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Stochastic Multidisciplinary Analysis with High Dimensional Coupling

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1. Abstract
This paper combines efficient uncertainty propagation and sampling techniques for uncertainty quantification of aerospace structures. A new method is developed for including uncertainty quantification in multi-disciplinary system analysis (MDA) that usually requires iterative analyses with a large number of coupling variables. The methodology for MDA estimates the probability distributions of the coupling variables based on computing the probability of satisfying the inter-disciplinary compatibility equations. This idea is used to develop a joint distribution approach for multidisciplinary analysis. Using the distributions of the feedback variables, the bi-directional coupling can be reduced to unidirectional coupling, while still preserving the mathematical relationship between the coupling variables. In realistic structures, the number of coupling variables is potentially large. Therefore, principal component analysis (PCA) is adopted to decrease the number of coupling variables. For computational efficiency, a copula-based sampling is employed to implement probabilistic multi-disciplinary analysis. The proposed methods are illustrated using 3-D aero-elasticity analysis of an aircraft wing.
2. Keywords: MDA, Likelihood, Principal Component Analysis, Copula

3. Introduction
Uncertainty quantification and design under uncertainty of realistic structures requires multiple functional evaluations in the presence of natural variability, data uncertainty and model uncertainty. When the structural system requires high-dimensional coupled multi-disciplinary analysis between several disciplines, the computational effort multiplies tremendously, creating difficulties in implementing reliability analysis techniques. This paper combines efficient uncertainty propagation and sampling techniques to overcome such challenges.
In this paper, we address the issue of uncertainty propagation analysis in multi-level, multi-disciplinary systems using a joint distribution method aspired by a likelihood-based methodology (LAMDA) [1]. This approach requires no coupled system analysis, thereby improving the computational cost.
In realistic multidisciplinary analyses, the amount of data exchanged could be extremely large. (For example, in aero-elasticity analysis shown in Fig.1 and Fig. 2, the aerodynamic response, i.e. pressure distributed on the airfoil, and structural responses, i.e. displacement of the nodes of the wing, are not independent and the sizes are tremendous). As the dimension increases, the computational expense of MDA grows dramatically.

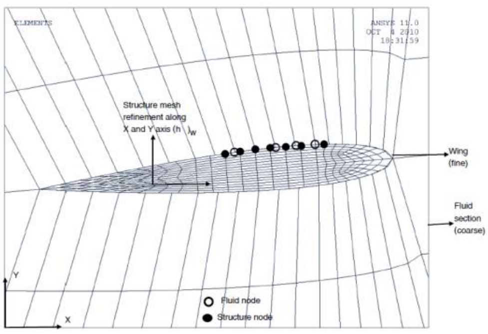


Figure 1: 3-D Wing with NACA 0012 Airfoil

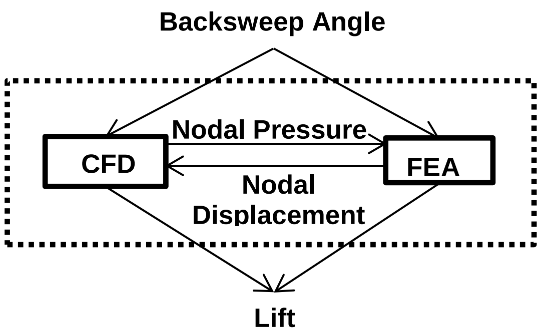


Figure 2: Functional Relationships between Two Analyses

Meanwhile, in the aero-elasticity problem, the coupling variables in each direction are inherently coupled. Thus, a model reduction technique together with a fast sampling technique is required to overcome the computational challenge. A principal component analysis (PCA) [2] is adopted to transform the coupling variables into a much smaller number of independent variables. Within the principal component space, a Bayesian Network along with copula-based sampling [3,4,5,6] is implemented to construct the probability function of the variables.
The contributions of this paper can be summarized as follows:

1. Extension of LAMDA to problems with a large number of correlated coupling variables;
2. Uncertainty quantification in model output, through probability distributions, due to variability;

3. Use of a Bayesian Network together with copula-based sampling to improve the computational efficiency of probabilistic MDA; and
4. Illustration of the proposed methodology with an aircraft wing analysis problem.

4. Unidirectional Decoupling for Feedback Multidisciplinary Analysis

Consider the multi-disciplinary system in Fig. 3. Given the probability distributions of the inputs $\boldsymbol{x}$, the target problem is to estimate the distribution of each single disciplinary outputs g_1, g_2, and final system output *f*. An intermediate step is to calculate the PDFs of the coupling variables u_{12} and u_{21} for uncertainty propagation. For the deterministic problem of estimating the converged u_{12} and u_{21} values corresponding to given values of $\boldsymbol{x}$ in the fixed point iteration approach, an arbitrary value of u_{12} is taken as input to "Analysis 2" for start, and the resultant value of u_{21} serves as input to "Analysis 1". This proceeds recursively until the inter-disciplinary compatibility is satisfied, which means the output from "Analysis 1" is the same as the initial u_{12}. Then the analysis is converged and terminated.

