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Phase Field Approach to Topology Optimization of Contact Problems

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1. Abstract

The paper deals with a phase field model for formulation and solution of the topology optimization problems of bodies in unilateral contact consisting in the normal contact stress minimization. The cost functional includes also surface and bulk energy terms. These terms allow to control global perimeter constraint and the occurence of the intermediate solution values. The optimal topology is obtained as the steady state of the phase transition governed by the Cahn-Hilliard equation. The optimization problem is solved using the finite element method. Numerical examples are provided.

2. Keywords: topology optimization, contact problems, phase field regularization.

3. Introduction

The paper deals with the topology optimization for an elastic body in unilateral contact with a rigid foundation. This structural optimization problem consists in finding such material distribution in a given design domain occupied by the body and/or the shape of its boundary that the normal contact stress along the boundary of the body is minimized.

Many successful numerical methods have been proposed to solve topology optimization problems, including Simple Isotropic Material Penalization metod (SIMP) and Evolutionary Structural Optimization (ESO) approach [6] or topology derivative method [15]. Recently the use of the level set methods has been proposed [1, 7, 11, 18] to solve the topology optimization problems. In structural optimization the level set method [1, 11] is employed in numerical algorithms for tracking the evolution of the domain boundary on a fixed mesh and finding an optimal domain. This method is based on an implicit representation of the boundaries of the optimized structure, i.e., the position of the boundary of the body is described as an isocountour of a scalar function of a higher dimensionality. In standard level set approach the evolution of the domain boundary is governed by Hamilton - Jacobi equation. The speed vector field driving the propagation of the level set function is given by the Eulerian derivative of the cost functional with respect to the variations of the free boundary. In binary or piecewise constant level set approach [13], based on two phase formulation, the evolution of the domain boundary is governed by the flow equation. The topology optimization problem in multiphase setting can be transformed further into a phase field problem where the optimal topology is characterized as the steady state of the phase transition. Phase field models in the form of Cahn-Hilliard or Allen-Hillard equations have been first introduced [4, 5, 6, 8, 10, 17] in metalurgy to describe phase separation in binary alloy systems. Next these approaches have been used to provide mathematical models in different areas, including crack propagation, image processing, tumor growth. Phase field models have many similarities with the level set approach. While level-set methods have become an accepted tool in structural topology optimization the use of phase field methods in this field has not yet become popular.

In the paper phase field approach is proposed to regularize two phase topology optimization problem for unilateral elastic contact system and to solve it. Material density function is a variable subject to optimization. This approach consists in using Ginzburg Landau free energy term [8, 14, 17, 18] as the regularization term rather than the perimeter constraint term [11, 15]. Although the proposed regularization for topology optimization of contact problems is more complicated than the perimeter one it has advantages comparing to the standard one [4, 15]. The derivative formula of the cost functional with respect to the material density function is calculated and is employed to formulate a necessary optimality condition for the topology optimization problem. This necessary optimality condition takes the form of the generalized Cahn-Hilliard equation where the potential function depends on the derivative of the cost functional. Moreover the cost functional derivative is employed to calculate a descent direction in the numerical algorithm. Finite element method is used as the approximation method. Numerical examples

are provided and discussed.

4. Problem Formulation

Consider deformations of an elastic body occupying two – dimensional domain Ω with the smooth boundary Γ (see Fig. 1). Assume $\Omega \subset D$ where D is a bounded smooth hold – all subset of R^2. The body is subject to body forces $f(x) = (f_1(x),\, f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. We assume, that the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset,\ i \neq j,\ i, j = 0, 1, 2,\ \Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

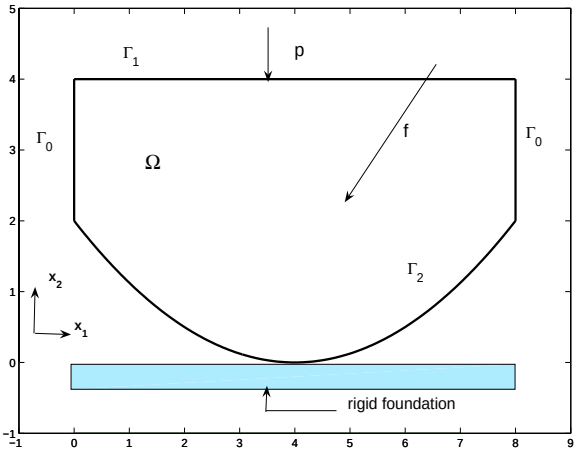

Figure 1: Initial domain Ω.

