance and contrary to classical Michell solutions, the bars in the optimal multi-load truss have not to be orthogonal nor fully stressed and the optimal trusses are often statically indeterminate structures. In the present paper we focus on new properties of multi-load trusses.

4. Primal and Dual Formulations of Multi-Load Stress-Based Optimization of Trusses
We start the discussion by reviewing various formulations of the stress-based multi-load truss optimization problem and then derive the formulae needed to determine which new candidate bars should be activated in a subsequent iteration. These formulae define adjoint strains (or elongations) in the optimal truss and can be treated

as generalized Michell optimality criteria [9]. They allow also defining a proper stop condition for the method developed, see Sec. 5.

Any optimization problem can be written in different forms which are mathematically equivalent but can lead to significantly different calculation times using the given optimization method (see [1-3]). In other words, the formulation of the optimization problem should be matched to the method applied.

According to well known duality principles, the plastic design optimization problem can be written as well in primal as dual form. Both of them play an important role in the proposed method and should be considered together.

The primal (lower bound) form of the truss topology optimization problem for different stress limits for tension σ_T and compression σ_C, and for multiple load conditions can be defined as follows (c.f. [3]):

$$\begin{aligned} &\min_{\mathbf{A},\mathbf{S}_{(l)}\in\mathbf{R}^M} V = \mathbf{L}^T\mathbf{A} \\ &s.t. \quad \mathbf{B}^T\mathbf{S}_{(l)} = \mathbf{P}_{(l)} \\ &\quad -\mathbf{A}\sigma_C \le \mathbf{S}_{(l)} \le \mathbf{A}\sigma_T \\ &\quad \text{for } l = 1,2,\ldots,nlc \end{aligned} \tag{1}$$

The goal of this problem is to find the truss of minimal volume of structural material and subject to equilibrium equations (defining the sets of statically admissible member forces) and to inequality constraints arising from the allowable stresses in tension and compression. Thus in this natural form, the objective function V is the total volume of bars of the truss; M denotes the number of potential bars; $\mathbf{L}$ is the vector of bar lengths; $\mathbf{A}$ is the vector of member cross-section areas (the main design variables); $\mathbf{B}$ is the geometric matrix including directional cosines of bars; $\mathbf{P}_{(l)}$ are the vectors of nodal loads, independent for every load case $l = 1, 2, \ldots, nlc$, where nlc denotes the number of load conditions; and $\mathbf{S}_{(l)}$ are the vectors of member forces corresponding to loads $\mathbf{P}_{(l)}$.

Note that (1) belongs to *worst-case formulations* which define 'true designs' for multiple and independent load conditions because the constraints $(1)_3$ simply define the smallest required cross-section area of every bar:

$$A_i \ge \max_{l=1,2,\ldots,nlc}\left(\frac{S_{(l),i}}{\sigma_T}, \frac{-S_{(l),i}}{\sigma_C}\right) \quad \text{for } i = 1,2,\ldots,M \tag{2}$$

It is to be remarked that the more popular (easier to use) *weighted-average formulations* cannot strictly satisfy (2).

The concise form (1) is not convenient for direct use in linear programming methods because the design variables can be positive or negative. These methods prefer a standard form of the optimization problem in which all design variables are greater than or equal to zero. In addition, the form (1) cannot be applied in the adaptive ground structure method, discussed in the next section. The reason of this can be understood by examining the dual form of the problem (1)

$$\begin{aligned} &\max_{\mathbf{u}_{(l)}\in\mathbf{R}^N,\, \mathbf{e}^+_{(l)},\mathbf{e}^-_{(l)}\in\mathbf{R}^M} W = \sum_l \mathbf{P}^T_{(l)}\mathbf{u}_{(l)} \\ &s.t. \quad \sum_l (\sigma_T\mathbf{e}^+_{(l)} + \sigma_C\mathbf{e}^-_{(l)}) = \mathbf{L} \\ &\quad \mathbf{B}\,\mathbf{u}_{(l)} - \mathbf{e}^+_{(l)} + \mathbf{e}^-_{(l)} = \mathbf{0} \\ &\quad \mathbf{e}^+_{(l)} \ge \mathbf{0}, \quad \mathbf{e}^-_{(l)} \ge \mathbf{0} \\ &\quad \text{for } l = 1,2,\ldots,nlc \end{aligned} \tag{3}$$

in which vectors $\mathbf{u}_{(l)}$, $\mathbf{e}^+_{(l)}$ and $\mathbf{e}^-_{(l)}$ denote appropriate Lagrange multipliers. Note that they are independent for every load condition. The physical interpretation of these multipliers is as follow: $\mathbf{u}_{(l)}$ denote the *adjoint nodal displacements*; $\mathbf{e}^+_{(l)}$ and $\mathbf{e}^-_{(l)}$ are the vectors of *adjoint elongations of members*, both considered in positive manner for tension and compression. The objective function W is the sum of works of the forces $\mathbf{P}_{(l)}$ over $\mathbf{u}_{(l)}$ so we can loosely say that we maximize the 'total compliance' over admissible adjoint displacements and member elongations.

