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Structural Optimization of Thin-Walled Tubular Structures using Weighted Multi-Objective Approach.

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1. Abstract
This paper focuses on improvement of the crashworthiness of tubular structures using compliant mechanism approach to intelligently tailor local mechanical properties throughout the structure. The ideal crashworthy performance for energy absorbing tubular structures can be achieved by promoting progressive buckling throughout the length of the tube to occur. Progressive buckling is typically achieved when uniform tubular structures are subjected to an axial impact however in practical applications the tubes have geometric imperfections and are subjected to oblique impacts for which Euler buckling / Global Bending deformation is observed. This design methodology relies on the ability of a compliant mechanism to transfer displacement and/or force from an input to desired output port locations. The suitable output port locations are utilized to enforce desired buckle zones, mitigating the natural Euler-type buckling effect. However compliant mechanism approach increases the compliance of the structures/decrease the stiffness of the structures. This paper uses a weighted linear combination of two objective functions of maximizing the MPE (Mutual Potential Energy) and minimizing the SE (Strain Energy). The objective is to find the distribution of the value of thickness over the entire tubular structure to achieve progressive buckling while improving the energy absorption and limiting the peak force. A nonlinear explicit finite element code LS-DYNA is used to simulate tubular structures under impact loading. Biologically inspired hybrid cellular automaton (HCA) method is used to drive the design process. This paper illustrates the use of this design methodology on representative tubular structures which are typical components of vehicle structures.
2. Keywords: Crashworthiness, structural optimization, compliant mechanism.

3. Introduction
Automotive industry have a wide application of thin-walled tubular components because of their low cost, good energy absorption capability and relatively low density. They have the ability to absorb the kinetic energy of the impacting body in the form of plastic deformations, hence protecting the structure and passengers involved. These structures can be used in various loading conditions such as axial crushing, bending, oblique impact, transverse loading, among others. Tubular structures show significant energy absorption for long strokes in an axial crushing mode and hence, present an attractive option in crashworthiness designs.
A great deal of research has been done to study the axial crushing of thin-walled tubes since the pioneering work of Pugsley [1] and Alexander [2] in the 1960s. Later work from Abramowicz and Jones on both the static and dynamic responses of tubes focused on theoretical and experimental studies [3-6]. The concept of a superfolding element was introduced by Wierzbicki and Abramowicz [7] to better understand the mechanics of the crushing of thin-walled structures. With the advancement in computing power and the numerical implementation of finite element methods in the last two decades, considerable work has been done in creating equivalent numerical models for tube crushing and their validation with experimental tests [8-11].
Various experimental and numerical studies revealed three dominant modes of deformation during the axial crushing of thin-walled structures: progressive buckling, global or Euler-type buckling, and dynamic plastic buckling [12]. Of these three, the progressive buckling mode is desired for crashworthy designs because of its efficient energy absorption and better force-displacement behavior. In progressive buckling, crushing starting at one end (often the end close to the impact) and progressing systematically toward the other end of the structure is preferred as it utilizes the maximum possible material for plastic deformation without jamming. Collapse initiators, also known as triggers, stress concentrators, or imperfections, can be used to 1) initiate a specific axial collapse mode, 2) stabilize the collapse process, and 3) reduce the peak force during the axial crush. Many researchers came up with ideas for introducing buckle initiators or surface patterns to enhance energy absorption and buckling behavior [13]. Crush zones in the structure have been used to enforce progressive buckling [14]. Chamfering and other triggering mechanisms have been investigated on quasi-static axial crush of square aluminum tubes [15]. The effect of triggering dents on the energy absorption capacity of axially compressed aluminum tubes has been also investigated [16]. An overview of techniques of geometric and material modifications to improve the buckling

behavior and energy absorption characteristics of thin-walled tubes under axial crushing can be found in a review article by Yuen and Nurick [17].