Let $\rho = \rho(x) : \Omega \to R$ denote the material density function at any generic point x in a design domain Ω. It is a phase field variable taking value close to 1 in the presence of material, while $\rho = 0$ corresponds to regions of Ω where the material is absent, i.e. there is a void. In the phase field approach the interface between material and void is described by a diffusive interfacial layer of a thickness proportional to a small lenght scale parameter $\epsilon > 0$ and at the interface the phase field ρ rapidly but smoothly changes its value [6]. We require that $0 \leq \rho \leq 1$. The ρ values outside this range do not seem to correspond to admissible material distributions. The elastic tensor $\mathcal{A}$ of the material body is assumed to be a function depending on density function ρ:

$$\mathcal{A} = g(\rho)\mathcal{A}_0, \quad \mathcal{A}_0 = \{a_{ijkl}\}_{i,j,k,l=1}^{2} \tag{1}$$

and $g(\rho)$ is a suitable chosen function [2, 4, 6, 15].

We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$

$$\sigma_{ij}(u(x)) = g(\rho)a_{ijkl}(x)e_{kl}(u(x)). \tag{2}$$

We use here and throughout the paper the summation convention over repeated indices [9]. The strain $e_{kl}(u(x))$, $k, l = 1, 2$, is defined by:

$$e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \tag{3}$$

where $u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}$. The stress field σ satisfies the system of equations in the domain Ω [9]

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad x \in \Omega, i, j = 1, 2, \tag{4}$$

where $\sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}$, $i, j = 1, 2$. The following boundary conditions are imposed on the boundary $\partial\Omega$

$$u_i(x) = 0 \quad \text{on} \quad \Gamma_0, \quad i = 1, 2, \tag{5}$$

$$\sigma_{ij}(x)n_j = p_i \quad \text{on} \quad \Gamma_1, \quad i, j = 1, 2, \tag{6}$$

$$u_N \leq 0, \quad \sigma_N \leq 0, \quad u_N\sigma_N = 0 \quad \text{on } \Gamma_2, \tag{7}$$

$$\mid \sigma_T \mid \leq 1, \quad u_T\sigma_T + \mid u_T \mid = 0 \quad \text{on } \Gamma_2, \tag{8}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i, j = 1, 2$, represent the normal components of displacement u and stress σ, respectively. The tangential components of displacement u and stress σ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. $\mid u_T \mid$ denotes the Euclidean norm in R^2 of the tangent vector u_T. The results concerning the existence of solutions to (4) - (8) can be found in [9].

4.1 Variational Formulation of Contact Problem

Let us formulate contact problem (4)-(8) in the variational form. Denote by V_{sp} and K the space and the set of kinematically admissible displacements:

$$V_{sp} = \{z \in [H^1(\Omega)]^2 \ : \ z_i = 0 \text{ on } \Gamma_0, \ i = 1, 2\}, \quad K = \{z \in V_{sp} \ : \ z_N \leq 0 \ \text{ on } \ \Gamma_2\}. \tag{9}$$

$H^1(\Omega)$ denotes Sobolev space of square integrable functions and their first derivatives [9, 16]. $[H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega)$. Denote also by Λ the set

$$\Lambda = \{\zeta \in L^2(\Gamma_2) \ : \mid \zeta \mid \leq 1\}.$$

Variational formulation of problem (4)-(8) has the form: *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying*

$$\int_\Omega g(\rho) a_{ijkl} e_{ij}(u) e_{kl}(\varphi - u) dx - \int_\Omega f_i(\varphi_i - u_i) dx - \int_{\Gamma_1} p_i(\varphi_i - u_i) ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T) ds \geq 0 \ \ \forall \varphi \in K, \tag{10}$$

$$\int_{\Gamma_2} (\zeta - \lambda) u_T ds \leq 0 \quad \forall \zeta \in \Lambda, \tag{11}$$