In the problem (3) all dual variables are coupled through equality constraints and must be perfectly matched what makes trouble with the convergence of the optimization process. Moreover this formulation is not applicable in the adaptive ground structure method (discussed later). Thus it is recommended to explicitly specify that the cross-section areas $\mathbf{A}$ are non-negative

$$
\begin{aligned}
&\min_{\mathbf{A},\mathbf{S}_{(l)}\in\mathbf{R}^M} V = \mathbf{L}^T\mathbf{A} \\
&s.t. \qquad \mathbf{B}^T\mathbf{S}_{(l)} = \mathbf{P}_{(l)} \\
&\qquad -\mathbf{A}\sigma_C \le \mathbf{S}_{(l)} \le \mathbf{A}\sigma_T \\
&\qquad \mathbf{A} \ge \mathbf{0} \\
&\qquad \text{for } \; l = 1,2,\ldots,nlc
\end{aligned}
\tag{4}
$$

Formally, the constraints $(4)_4$ are not necessary because they follow directly from $(1)_3$. Nevertheless it is worth adding them explicitly to obtain a more relaxed dual form

$$
\begin{aligned}
&\max_{\mathbf{u}_{(l)}\in\mathbf{R}^N,\mathbf{e}^+_{(l)},\mathbf{e}^-_{(l)}\in\mathbf{R}^M} W = \sum_l \mathbf{P}^T_{(l)}\mathbf{u}_{(l)} \\
&s.t. \quad \sum_l (\sigma_T\mathbf{e}^+_{(l)} + \sigma_C\mathbf{e}^-_{(l)}) \le \mathbf{L} \\
&\qquad \mathbf{B}\,\mathbf{u}_{(l)} - \mathbf{e}^+_{(l)} + \mathbf{e}^-_{(l)} = \mathbf{0} \\
&\qquad \mathbf{e}^+_{(l)} \ge \mathbf{0}, \quad \mathbf{e}^-_{(l)} \ge \mathbf{0} \\
&\qquad \text{for } \; l = 1,2,\ldots,nlc
\end{aligned}
\tag{5}
$$

Here, instead of equations $(3)_2$ we have inequality constraints $(5)_2$ letting dual variables to be easier adjusted. Moreover, formulation (5) enables us to derive the optimality criteria suitable for the adaptive ground structure approach (they are however discussed later).
The formulations (4) and (5) were applied in the first version of the program developed. It enabled finding accurate solutions of some relatively simple multi-load case problems for which the ground structure did not have to be very dense. Unfortunately, the larger problems quickly revealed that the convergence of the interior point method based on the form (4) is rather slow. Therefore, in the next stage of the study we decided to convert (4) to a more applicable (standard) form. It can be done by separating member forces **S** into tension and compression forces: **T** and **C** respectively (cf. [18]). This leads to the following primal form

$$
\begin{aligned}
&\min_{\mathbf{A},\mathbf{T}_{(l)},\mathbf{C}_{(l)}\in\mathbf{R}^M} V = \mathbf{L}^T\mathbf{A} \\
&\text{s.t.} \quad \mathbf{B}^T\mathbf{T}_{(l)} - \mathbf{B}^T\mathbf{C}_{(l)} = \mathbf{P}_{(l)} \\
&\qquad \sigma_T\mathbf{A} - \mathbf{T}_{(l)} \ge \mathbf{0} \\
&\qquad \sigma_C\mathbf{A} + \mathbf{C}_{(l)} \ge \mathbf{0} \\
&\qquad \mathbf{A} \ge \mathbf{0}, \; \mathbf{T}_{(l)} \ge \mathbf{0}, \; \mathbf{C}_{(l)} \ge \mathbf{0} \\
&\qquad \text{for } l = 1,2,\ldots,nlc
\end{aligned}
\tag{6}
$$