During an actual crash event, the tubular structure will seldom be subjected to pure axial loads. For example, thin-walled structures are subjected to both axial forces and bending moments in an oblique crash. A phenomenon of onset of global bending or Euler-type buckling was observed in thin-walled tubes under oblique impacts if the load angle was higher than a critical value [18, 19]. This onset of global bending severely reduces the energy absorption capability of tube structures. In the methodology presented here, the compliant mechanism design approach is used for thickness-based (topometry) design of thin-walled square tubes under axial compression [20]. The ability of compliant mechanisms to transfer motion and forces from an input load location to the desired points in the structure is utilized to achieve the desired buckle zones in the axial member. By suitably defining the output port locations and desired displacement directions, progressive buckling can be initiated at the desired locations. Moreover, using this method, thin-walled structures can be designed to show progressive buckling even in cases of oblique impact at angles higher than the critical value at which bending collapse dominates the axial collapse and leads to poor energy absorption. However the compliant mechanism synthesis would reduce the overall stiffness of the structure. In Automotive application for crashworthiness we want the structure to be at least stiff enough to sustain low impact events. Also lower value of stiffness for a structure with imperfection would make the structure more likely to fail in the area f imperfection. We have therefore used a weighted multi-objective optimization problem where we maximize the MPE to have structure compliant and minimize compliance to increase stiffness. To perform thickness-based topometry design of mechanisms, the HCA-based method for compliant mechanism synthesis [21] is modified by using thickness as the design variable.

4. Compliant mechanism approach

A compliant mechanism is a structure which is a single-piece flexible structure that drives the desired motion by undergoing elastic deformation as opposed to the rigid body motions in a conventional mechanism. Using this approach we can efficiently transfer the forces and motion from an input actuation location (referred to as input port I/P) to the desired output locations (referred to as output ports O/P) in a structure. An appropriate objective function can be defined according to the desired choice of force, displacement or a combination of both. We are optimizing the thickness over the entire design space. We can formulate the optimization problem as:

$$\begin{aligned} \text{find}: \quad & t_i \\ \text{maximize}: \quad & u_{out} \\ \text{subject to}: \quad & r = 0 \\ & \sum_{i=1}^{N} t_i A_e \rho = M \end{aligned} \tag{1}$$

where u_{out} is the output displacement, r the finite element residual for the analysis problem with the applied input load f_{in}, t_i is the thickness of every element, A_e area of the element, ρ the density and M the total mass of the design space. Given the input displacement u_{in}, the objective is to maximize u_{out}. Assuming a fictitious "dummy" unit input force vector $\boldsymbol{F}_d$ that acts only O/P in the direction of the output displacement, d_{out} can be expressed as the scalar product. $\boldsymbol{U}_1$ is the displacement vector corresponding to the input loading condition.

$$u_{out} = [0 \cdots 0 \quad 1 \quad 0 \cdots 0] \begin{bmatrix} \vdots \\ d_{out} \\ \vdots \end{bmatrix} = \boldsymbol{F}_d^T \boldsymbol{U}_1 \tag{2}$$

The product $\boldsymbol{F}_d^T \boldsymbol{U}_1$ is termed mutual potential energy MPE or the complementary virtual work. For a discrete finite element model, MPE can be expressed as the sum of element quantities.

$$MPE = \boldsymbol{F}_d^T \boldsymbol{U}_1 = \sum_{i=1}^{n} \boldsymbol{\sigma}_{d_i}^{\ T} \boldsymbol{\epsilon}_{1_i} \tag{3}$$

where $\boldsymbol{\sigma}_{d_i}$ and $\boldsymbol{\epsilon}_{1_i}$ are the element stress and strain vector corresponding to the dummy and the input loads, respectively.

The overall optimization problem then can be stated as:

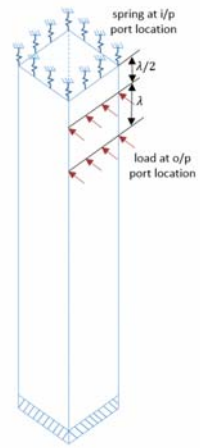
$$\begin{aligned} \text{find}: \quad & t_i \\ \text{maximize}: \quad & MPE = \sum_{i=1}^{n} \boldsymbol{\sigma}_{d_i}^{T} \boldsymbol{\epsilon}_{1_i} \\ \text{subject to}: \quad & r = 0 \\ & \sum_{i=1}^{N} t_i A_e \rho = M \\ & t_{min} \leq t_i \leq t_{max} \end{aligned} \tag{4}$$

The compliant mechanism optimization problem was extended to the tubular structure. The 2 load cases for this approach are shown as follows:

Load case 1: Input load model

(a)

Load case 2: Dummy load model



(b)

Figure 1: Load case's for tubular compliant mechanism.