$i, j, k, l = 1, 2$. Function λ is interpreted as a Lagrange multiplier corresponding to term $\mid u_T \mid$ in equality constraint in (8) [9]. This function is equal to tangent stress along the boundary Γ_2, i.e., $\lambda = \sigma_{T_{\mid \Gamma_2}}$. Function λ belongs to the space $H^{-1/2}(\Gamma_2)$, i.e., the space of traces on the boundary Γ_2 of functions from the space $H^1(\Omega)$. Here following [9] function λ is assumed to be more regular, i.e., $\lambda \in L^2(\Gamma_2)$. The results concerning the existence of solutions to system (10)-(11) can be found, among others, in [9, 16].

4.2 Structural Optimization Problem

Before formulating a structural optimization problem for (10)-(11) let us introduce the set U_{ad} of admissible domains. Denote by $Vol(\Omega)$ the volume of the domain Ω equal to

$$Vol(\Omega) = \int_\Omega \rho(x) dx. \tag{12}$$

Domain Ω is assumed to satisfy the volume constraint of the form

$$Vol(\Omega) - Vol^{giv} \leq 0, \tag{13}$$

where the constant $Vol^{giv} = const_0 > 0$ is given. In a case of shape optimization of problem (10) - (11) the optimized domain Ω is assumed to satisfy equality volume condition, i.e., (13) is assumed to be satisfied as equality. In a case of topology optimization Vol^{giv} is assumed to be the initial domain volume and (13) is satisfied in the form $Vol(\Omega) = r_{fr} Vol^{giv}$ with $r_{fr} \in (0, 1)$ [15]. The set U_{ad} has the following form

$$U_{ad} = \{\Omega \ : \ E \subset \Omega \subset D \subset R^2 \ : \ \Omega \text{ is Lipschitz continuous, } \Omega \text{ satisfies condition (13)}\}, \tag{14}$$

where $E \subset R^2$ is a given domain such that Ω as well as all perturbations of it satisfy $E \subset \Omega$. The constant $const_1 > 0$ is assumed to exist. The set U_{ad} is assumed to be nonempty. In order to define a cost functional we shall also need the following set M^{st} of auxiliary functions

$$M^{st} = \{\eta = (\eta_1, \eta_2) \in [H^1(D)]^2 : \eta_i \leq 0 \ \text{ on } D, \ i = 1, 2, \quad \| \eta \|_{[H^1(D)]^2} \leq \ 1\}, \tag{15}$$

where the norm $\| \eta \|_{[H^1(D)]^2} = (\sum_{i=1}^{2} \| \eta_i \|^2_{H^1(D)})^{1/2}$. Recall from [11, 12] the cost functional approximating the normal contact stress on the contact boundary

$$J_\eta(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\eta_N(x)ds, \tag{16}$$

depending on the auxiliary given bounded function $\eta(x) \in M^{st}$. σ_N and η_N are the normal components of the stress field σ corresponding to a solution u satisfying system (10) - (11) and the function η, respectively.

Consider the following structural optimization problem: *for a given function $\eta \in M^{st}$, find a domain $\Omega^\star \in U_{ad}$ such that*

$$J_\eta(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\eta(u(\Omega)). \tag{17}$$

Adding to (14) a perimeter constraint $P_D(\Omega) \leq const_1$, where $P_D(\Omega) = \int_\Gamma dx$ is a perimeter of a domain Ω in D [5, 11, 16] and $const_1 > 0$ is a given constant the existence of an optimal domain $\Omega^\star \in U_{ad}$ to the problem (17) is ensured (see [4, 5, 16]).