which is almost two times greater than (4) but here all design variables are non-negative. This significantly improves the convergence of the interior point method and reduces the computation time.
After arranging the total vector of design variables in the order: {**A**, $\mathbf{T}_{(1)}$, $\mathbf{C}_{(1)}$, ..., $\mathbf{T}_{(nlc)}$, $\mathbf{C}_{(\mathrm{nlc})}$}, the coefficient matrix of the problem (6) can be written in the following block-matrix form:

$$
\mathbf{H} = \begin{bmatrix} \mathbf{F} & \mathbf{G} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{F} & \mathbf{0} & \mathbf{G} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{F} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G} \end{bmatrix}
\tag{7}
$$

where

$$
\mathbf{F} = \begin{bmatrix} \mathbf{0} \\ \sigma_T\mathbf{I} \\ \sigma_C\mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} \mathbf{B}^T & -\mathbf{B}^T \\ -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}
\tag{8}
$$

Note that all matrices defined in (7) and (8) are very sparse and this sparsity has to be utilized to achieve a good efficiency of computations. In addition, the main matrix **H** has a regular-repetitive form which also should be taken into account.
The dual (upper bound) form of the multi-load case problem (6) can be written as

$$\max_{\mathbf{u}_{(l)}\in\mathbf{R}^N,\mathbf{e}^+_{(l)},\mathbf{e}^-_{(l)}\in\mathbf{R}^M} W=\sum_l \mathbf{P}^T_{(l)}\mathbf{u}_{(l)}$$
$$\begin{aligned} s.t.\quad & \sum_l(\sigma_T\mathbf{e}^+_{(l)}+\sigma_C\mathbf{e}^-_{(l)})\le\mathbf{L}\\ & \mathbf{B}\mathbf{u}_{(l)}-\mathbf{e}^+_{(l)}\le\mathbf{0}\\ & -\mathbf{B}\mathbf{u}_{(l)}-\mathbf{e}^-_{(l)}\le\mathbf{0}\\ & \mathbf{e}^+_{(l)}\ge\mathbf{0},\quad \mathbf{e}^-_{(l)}\ge\mathbf{0}\\ & \text{for}\ \ l=1,2,\ldots,nlc \end{aligned}\tag{9}$$

It is worth noting that using the primal-dual interior point method [7,25] it is enough to solve the problem (6) because we get the dual variables of the problem (9) for free. They are necessary to derive a suitable recipe of activating new bars in the ground structure.
Let us consider for the moment only one, i-th bar of the truss. Its elongation for load case l is equal to

$$e_{(l),i}=\mathbf{B}_i\,\mathbf{u}_{(l)}\tag{10}$$

where $\mathbf{B}_i$ denotes i-th row of matrix $\mathbf{B}$. Now one can easily deduce that in an optimal solution one of the constraints $(9)_3$ or $(9)_4$ has to be active and three scenarios are possible:

$$\begin{aligned} &\text{if}\ \ e_{(l),i}>0\ \ \text{than}\ \ e^+_{(l),i}=e_{(l),i}\ \ \text{and}\ \ e^-_{(l),i}=0\\ &\text{if}\ \ e_{(l),i}<0\ \ \text{than}\ \ e^+_{(l),i}=0\ \ \text{and}\ \ e^-_{(l),i}=-e_{(l),i}\\ &\text{else}\ \ e^+_{(l),i}=e^-_{(l),i}=e_{(l),i}=0 \end{aligned}\tag{11}$$

which can be reduced to a more concise form

$$e^+_{(l),i}=\max(e_{(l),i},0)\ \ \text{and}\ \ e^-_{(l),i}=\max(-e_{(l),i},0)\tag{12}$$

After dividing the elongations $e^+_{(l),i}$ and $e^-_{(l),i}$ by the length L_i one can easily replace (11) and (12) by similar formulas based on adjoint member strains

$$\varepsilon^+_{(l),i}\ =\ e^+_{(l),i}\,/\,L_i\quad\text{and}\quad \varepsilon^-_{(l),i}\ =\ e^-_{(l),i}\,/\,L_i\tag{13}$$

The constraints $(9)_{2,3,4}$ determine the domain of feasible adjoint member elongations and allow to formulate the following theorem of generalized Michell optimality criteria for multi-load trusses.