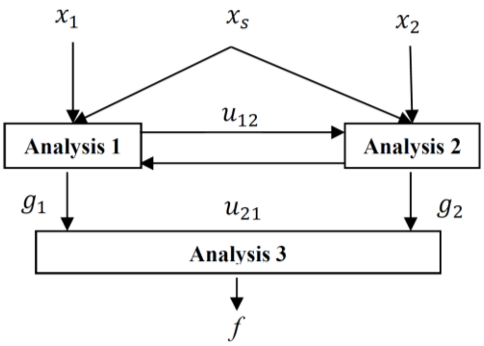


Figure 3: Multidisciplinary System

A unidirectional coupling is formulated by severing the coupling from analysis 1 to analysis 2. A function *G* is defined such that: its input is the coupling variable u_{12}, in addition to $\boldsymbol{x}$, and the output is denoted by U_{12}, which is obtained by propagating the input through Analysis 2 followed by Analysis 1, as shown in Fig. 4.

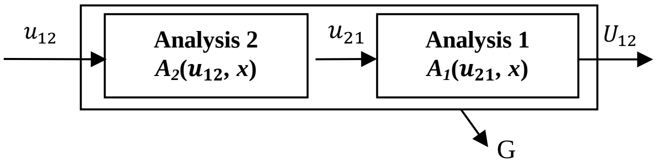


Figure 4: Definition of G

If the multi-disciplinary compatibility is satisfied, then $u_{12} = U_{12}$. A likelihood-based approach to calculate this probability is formulated in [1]. This method employs first-order reliability method [7] to evaluate the PDF function of the coupling variables.

In the case where the coupling variables are scalars, each likelihood evaluation requires two FORM analyses to obtain the PDF. As the number of coupling variables increases, the integration procedure causes the computational cost to increase exponentially. In the aero-elasticity problem, the coupling analysis between FEA and CFD is associated with the exchange of nodal pressures and nodal displacements, both of which are field quantities. LAMDA using FORM will be computationally expensive to implement. Moreover, the fluid-structure analysis for an aircraft wing takes large effort to converge even for one input realization; this makes Monte Carlo sampling prohibitive. Therefore, a more efficient sampling and PDF evaluation technique is required.

5. Model Reduction and Sampling Techniques

For every given value of $\boldsymbol{x}$, there is only one converged value of u_{12}, which suggests a functional dependence between the input and coupling variable. However, due to the uncertainty from the input and measurement noise from the output, a statistical dependence describes the relationship in between better. Based on this idea, a Bayesian network is introduced to represent this relationship.

Consider a Bayesian network over a set of nodes: $U = \{X_1 ... X_n\}$. The joint probability of $\{X_1 ... X_n\}$ is given as:

$$P(U) = P\{X_1 \dots X_n\} = \prod_{i=1}^{n} P(X_i|\pi_i) \quad (1)$$

where π_i is the set of parents of node X_i. The marginal probability function of X_i is denoted as:

$$P(X_i) = \sum_{X_{\sim i}} P(U) \tag{2}$$

A common usage of Bayesian network is to infer the beliefs of events given the observation of other events, which is called evidence. The joint probability of events U given evidence e could be evaluated based on Bayes theorem:

$$P(U|e) = \frac{P(U,e)}{P(e)} = \frac{P(U,e)}{\sum_U P(U,e)} \tag{3}$$

Let ε denote the difference between the input u_{12} and output U_{12} of G, $\varepsilon = U_{12} - u_{12}$. The evidence e, which is the interdisciplinary compatibility is satisfied, could be represented as $\varepsilon = 0$. Therefore, the PDF of U_{12} given the compatibility condition is:

$$f_{U_{12}}(U_{12}|U_{12} = u_{12}) = f_{U_{12}}(U_{12}|\varepsilon = 0) \tag{4}$$

To obtain the PDF of the coupling variable given the compatibility condition, a copula-based sampling technique is introduced. Consider n random variables: X_i $(i = 1 \ldots n)$, all of which have continuous cumulative distribution functions (CDFs): $F_i(x_i)$. An n-dimensional copula function relating to X_i is defined as:

$$C = P[\bigcup_{i=1}^{n} F_i(x_i) \leq u_i] \tag{5}$$

where u_i denotes the CDF value of X_i. A copula is naturally a joint CDF of the CDF of the variables. Samples of u_i are drawn from the copula, the inverse CDF of the samples is taken to retrieve the corresponding values of x_i.