5. Phase field model

In the paper we use Ginzburg-Landau energy functional [2, 3, 4, 6, 10, 18] as a regularization term rather than standard perimeter term. Consider a two phase problem. To indicate the evolution of the material density function ρ let us assume this function depends not only on $x \in \Omega$ but also on the artificial time variable $t \in [0, T)$, $T > 0$ given, i.e. $\rho = \rho(t, x)$. Function ρ describes the concentration of one of the phases in the domain Ω. The other phase is obtained as $(1 - \rho)$. This variable is used to describe the phase transition.

Recall [2, 3, 4, 5, 6, 7, 17] the Ginzburg-Landau free energy is expressed as

$$E(\rho) = \int_\Omega \psi(\rho)d\Omega, \tag{18}$$

with the total free energy function $\psi(\rho)$ in the form

$$\psi(\rho) = \frac{\gamma\epsilon}{2} \mid \nabla\rho \mid^2 + \frac{\gamma}{\epsilon}\psi_B(\rho), \tag{19}$$

where $\epsilon > 0$ is a constant, $\gamma > 0$ is a parameter related to the interfacial energy density and $\psi_B(\rho)$ is a double-well potential which characterizes the two phases [2, 6]. Usually it is taken as an even-order polynomial of the form [10]

$$\psi_B(\rho) = \rho^2(1 - \rho^2). \tag{20}$$

The first term in (19) is called the interface energy. It represents [6] a measure of the perimeter of the interfaces between the phases and in this sense it is the relaxed version of the global perimeter constraint. The term (20) is called the bulk energy. It is a non-convex smooth function attaining minimum in the pure phases $\rho =$ and $\rho = 1$. The values assumed by $\psi_B(\rho)$ for intermediate values of ρ are larger than for pure phases and are not prefered in the optimization process. Parameter ϵ measures the width of the transition zone. The definition of the phase transition model is based on the concept of the flow of the gradient ∇E of the functional (18) with respect to ρ in the norm of a suitable chosen Hilbert space H:

$$\frac{\partial\rho}{\partial t}(t, x) = -\frac{\partial E}{\partial\rho}(\rho) \quad \text{in} \quad \Omega, \quad \forall t \in [0, T), \tag{21}$$

and the initial condition

$$\rho(0, x) = \rho_0(x) \;\; t = 0 \;\; x \in \Omega. \tag{22}$$

Denoting by $(\cdot, \cdot)_{L^2(\Omega)}$ the scalar product in the space $L^2(\Omega)$ and selecting the space H [10] as the subspace of a space $[H^1(\Omega)]'$ dual to $H^1(\Omega)$

$$H = \{\zeta \in [H^1(\Omega)]' : (\zeta, 1)_{L^2(\Omega)} = 0\},$$

the Cahn-Hilliard gradient flow is given by [2, 6, 10, 17]

$$\frac{\partial E}{\partial\rho}(\rho) = -\nabla \cdot (M(\rho)\nabla\varphi_E(\rho)), \tag{23}$$

where $M(\rho) \geq 0$ is a sufficiently regular function called mobility and $\varphi_E(\rho)$ is the potential associated to the total free energy function $\psi(\rho)$. The mobility function $M(\rho)$ usually is taken either as a constant function M_0 or a degenerate function $M_0\rho(1-\rho)$. For ρ sufficiently regular and satisfying $\nabla\rho \cdot n = 0$ on the boundary $\partial\Omega$ of domain Ω the potential function $\varphi_E(\rho)$ is given by

$$\varphi_E(\rho) = \frac{\partial\psi(\rho)}{\partial\rho} = \gamma\epsilon \triangle \rho + \frac{\gamma}{\epsilon}\frac{d\psi_B(\rho)}{d\rho}. \tag{24}$$