Theorem:
In the stress-based multi-load truss optimization problem the optimal solution has to satisfy the following conditions:
1) for every bar of the truss the adjoint multi-load strains are restricted by

$$\forall i=1:M\ \ \text{and}\ \ \forall l=1:nlc\quad \sum_l(\sigma_T\varepsilon^+_{(l),i}+\sigma_C\varepsilon^-_{(l),i})\le 1\tag{14}$$

where

$$\begin{aligned}\varepsilon^+_{(l),i}&=\max(\varepsilon_{(l),i},0)\\ \varepsilon^-_{(l),i}&=\max(-\varepsilon_{(l),i},0)\end{aligned}\tag{15}$$

and

$$\varepsilon_{(l),i}=\mathbf{B}_i\,\mathbf{u}_{(l)}/L_i\tag{16}$$

2) moreover, the non-zero cross-section area A_i is needed only for '*fully strained*' bar:

$$\text{if}\ \ A_i>0\ \ \text{then}\ \ \sum_l(\sigma_T\varepsilon^+_{(l),i}+\sigma_C\varepsilon^-_{(l),i})=1\tag{17}$$

The conditions (14) define the domain of feasible adjoint strain fields and can be utilized in the adaptive ground structure method discussed in the next section.

5. The Method of Adaptive Ground Structures with Selective Subsets of Active Bars

In the proposed method the solution is obtained and improved iteratively. Each iteration begins with a properly adjusted set of active bars using the information from the previous solution. Inactive bars are assumed to have zero cross-section areas and zero member forces and are eliminated from the system. Then the significantly reduced

problem (6) for m active bars ($m << M$) is solved using the primal-dual version of the interior point method. After solution the dual variables of the problem (9) are extracted without any additional cost. The optimality criteria discussed in the previous section serve as a hint for activating or deactivating new bars and also determine the stop condition (after obtaining the globally optimal solution).
The step by step procedure for the proposed method can be described as follows:

First iteration:

1. Set $iter = 1$, $d = 1$ and generate the initial ground structure $N_x \times N_y$:1×1 with bars connecting the neighbouring nodes (here only horizontal, vertical and 45° slope bars are included, see [19]). These bars will always be active (even for zero cross sectional area). They play a role of groundwork to obtain nodal displacements even in the voids.
2. Solve the problem (6) for this initial ground structure and get the dual variables $\mathbf{u}_{(l)}^{(1)}$.

Next iterations:

3. Increment the number of iteration $++iter$.
4. Increment the distance of connections $d := \max(d_{max}, d+1)$, $d_x := \max(d_{xmax}, d_x+1)$, $d_y := \max(d_{ymax}, d_y+1)$.
5. Select the new set of active bars in the ground structure $N_x \times N_y : d_x \times d_y$:
   - for every new bar compute normalized strain using the displacement fields from the previous iteration:
     $\mathbf{u}_{(l)}^{(iter-1)} \Rightarrow \bar{\bar{\varepsilon}}_i = \sum_l (\sigma_T \varepsilon_{(l),i}^{+} + \sigma_C \varepsilon_{(l),i}^{-})$ (see Eqs (10), (12), (13)),
   - if $\bar{\bar{\varepsilon}}_i \geq 1 - tol$, then activate (add) i-th bar,
   - otherwise, if $\bar{\bar{\varepsilon}}_i < 0.3$ and $d < d_{max}$ then deactivate (remove) bar,
   - if $d < d_{max}$ and the number of added bars is too small then go to step 4.
6. Check the stopping criterion:
   - if $d = d_{max}$ and there are no new bars added then finish (we approach the optimum solution because for all potential bars $i = 1{:}M$ the constraints (14) are satisfied and the solution can not be further improved)
7. Solve primal problem (6) and get dual variables $\mathbf{u}_{(l)}^{(iter)}$.
8. Repeat from step 3.

The program implementing the above algorithm has been written in Mathematica [24].

6. Examples

Let us consider a two-load case problem of two symmetrically distributed but independent loads $\mathbf{P}_{(1)}$ and $\mathbf{P}_{(2)}$, see Fig. 1. In this example we assume that: a) $\sigma_T = \sigma_C = \sigma_0$; b) the magnitudes of the loads are equal: $|\mathbf{P}_{(1)}| = |\mathbf{P}_{(2)}| = P$; c) the distances of applied loads measured from the supports are defined by $d/l = \sqrt{2}/(1+\sqrt{2}) \approx 0.585786$. The optimal volumes given in subsequent figures are scaled by a reference volume defined as $V_0 = Pl/\sigma_0$.
In Figs 2a and b the solutions of related one-load case problems are presented just to show that they do not have much in common with the two-load case solution of Fig. 3a. They are rather special cases of the three forces problem, discussed in Sokół and Lewiński [21].