The Figure 1 shows the load cases used for compliant mechanism. Figure 1(a) is the input load model with springs at the output port location. Figure 1(b) is the dummy load having force at the o/p port locations. The output port locations can be naturally assigned by the wavelength λ of the progressive buckling corresponding to an ideal axial crushing condition.

5. Model

5.1. Tube with geometric imperfection

We have used a tubular structure with geometric imperfection as our example. We have designed a square tube of mesh size 40x100 on each side of the tube. The square tube is 1m long, 0.5m in width and the thickness of the tube is 4mm. The geometric imperfection starts from 0.125m from bottom and has a length of 0.125m. The geometry is shown in the Figure 2: The geometry of the tubular structure with imperfection. A rigid solid plate is used to crush the tube which is inclined at the θ_{cr} angle which is obtained from the benchmark procedure. The rigid plate is given a velocity of 5m/s for a prescribed displacement.

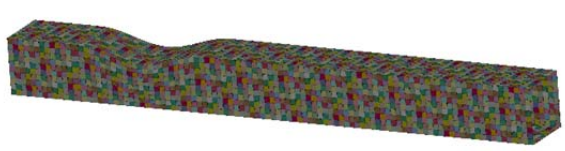


Figure 2: The geometry of the tubular structure with imperfection.

5.2. Numerical model

Explicit nonlinear finite element code LS-DYNA is used to perform dynamic simulations of axial crushing of square tubes. A linear elastic, piecewise linear plastic material (*MAT24) is used for modeling steel tubes with properties shown in Table 1. Generally, thin-walled structures under axial compression are modeled with plane stress shell elements. In several investigations reasonable agreement has been found between shell-based finite element simulations and experimental results [10, 48]. A very efficient plane stress shell element formulation from LS-DYNA (ELFORM=16) that is not subject to hourglassing (spurious strain energy modes) due to 4 in-plane integration points is used to model tubes in this work. Five integration points are used throughout the thickness in order to accurately capture the local element bending. This element is a fully integrated shell with assumed strain interpolants used to alleviate locking and enhance in-plane bending behavior [49].

Table 1: Material properties of steel used for tube models

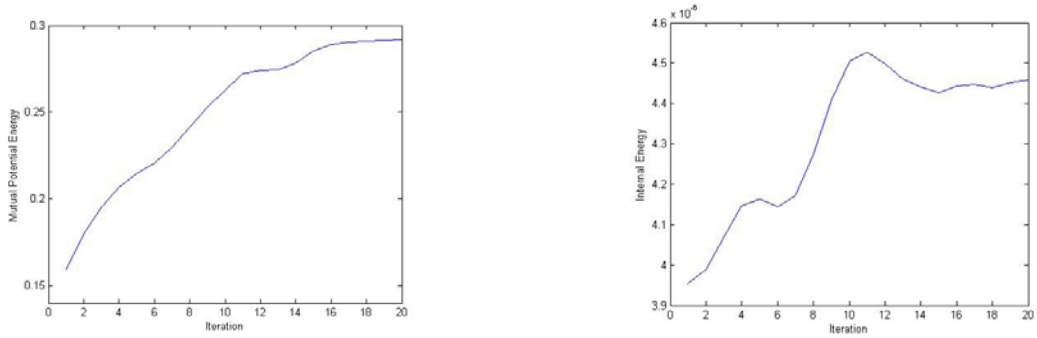
Property	Value
Density	7800 kg/m^3
Elastic Modulus	207 GPa
Poisson's Ratio	0.29
Yield Stress	253 MPa

Effective plastic strain	Effective stress (MPa)
0.000	253
0.048	367
0.108	420
0.148	442
0.208	468
0.407	524
0.607	561
0.987	608

The tube is subjected to an impact by a rigid plate with a velocity of 5 m/s. The contact between the rigid plate and tube is modeled using a constraint algorithm with a friction coefficient of 0.3 to allow sliding movement. To account for the contact between the lobes (folds) during deformation, a single surface contact algorithm with a coefficient of friction 0.1 is used.

6. Example of a compliant mechanism approach

The example bellow is a tube with geometric imperfection described above.



(a) (b)

Figure 3 : Change in MPE and SE as the optimization progresses.