A standard formulation of the Cahn-Hilliard equation in operator form is:

$$\frac{\partial\rho}{\partial t} = \nabla \cdot (M(\rho) \cdot \nabla\varphi_E(\rho)) \quad \text{in } \Omega,\ \forall t \in [0,T), \tag{25}$$

$$M(\rho)\nabla\varphi_E(\rho) \cdot n = 0 \quad \text{on } \partial\Omega,\ \forall t \in [0,T), \tag{26}$$

$$\nabla\rho \cdot n = 0 \quad \text{on } \partial\Omega,\ \forall t \in [0,T), \tag{27}$$

$$\rho(0,x) = \rho_0(x) \quad \text{in } \Omega,\ t = 0. \tag{28}$$

In order to formulate weak Cahn-Hilliard equation let us introduce the function space $\mathcal{B} = \{\zeta \in H^2(\Omega) : \nabla\zeta \cdot n = 0, \text{ on } \partial\Omega\}$. The weak Cahn-Hilliard equation consists in finding such function ρ that is a C^1 continuos mapping from time interval $[0,T)$ into the space $\mathcal{B}$, i.e., $\rho \in C^1([0,T);\mathcal{B})$ satisfying

$$\int_\Omega \frac{\partial\rho}{\partial t}\zeta d\Omega + \int_\Omega M(\rho)\nabla\varphi_E(\rho) \cdot \nabla\zeta d\Omega = 0, \ \ \forall\zeta \in \mathcal{B}, \ \ \forall t \in [0,T), \tag{29}$$

and the initial condition (28). The Cahn-Hilliard equation was originally designed to study phase separation of binary fluids [6]. In topology optimization this model is interpreted in terms of the material density function representing a phase fraction. For constant mobility function M, $\psi_B(\rho)$ a polynomial of order $2p$ and $\rho_0 \in L^2(\Omega)$ the equation (29) has a unique solution [10]. Moreover this equation is mass conservative, i.e., area or volume V covered by the phases in domain Ω is constant in time. Setting $\zeta = 1$ in (29) we obtain:

$$\frac{dV}{dt} = 0 \iff V = \int_\Omega \rho_0 d\Omega, \ \ \forall t \in [0,T).$$

For a given ρ_0 and analytic bulk function the unique solution ρ to (29) converges to a steady state and its time derivative converges to 0 for $t \to +\infty$ in the topology of the corresponding function spaces. It indicates that the steady state solution is a critical one for the total free energy (18). The original Cahn-Hilliard equation has an energy term that serves as a Lyapunov function, i.e., from (29) it follows:

$$\frac{dE}{dt}(\rho) = -\int_\Omega M(\rho)\nabla\varphi_E(\rho) \cdot \nabla\varphi_E(\rho) d\Omega \leq 0, \quad \forall t \in [0,T).$$

It results that the phase transition occurs in such a way that the energy associated to the Cahn-Hilliard equation is bounded at every finite time interval $[0,T]$.

6. Phase field based topology optimization problem

Let us introduce the regularized cost functional $J(\rho,u)$ in the form:

$$J(\rho,u) = J_\eta(u) + E(\rho), \tag{30}$$

where the functionals $J_\eta(u)$ and $E(\rho)$ are given by (16) and (18)-(19), respectively. The structural optimization problem (17) takes the form: *find* $\rho^\star \in U^\rho_{ad}$ *such that*

$$J(\rho^\star,u^\star) = \min_{\rho \in U^\rho_{ad}} J(\rho,u), \tag{31}$$

where $u^\star = u(\rho^\star)$ denotes a solution to the state system (10)-(11) depending on $\rho^\star$ and $U^\rho_{ad} = \{\rho : Vol(\Omega) = Vol^{giv}\}$ denotes the set of admissible material density functions.
In order to compute the first variation of the cost functional (30) we apply a formal Lagrangian approach combine with Cahn-Hilliard approach. Let us introduce the Lagrangian $L(\rho) = L(\rho,u,\lambda,p^a,q^a)$:

$$L(\rho,u,\lambda,p^a,q^a) = J_\eta(u) + E(\rho) + \int_\Omega g(\rho)a_{ijkl}e_{ij}(u)e_{kl}(p^a)dx - \tag{32}$$
$$\int_\Omega f_i p^a_i dx - \int_{\Gamma_1} p_i p^a_i ds + \int_{\Gamma_2} \lambda p^a_T ds + \int_{\Gamma_2} q^a u_T ds,$$

where $(p^a, q^a) \in K_1 \times \Lambda_1$ denotes an adjoint state defined as follows:

$$\int_\Omega g(\rho) a_{ijkl} e_{ij}(\eta + p^a) e_{kl}(\varphi) dx + \int_{\Gamma_2} q^a \varphi_T ds = 0 \quad \forall \varphi \in K_1, \tag{33}$$

and

$$\int_{\Gamma_2} \zeta (p^a_T + \eta_T) ds = 0 \ \ \forall \zeta \in \Lambda_1. \tag{34}$$