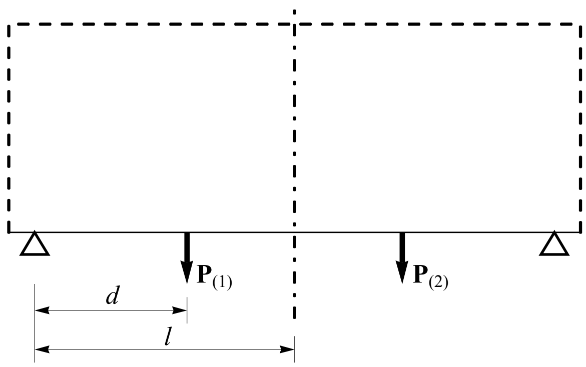

Figure 1: The two-load case problem

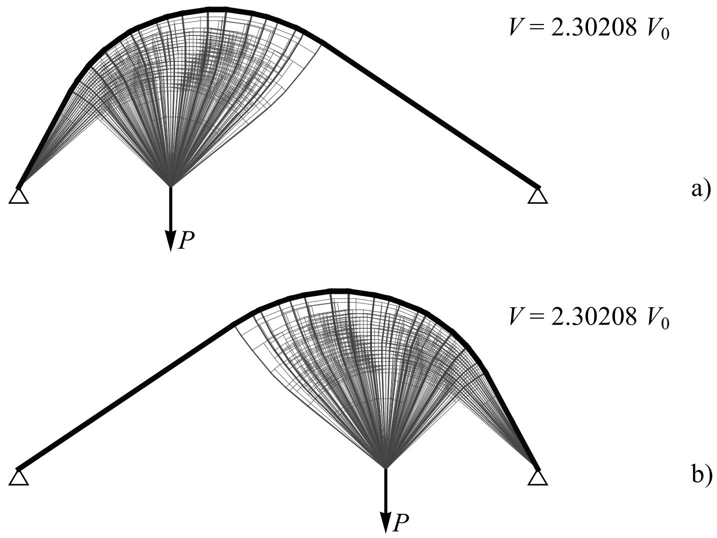

Figure 2: Numerical solutions of one-load case problems relevant to the problem of Fig. 1

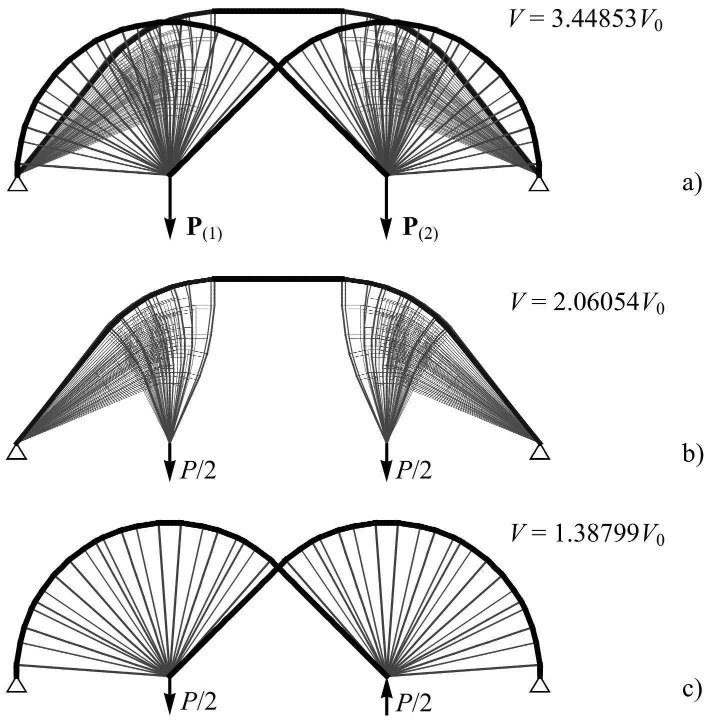

Figure 3: Numerical solutions of: a) two-load case problem, b) symmetric and c) asymmetric component loads

Numerical solution of Fig. 3a suggests that the optimal truss is composed of two almost independent trusses (the common nodes are located only at the supports and at the points of application of forces). This rather surprising solution can be better understood if we refer to the concept of *component loads*, see Rozvany and Hill [14] and Rozvany et al. [15,16]. It was checked numerically that the layout and volume shown in Fig. 3a can be obtained by superposition of solutions of Figs 3b and c which were obtained using the older, one-load case version of the program developed by the first author [20].