We can clearly see that we have achieved our objective by maximizing the mutual potential energy. However we have also increased the internal energy of the system. This increase in the internal energy would increase the

compliance of the structure or make it more flexible. There is a need that the structure is adequately stiff to resist lower impacts.
The optimization problem described above maximizes the M.P.E. however while doing so it also maximizes the compliance of the structure, i.e. it reduces the stiffness.

7. Minimum compliance problem

For a given vector of load f and output displacement vector u obtained from the residual i.e. the structural equilibrium condition. The compliance is defined as the scalar product of the structure's finite-element force and displacement vectors, $\boldsymbol{F}$ and $\boldsymbol{U}$.
The optimization problem can be formulated as:

$$\begin{aligned} \text{find :} \quad & t_i \\ \text{minimize :} \quad & f^T u \\ \text{subject to :} \quad & r = 0 \\ & \sum_{i=1}^{N} t_i A_e \rho = M \end{aligned} \tag{5}$$

Compliance of a structure can also be represented as:

$$f^T u = \sum_{i=1}^{N} \frac{\sigma_i \boldsymbol{\epsilon_i}}{\mathbf{2}} = \sum_{i=1}^{N} \frac{(\boldsymbol{SE})_{\boldsymbol{i}}}{\mathbf{2}} \tag{6}$$

Where σ_i and ϵ_i are the elemental stress and strain values. $(SE)_i$ is the strain energy corresponding to every element. Using the above equation is used in the optimization problem definition in equation.

$$\begin{aligned} \text{find :} \quad & t_i \\ \text{minimize :} \quad & \frac{\sigma_i \boldsymbol{\epsilon_i}}{\mathbf{2}} \\ \text{subject to :} \quad & r = 0 \\ & \sum_{i=1}^{N} t_i A_e \rho = M \end{aligned} \tag{7}$$

8. Weighted multi-objective approach

Tubular structures used in automotive industry for crashworthiness needs to be compliant so as to have progressive buckling but while doing so the stiffness of the structure reduces. Such structures also need to be stiff enough to withstand low impact and enforce the weaker areas due to geometric imperfections. We are using a weighted approach between minimizing compliance and maximizing the mutual potential energy. The weight w is assigned to the objective function which maximizes the MPE. $(1-w)$ is the weight assigned to the minimum compliance objective function. The combined problem statement is defined as:

$$\begin{aligned} \text{find :} \quad & t_i \\ \text{minimize :} \quad & -w\left(\sum_{i=1}^{n} \boldsymbol{\sigma}_{d_i}^{T} \boldsymbol{\epsilon}_{1_i}\right) + (1-w)\left(\sum_{i=1}^{n} \frac{\sigma_i \boldsymbol{\epsilon_i}}{\mathbf{2}}\right) \\ \text{subject to :} \quad & r = 0 \\ & \sum_{i=1}^{N} t_i A_e \rho = M \end{aligned} \tag{8}$$

8.1. KKT conditions on weighted multi-objective approach

A compliant mechanism is required to meet both flexibility and stiffness requirements. The Lagragian from the optimization problem defined before.

$$\mathcal{L} = -w\left(\sum_{i=1}^{n} \boldsymbol{\sigma}_{d_i}^{T}\, \boldsymbol{\epsilon}_{1_i}\right) + (1-w)\left(\sum_{i=1}^{n} \frac{\sigma_i\, \boldsymbol{\epsilon_i}}{\mathbf{2}}\right) + \sum_{i=1}^{n} \lambda_{1i}(t_i - ub + s_{1i}^2) + \sum_{i=1}^{n} \lambda_{2i}(lb - t_i + s_{0i}^2)$$
$$+ \sum_{i=1}^{n} \lambda_v(t_i A_e \rho - M + s_v^2) \tag{9}$$

where λ_{1i}, λ_{2i} and λ_v are the Lagrange multipliers while s_{1i}, s_{0i} and s_v are the slack variables. The necessary conditions then can be obtained as:

$$\begin{aligned} \frac{d\mathcal{L}}{dt_i} &= 0 \quad \text{for} \quad i = 1,2,\dots N \\ \frac{d\mathcal{L}}{d\lambda_{1i}} &= 0 \quad \text{for} \quad i = 1,2,\dots N \\ \frac{d\mathcal{L}}{d\lambda_{0i}} &= 0 \quad \text{for} \quad i = 1,2,\dots N \\ \frac{d\mathcal{L}}{d\lambda_v} &= 0 \end{aligned} \tag{10}$$