The sets K_1 and Λ_1 are given by

$$K_1 = \{\xi \in V_{sp} \ : \ \xi_N = 0 \ \text{ on } A^{st} \ \}, \tag{35}$$

$$\Lambda_1 = \{\zeta \in \Lambda \ : \ \zeta(x) = 0 \text{ on } B_1 \cup B_2 \cup B_1^+ \cup B_2^+\}, \tag{36}$$

while the coincidence set $A^{st} = \{x \in \Gamma_2 \ : \ u_N + v = 0\}$. Moreover $B_1 = \{x \in \Gamma_2 : \lambda(x) = -1\}$, $B_2 = \{x \in \Gamma_2 : \lambda(x) = +1\}$, $\tilde{B}_i = \{x \in B_i : u_N(x) + v = 0\}$, $i = 1, 2$, $B_i^+ = B_i \setminus \tilde{B}_i$, $i = 1, 2$. The derivative of the Lagrangian L with rescpect to ρ has the form:

$$\int_\Omega \frac{\partial J}{\partial \rho}(\rho, u)\zeta dx = \int_\Omega \frac{\partial L}{\partial \rho}(\rho, u, \lambda, p^a, q^a)\zeta dx = \int_\Omega [\gamma \epsilon \nabla \rho \cdot \nabla \zeta + \frac{\gamma}{\epsilon} \psi'_B(\rho)\zeta] dx + \int_\Omega [g'(\rho) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a + \eta) - f(p^a + \eta)]\zeta dx, \ \ \forall \zeta \in H^1(\Omega) \tag{37}$$

Using (37) we formulate a modified Cahn-Hilliard equation as gradient flow dynamics with an artificial time variable. It leads to a pseudo time stepping approach. This problem is as follows: *find sufficiently regular* $(\rho, u, \lambda, p^a, q^a)$ *satisfying (10)-(11), (33)-(34) as well as*

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (M(\rho) \cdot \nabla \varphi_E(\rho)) \qquad \text{in } \ \Omega, \ \forall t \in [0, T), \tag{38}$$

$$M(\rho) \nabla \varphi_E(\rho) \cdot n = 0 \qquad \text{on } \ \partial\Omega, \ \forall t \in [0, T), \tag{39}$$

$$\nabla \rho \cdot n = 0 \qquad \text{on } \ \partial\Omega, \ \forall t \in [0, T), \tag{40}$$

$$\rho(0, x) = \rho_0(x) \qquad \text{in } \ \Omega, \ t = 0. \tag{41}$$

where the potential function φ_E is given by

$$\varphi_E = -\gamma \epsilon \triangle \rho + \frac{\gamma}{\epsilon} \psi'_B(\rho) - g'(\rho) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a + \eta) - f(p^a + \eta), \quad \text{a.e. in } \ \Omega, \tag{42}$$

The necessary optimality condition to optimization problem (32) has the form: if $(\rho^\star, u^\star, \lambda^\star, p^{a\star}, q^{a\star})$ is an optimal solution to structural optimization problem (32) than it satisfies (10)-(11), (33)-(34) and (38)-(42).