The solutions of Figs 2 and 3 were performed using the ground structures of the same densities 140x49 with about 15 million potential bars. The one-load case solutions of Figs 2ab and 3bc were obtained after 10 iterations with the execution time less than two minutes. The two-load case solution of Fig. 3a was obtained after 12 iterations and took about 25 minutes. This significantly longer time of computation results from the bigger size of the two-load case problem which required about 75 million design variables (it is 2.5 times greater than in the one-load case problems). Nevertheless, it was also noted that the convergence of the interior point method implemented in

Mathematica is slower for two-load case. This topic is currently investigated and will be reported during the conference.
The numerical solutions of Fig. 3 can be refined to obtain the exact analytical solutions by superposition (Rozvany and Hill [14]), see Fig. 4. The solution of Fig. 4a can be derived after Sokół and Lewiński [21]. Using the notation from this paper and after solving the transcendental system (5.10 in [21]) one can obtain the three angles defining the topology:

$$\gamma_2 = 24.1351°,\ \theta_2 = 27.5555° \text{ and } \theta_1 = \gamma_2 + \theta_2 = 51.6906° \tag{18}$$

The height of this structure is equal to h = 0.630069l and the total volume is equal to

$$V_s = 2.05757V_0 \tag{19}$$

The solution of Fig. 4b is easy to obtain due to circular fans of angles ¾π. The member forces in the external chords are equal to vertical reactions (horizontal reactions vanish due to antisymmetric load). The magnitude of the reactions is equal to $R = P/[2(1+\sqrt{2})]$. After simple algebraic calculations we obtain

$$V_a = (3/\sqrt{2} - 2)(2 + 3\pi)V_0 \approx 1.38606V_0 \tag{20}$$

Finally, the exact solution presented in Fig. 4c is just the superposition of Figs 4a and b with the total volume equal to $V = V_s + V_a \approx 3.44363V_0$. The numerical volume of Fig. 3a is only 0.14% worse. Obviously these volumes (Figs 3a and 4c) are less than the sum of the volumes of structures presented in Fig. 2.

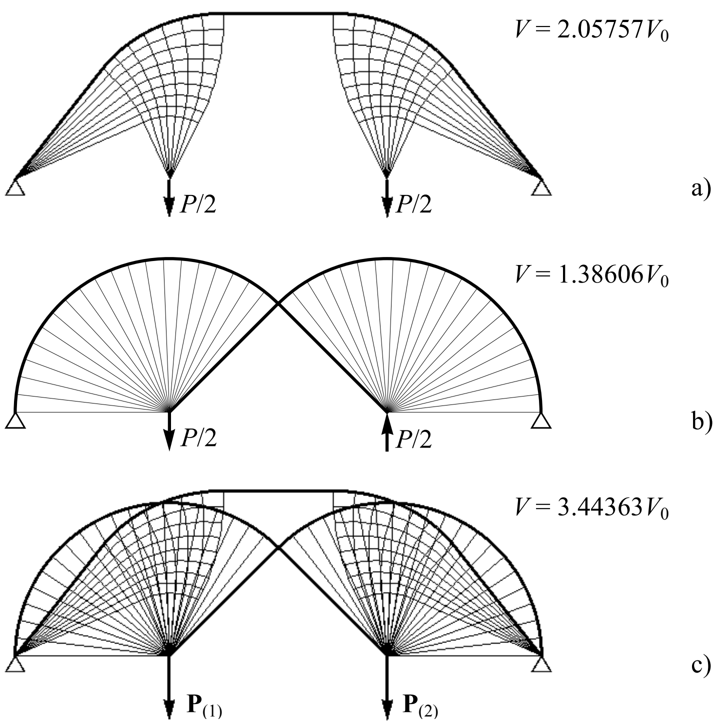

Figure 4: Exact layouts and volumes of the problems of Fig. 1

The program presented in previous section was written for a general multi-load case problem with any number of load cases. The more complex examples of 'modified bridge problems' with nine-load cases are presented in Figs 5 and 6. They differ only in supports: pin-pin (Fig. 5) and pin-roller (Fig. 6). To broaden the discussion we present also in Figs 5a and 6a the numerical solutions of one-load case problems for continuous and uniformly distributed load $q = P/l$, where l denotes the half span length. The solution in Fig. 5a has also been derived by Pichugin et al. [12]. These two solutions were obtained using the ground structures 105x100 with almost 35 million bars and using the symmetry of the problem (right part of the structure was analyzed and then symmetrically reflected to the left). It is worth nothing that the problem of Fig. 5a was also calculated for much denser ground structure 400x280 with over 3.5 billion potential bars giving the optimal volume equal to 3.15257V_0 which is similar to πV_0 (this is suspected value of the exact solution not proved yet).