Differentiating with respect to slack variables will give switching conditions (which tells if inequality constraints are active or not) as

$$\frac{d\mathcal{L}}{ds_{1i}} = 2\lambda_{1i} s_{1i} = 0$$
$$\frac{d\mathcal{L}}{ds_{0i}} = 2\lambda_{0i} s_{0i} = 0$$
$$\frac{d\mathcal{L}}{ds_v} = 2\lambda_v s_v = 0$$

Solving the above equations we get:

$$-\frac{w(\sum_{i=1}^{n}(MPE)_i)}{t_i A_e \rho} + \frac{(1-w)(\sum_{i=1}^{n}(SE)_i)}{t_i A_e \rho} = -\frac{w(MPE)_g + (1-w)(SE)_g}{M} \tag{11}$$

where $(MPE)_i = \boldsymbol{\sigma}_{d_i}^{T} \boldsymbol{\epsilon}_{1_i}$, $(SE)_i = \frac{\sigma_i\, \boldsymbol{\epsilon_i}}{\mathbf{2}}$, $(MPE)_g$ and $(SE)_g$ are the global mutual potential energy ad strain energy. The field variable and the target values are defined from the above equation as:

$$Field\ variable = S_i = -\frac{w(\sum_{i=1}^{n}(MPE)_i)}{t_i A_e \rho} + \frac{(1-w)(\sum_{i=1}^{n}(SE)_i)}{t_i A_e \rho} \tag{12}$$

$$Target\ value = S^* = -\frac{w(MPE)_g + (1-w)(SE)_g}{M} \tag{13}$$

9. Hybrid Cellular Automaton

The hybrid cellular automaton (HCA) method is a computational technique that can be used to synthesize optimal topologies. While the HCA method is not explicitly an optimization technique, the local rules produce this behavior. This approach is inspired by the biological process of bone remodeling as presented by Tovar.14 that is, material is appropriated to uniformly distribute the effects of a specified mechanical stimulus, or field variable, throughout the design domain.

The hybrid cellular automaton (HCA) method combines the CA paradigm with finite element analysis (FEA) for structural optimization. As done in traditional topology optimization methods, the structural design domain is discretized into material elements. To use the finite element method for structural analysis, the design domain is represented using a finite element model that is discretized using continuum finite elements (FE). The states of the material elements in the design domain are represented using a lattice of CAs, where a one-to-one correspondence

between CAs and FEs generally exist, although this is not a requirement. However, uniformity in the CA discretization is required. A set of local rules is used to determine material distribution. These rules are applied to the local information collected in the neighborhood of each CA. At a discrete position i and time/iteration k, a CA is defined by a set of states that are operated on by a set of rules belonging to a given neighborhood of the CA.
The state of each CA, β i is defined by design variables xi (e.g., density, thickness) and field variables Si (e.g., stress, strain, strain energy density, mutual potential energy or functions of these quantities). The field variables are computed by a finite element analysis; hence, this is a hybrid approach since each cellular automaton is provided global information. The complete state of each cell at a time/iteration k is expressed by:

$$\beta_i^k = \begin{pmatrix} S_i^k \\ t_i^k \end{pmatrix}$$

To obtain an optimal design, the material is redistributed throughout the design domain in order to get uniform distribution of field variables. Accordingly, a setpoint or target is uniformly applied to all CAs in the design domain. HCA attempts to drive the state of each CA to this value. For linear static case, the
KKT conditions as derived in the appendix can be restated as uniform distribution of field variables rule as:

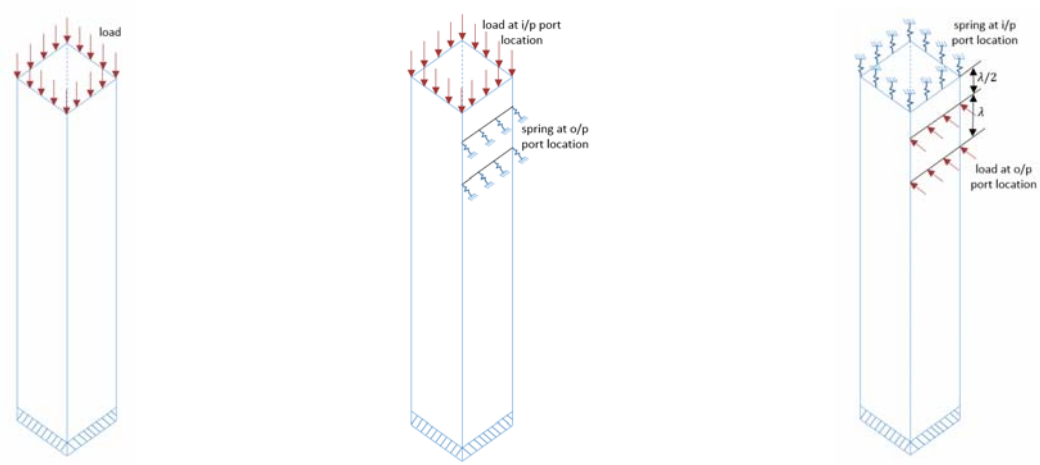
$$-\frac{w(\sum_{i=1}^{n}(MPE)_i)}{t_i A_e \rho} + \frac{(1-w)(\sum_{i=1}^{n}(SE)_i)}{t_i A_e \rho} = -\frac{w(MPE)_g + (1-w)(SE)_g}{M} \tag{14}$$

The linear static cases, gradient information can be stated as uniform field variable distribution rule in HCA. The idea of uniform distribution of field variables to get efficient designs (not necessarily optimal) comes from fully stressed design (FSD) method where the material in redistributed over the design such that all the elements contribute to the common goal efficiently. In this work, the same idea with some changes is extended to treat the non- linear compliant problems. Although, the final results cannot be claimed to be optimal for non-linear cases, however they present themselves as good candidate designs.

10. Implementation

There were 3 load cases o be used after the addition of the minimum compliance problem to the compliant mechanism approach. The following are the load cases:

Load case 1: Normal Load | Load case 1: Input load model | Load case 2: Dummy load model



(a) (b) (c)

Figure 4: Load case's for tubular weighted multi-objective approach.

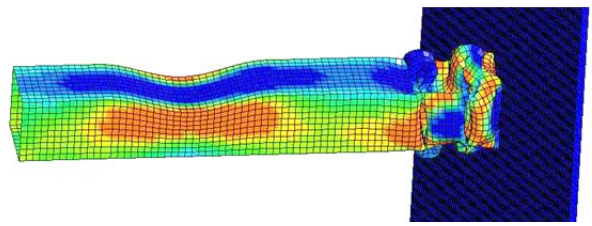
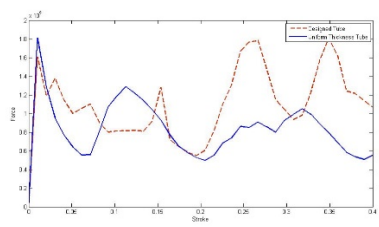
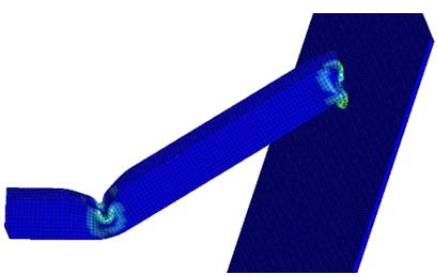
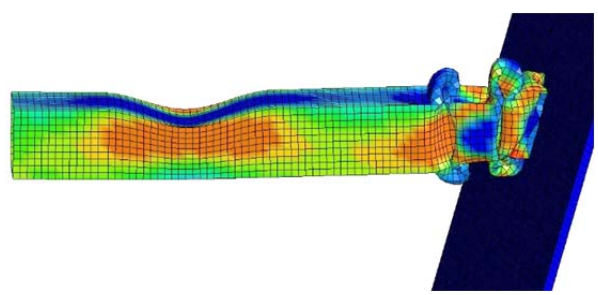
The fig shows the load cases used for weighted multi-objective approach. Figure 14 (a) is the load case used to calculate the Strain energy so as to obtain $(SE)_i$. Figure 14 (b) is the input load model with springs at the output port

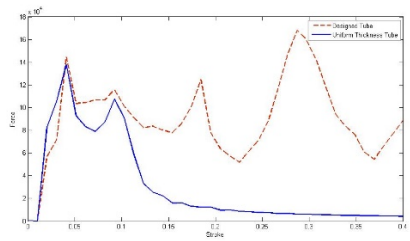
location. Figure 14(c) is the dummy load having force at the o/p port locations. 4(b) and 4(c) are the load cases used to obtain $(MPE)_i$.