7. Finite element approximation

Consider discretization of the system (38)-(42) [2, 9]. Let $\mathcal{T}_h$ be a regular triangulation of domain Ω into disjoint open elements T, i.e. $\Omega = \bigcup_{T \in \mathcal{T}_h} \bar{T}$. The discretization parameter h is defined as the maximal element size of $\mathcal{T}_h$, i.e. $h = \max_{T \in \mathcal{T}_h} \{diam T\}$. Associated with the family $\mathcal{T}_h$ are the piecewise constant and piecewise linear finite element spaces, respectively,

$$D_h^0 = \{\eta \in L^2(\Gamma) : \eta_{|T} \in P_0(T) \ \ \forall T \in \mathcal{T}_h\} \subset L^2(\Gamma) \tag{43}$$

$$D_h^1 = \{\eta \in C^0(\bar{\Omega}) : \eta_{|T} \in P_1(T) \ \ \forall T \in \mathcal{T}_h\} \subset H^1(\Omega) \tag{44}$$

where $P_k(T)$ denotes the set of all polynomials of order $k = 0, 1$ on the element T. Let us denote by $V_h \subset V$, $K_h \subset K_h$, $K_{h1} \subset K_1$, $\Lambda_h \subset \Lambda$, $\Lambda_{1h} \subset \Lambda_1$ finite element approximations of the space V and the sets K, K_1, Λ, Λ_1, respectively. Let ρ_h^n denotes piecewise linear approximation of ρ on Ω at n-th time step. Time interval denoted by $\tau = t_n - t_{n-1}$ is fixed. Time derivatives are approximated by the forward finite difference. By $(u_h^n, \lambda_h^n) \in K_h \times \Lambda_h$ and by $(p_h^{an}, q_h^{an}) \in K_{1h} \times \Lambda_{1h}$ we denote the discretized solutions to the state system (10)-(11) and to the adjoint system in the domain Ω_h for ρ_h^n rather than in Ω for ρ, respectively. $(\cdot, \cdot)_h$ denotes the lumped mass scalar product.

The finite element approximation of problem (38)-(42) takes the form: *find* $(\rho_h^n, u, \lambda_h^n, (p^a)_h^n, (q^a)_h^n)$ *satisfying*(10)-(11) and (33)-(34) *as well as*

$$(\frac{\epsilon}{\tau}(\rho_h^n - \rho_h^{n-1}, \zeta)_h + ((M(\rho_h^{n-1})\nabla\varphi_{Eh}^n, \nabla\zeta)_h = 0, \ \forall\zeta \in D_h^1, \tag{45}$$

$$(\varphi_{Eh}^n, \zeta)_h = \gamma\epsilon(\nabla\rho_h^n, \nabla\zeta)_h + \frac{\gamma}{\epsilon}(\psi_B'(\rho_h^{n-1}) + \psi_B''(\rho_h^{n-1})(\rho_h^n - \rho_h^{n-1}, \zeta)_h - (g'(\rho)a_{ijkl}e_{ij}(u_\epsilon)e_{kl}(p^a + \eta) - f(p^a + \eta), \zeta)_h, \ \ \forall\zeta \in D_h^1, \tag{46}$$

The equations (45)-(46) are the system of two second order equations to find ρ_h^n and the potential φ_{Eh}^n. The derivative of the double well potential is linearized with respect to ρ_h^n. Primal-dual active set method has been used to solve state and adjoint systems (10)-(11) and (33)-(34). Biconjugate gradient method has been used to solve (45)-(46).

8. Numerical results

The discretized structural optimization problem (45)-(46) is solved numerically. The numerical algorithms described in the previous sections have been used. The algorithm is programmed in Matlab enviroment. As an example a body occupying 2D domain

$$\Omega = \{(x_1, x_2) \in R^2 : 0 \le x_1 \le 8 \ \wedge \ 0 < v(x_1) \le x_2 \le 4\}, \tag{47}$$

is considered. The boundary Γ of the domain Ω is divided into three pieces

$$\begin{aligned} \Gamma_0 &= \{(x_1, x_2) \in R^2 : x_1 = 0, 8 \ \wedge \ 0 < v(x_1) \le x_2 \le 4\}, \\ \Gamma_1 &= \{(x_1, x_2) \in R^2 : 0 \le x_1 \le 8 \ \wedge \ x_2 = 4\}, \\ \Gamma_2 &= \{(x_1, x_2) \in R^2 : 0 \le x_1 \le 8 \ \wedge \ v(x_1) = x_2\}. \end{aligned} \tag{48}$$