The optimal multi-load trusses of Fig. 5b and 6b are quite complicated but indicate that the generalized Michel multi-load structures resemble multilayered laminates. This is very important conclusion, perhaps new in the literature.

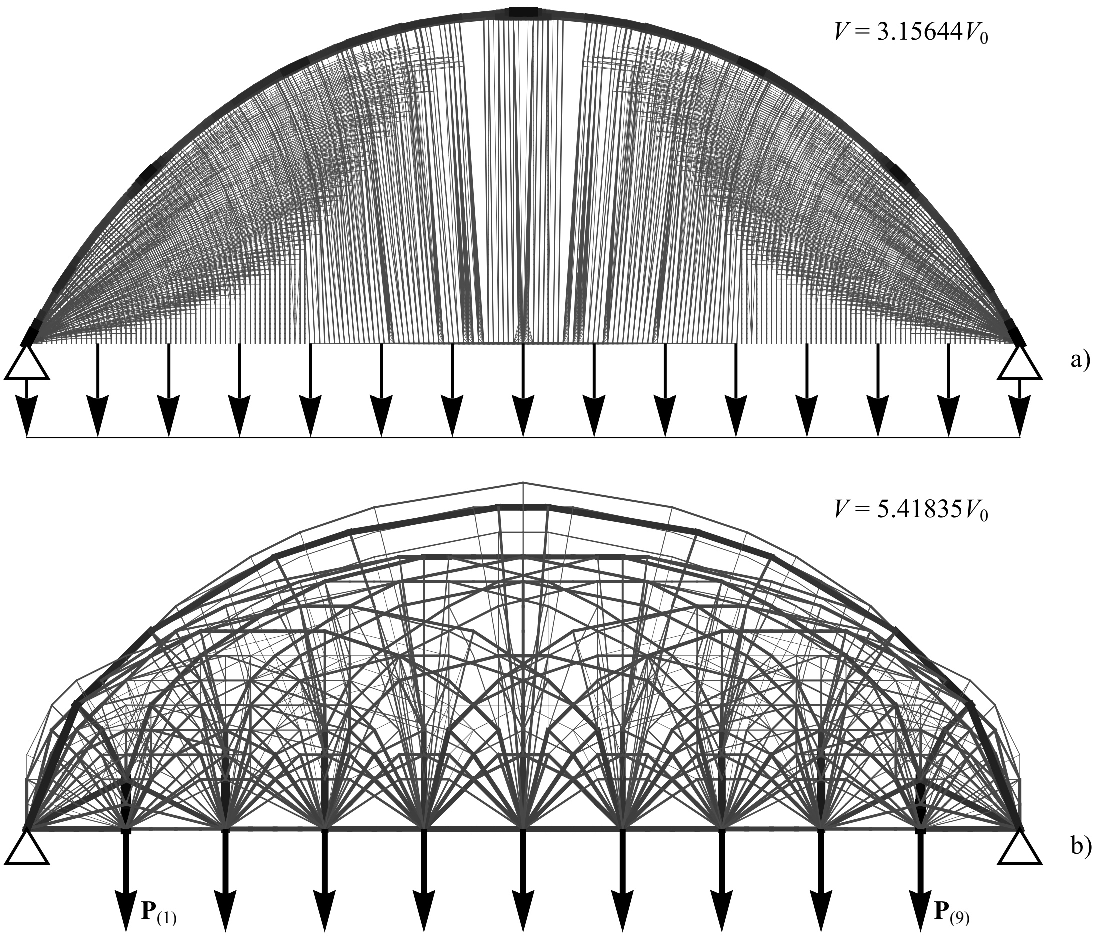

Figure 5: Numerical solutions of a 'bridge problem' for top half plane domain and two pin supports, a) one-load case with uniformly distributed load $q = P/l$, b) multi-load case: $|\mathbf{P}_{(l)}| = P$, $l = 1, 2, ..., 9$

7. Concluding Remarks

The method proposed in the present paper results in a significant reduction of the size of the problem because most of the unnecessary zero bars are eliminated a priori from the ground structure. The solution is obtained iteratively using suitably chosen small subsets of active bars, and instead of one large optimization problem a few much smaller problems are solved. Moreover, the convergence of the proposed method is very good because usually it is enough to perform 10 to 15 iterations. The program written by the first author is still being developed and improved but the preliminary results obtained in this paper indicate its high reliability. These results clearly indicate also that the optimal stress-based multi-load truss can resemble the multilayered laminate in which every layer is composed of the orthotropic material resulting from the classical one-load Michell solution obtained using the concept of component loads (Rozvany and Hill [14]). In addition, the generalized optimality criteria for multi-load trusses were derived in a concise form.