10.1. Numerical evaluation:

The simulations for uniform tubular structure with geometric imperfection was performed and compared with the design obtained from this methodology. The tubular structure's was subjected to axial and oblique impact. Peak force and energy absorbed are the considered to be one of the performance parameters for the comparison.

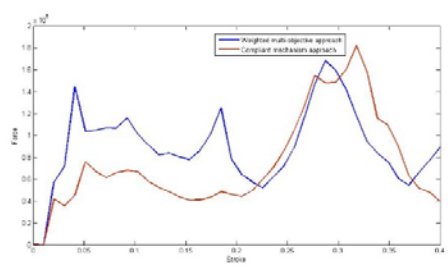
Table 2: Comparison between uniform thickness tube and designed tube.

Axial impact	
Uniform thickness tube	Designed tube
	
Force Displacement Curves	
	
Energy Absorbed = 38877.9751471072 Peak force = 1.8137e+005N	Energy Absorbed = 51360.1627432553 Peak force = 1.8079e+005
Oblique load at 15°	
	

Force Displacement Curves	
	
Energy Absorbed = 14157.8723353048 Peak force = 1.3778e+005N	Energy Absorbed = 41910.9276936808 Peak force = 1.6805e+005 N

We can clearly see from the above example that the designed tube outperforms the tube with uniform thickness in terms of deformation characteristics. We can see in case of tube with uniform thick, during both axial and oblique impact the tube starts to bend at the imperfection which is not desired. The designed tube has overall energy absorbed better as compared to the uniform thickness tube. However we have seen such results in the precious section while we used a compliant mechanism approach only. Following is a comparison of a designed tube from a compliant mechanism structure and a weighted multi-objective approach.

Table 3: Comparison between compliant mechanism approach and weighted multi-objective approach.

Compliant mechanism approach	Weighted multi-objective approach
Force Displacement Curves	
	
Energy Absorbed = 34765.3006865004 Peak force = 1.8172e+005 N	Energy Absorbed = 41910.9276936808 Peak force = 1.6805e+005 N

From the above Table 3 we can see that the overall energy absorption is increased from our design through weighted multi-objective approach. Progressive buckling and a good deformation characteristic is seen in both cases. If we have $w = 1$ in the weighted multi-objective approach the problem changes to the compliant mechanism approach. However by introducing a second objective function we have been successful in making the structure stiff and still have a progressive bucking. If we want the buckling to start at a lower impact then we can use $w = 1$ or else decrease the value of w to initiate buckling at higher impact. As we increase the weight w we are increasing the stiffness and hence bucking would start at a higher impacts. There is an upper and a lower bound for this approach. For the give mass constraint and tube geometry, at $w = 1$ gives us the minimum value for impact at which the buckling can initiate. Similarly as we decrease w after a certain value the tube will more stiff than compliant and would not show progressive buckling.

11. Conclusions

A design method for designing thin-walled tubular structures under axial and oblique impacts based on the multi-objective optimization so as to maximize the MPE to have structure compliant and minimize compliance to increase stiffness has been presented. Using this novel method, square cross-sectioned tubular structures can be designed to

exhibit buckling starting from the loading (impact) end and systematically progressing toward the rear end even in the presence of significant but reasonable geometric imperfections and asymmetries in the loading conditions. Using this methodology we can have a control over the impact intensity needed to initialize the buckling. In other words we have the flexibility to design structures for low impact and high impact events having the same mass.
The results of this research depict concept designs for tubular structures with variable local thickness throughout; however, the manufacturing challenges of such concept designs need to be addressed. One possible approach to address this is additive manufacturing methods. Alternatively, this proposed design methodology may be utilized in tubular structure design with variable local mechanical properties of hardness and strength. Such variable mechanical properties may be achieved by localized thermal treatments for conventionally manufactured metallic structures. This design approach is a topic of future research.

12. Acknowledgements

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