The domain Ω and the boundary Γ_2 depend on the function v. The initial position of the boundary Γ_2 is given as in Fig. 1. The computations are carried out for the elastic body characterized by the Poisson's ratio $\nu = 0.29$, the Young modulus $E = 2.1 \cdot 10^{11} N/m^2$. The body is loaded by boundary traction $p_1 = 0$, $p_2 = -5.6 \cdot 10^6 N$ along Γ_1, body forces $f_i = 0$, $i = 1, 2$. Auxiliary function η is selected as piecewise constant (or linear) on D and is aproximated by a piecewise constant (or bilinear) functions. The computational domain $D = [0, 8] \times [0, 4]$ is selected. Domain D is discretized with a fixed rectangular mesh of 80×40. Other parameters are: $\epsilon = 0.02$, $\gamma = 1$, $T = 0.200$. Following [17] $g(\rho) = \frac{\rho}{1+\exp(-40\rho)} + g_\epsilon$, $g_\epsilon = 0.02$, $\psi_B(\rho) = \rho^2(1-\rho)^2(\frac{1}{10}\exp(15(1/2-\rho)^2) + 1)$.

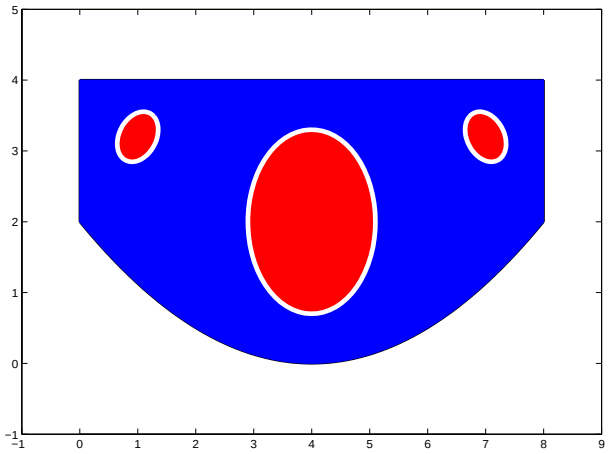

Figure 2: Optimal material density distribution in domain $\Omega^\star$.

Fig. 2 presents the optimal domain obtained by solving structural optimization problem (31) in the computational domain D using the optimality condition (45)-(46). The areas with low values of density function appear in the central part of the body and near the fixed edges. The obtained normal contact

stress is almost constant along the optimal shape boundary and has been significantly reduced comparing to the initial one (see Fig. 3).

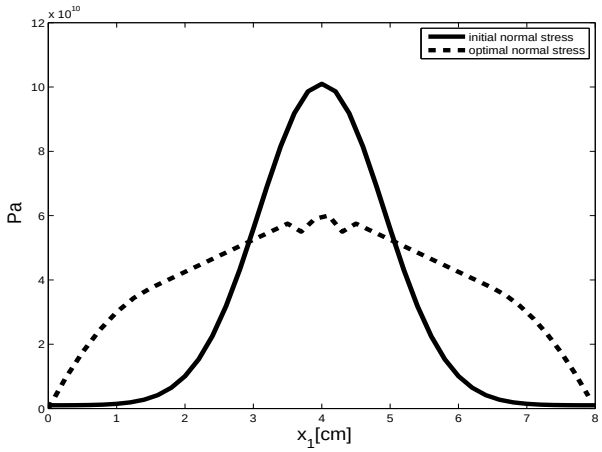

Figure 3: Initial and optimal normal contact stress.

9. Conclusions

The structural optimization problem for elastic contact problem with the prescribed friction is solved numerically in the paper. Obtained numerical results indicate that the proposed numerical algorithm allows for significant improvements of the structure from one iteration to the next.
Phase field approach based on the Cahn-Hilliard equation is flexible and can be easily combined with material density field. In this sense this approach follows SIMP method. On the other hand this approach can be also coupled with other physical fields allowing to consider different topology optimization problems. Since Cahn-Hilliard equation is the fourth order equation its numerical solution is rather costly, especially comparing to other existing methods.

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