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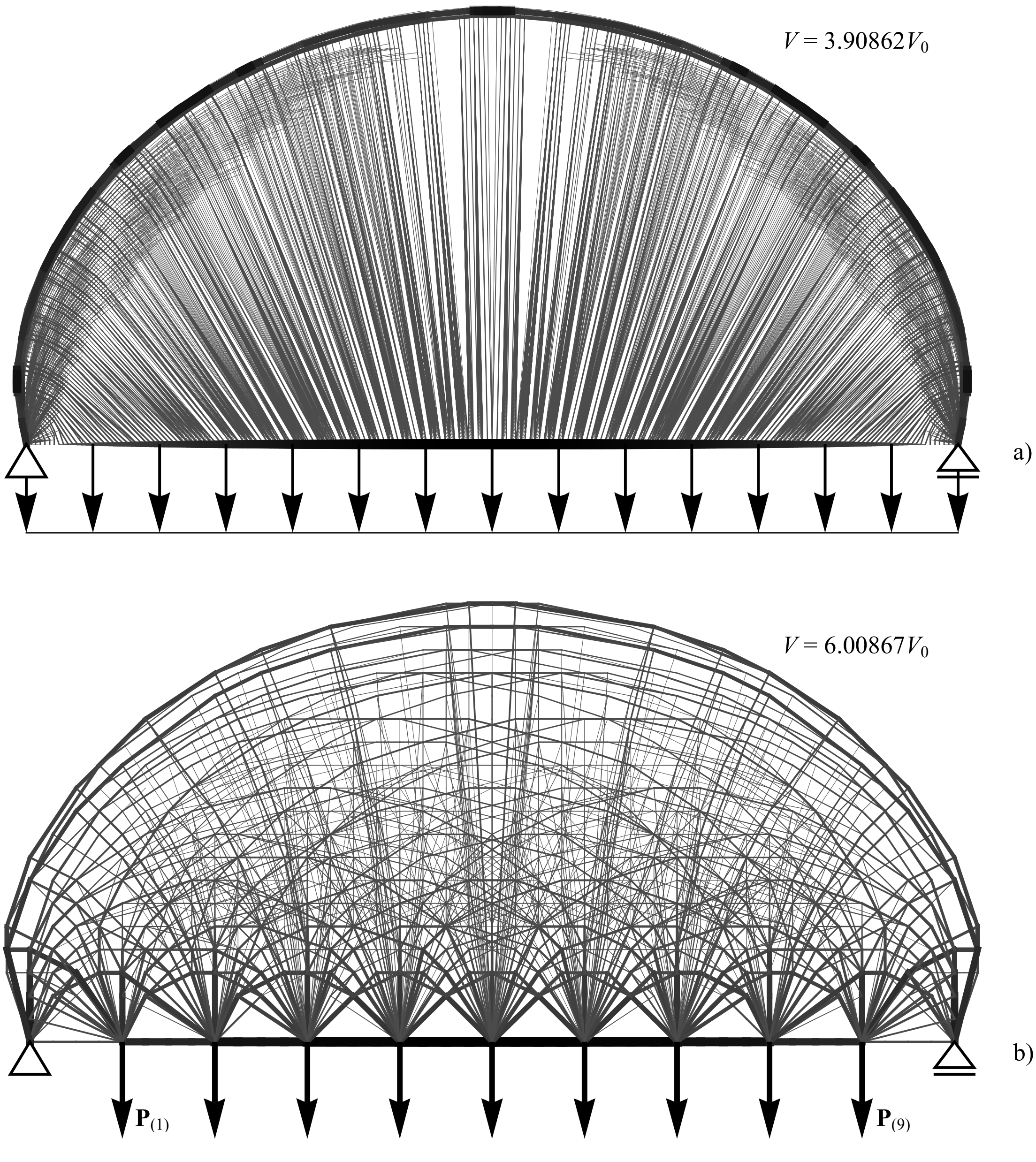

Figure 6: Numerical solutions of a bridge problem for top half plane domain and pin and roller supports, a) one-load case with uniformly distributed load $q = P/l$, b) multi-load case: $|\mathbf{P}_{(l)}| = P$, l = 1, 2, ..., 9

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