There are a variety of choices for copulas. They can be mainly categorized in to (a) families of copulas, which are parametric, and (b) empirical copulas, which are nonparametric. This paper uses the multivariate Gaussian copula (MGC) [8] for the sake of illustration. Let Σ be a symmetric, semi-positive definite matrix with $\text{diag}(\Sigma) = (1,1 \ldots 1)^{\mathrm{T}}$, and Φ_Σ the standardized multivariate normal distribution with correlation matrix Σ, the MGC is defined as:

$$C_{\Sigma}^{Gauss}(u) = \frac{1}{\sqrt{det(\Sigma)}} \exp\left(-\frac{1}{2} \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix} \cdot (\Sigma^{-1} - \mathrm{I}) \cdot \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix} \right) \tag{6}$$

where Φ^{-1} is the inverse of standard uni-variate normal distribution function Φ. Computational efficiency is the major advantage of copula-based sampling compare with MCMC-based sampling for Bayesian updating.

To implement this method, samples of coupling variables as input and as output need to be acquired beforehand. These samples could come from any two consecutive iterations of the MDA except for the initial samples and the result after first iteration, since the initial value of the coupling variable is assumed and is not an available piece of information to construct Bayesian network. Then the difference of the corresponding input/output is denoted as ε. Consequently, the samples of U_{12}, u_{12} and ε are used to build the network, and the PDF of the converged coupling variable u_{12} can be estimated by calculate the conditional PDF given $\varepsilon = 0$.

The Bayesian network together with copula sampling technique could largely simplify the MDA process for a high dimensional coupled system. However, when the coupling variables in one direction are correlated with each other, the effort to build the Bayesian network could be huge, and problems may rise particularly when two variables are highly correlated, which might lead to the singularity of the correlation matrix. Therefore, principal component analysis is employed to transform the correlated variables into an orthogonal space. Usually, the variances captured by the first 5 to 10 principal components could exceed 90% of the total variance of the original data. Therefore, PCA can help us to construct the Bayesian network using a smaller number of independent variables. A simple Bayesian network could be created for each principal component as shown in Fig. 5. $PC_{u_{12}}^{i}$and $PC_{U_{12}}^{i}$ denote the i^{th} column in the upper and lower part of *PC*. The collection of updated samples from the first several principal components is then used to transform into the original correlated samples. The method is illustrated for an aero-elasticity problem that requires the multidisciplinary analysis.

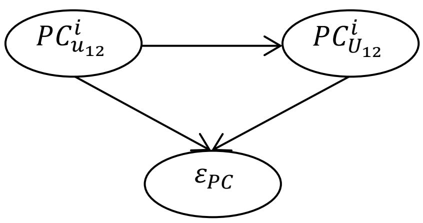
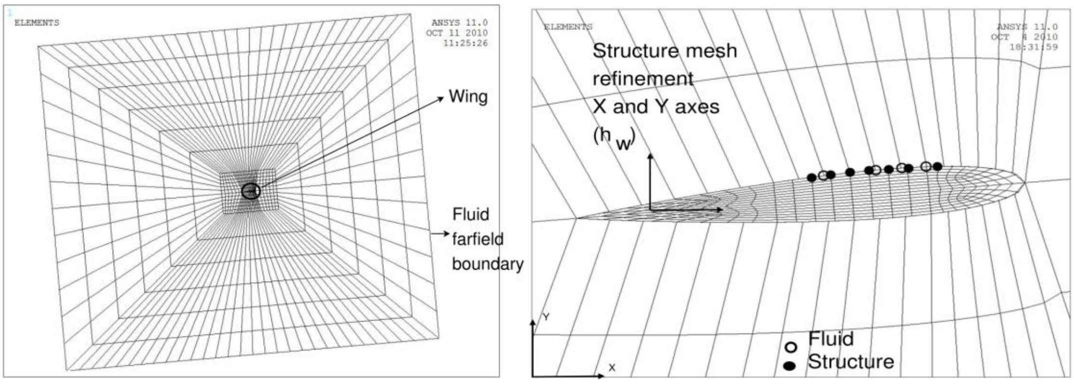


Figure 5: Bayesian Network for Multidisciplinary Analysis

6. Numerical Example

Three dimensional aero-elastic analysis of an aircraft wing are used to illustrate the proposed methods. A cantilevered wing with a NACA 0012 airfoil is adopted [8]. We use ANSYS to perform the fluid-structure interaction analysis of the wing. The fluid structure meshes are shown in Fig. 6.



(a) Overall view (b) Fluid and structure meshes

Figure 6: Non-matching fluid and structure meshes and refinement parameters

The focus of the MDA problem is the propagation of input variability through the fluid-structure system and its impact on the nodal pressure, which is the exchanging variable between the FEA-CFD analyses. The backsweep angle *bw* is chosen to be the input variable with natural variability. It is assumed to be normally distributed as $N(0.4, 0.04)$. 60 samples of backsweep angle are drawn to perform the analysis. 258 nodes are created after mesh. The analysis is terminated after 3 iterations, and the nodal pressure after iteration 2 and iteration 3 are recorded. The 3-iteration analysis is performed on desktop. For the purpose of comparison, converged analysis is performed by Vanderbilt University's ACCRE computer cluster. Five cases are discussed here: (1) LAMDA method using first 10 principal components. (2) LAMDA using 15 principal components. (3) LAMDA using 20 principal components. (4) Results after 2 iterations. (5) Results after 3 iterations.

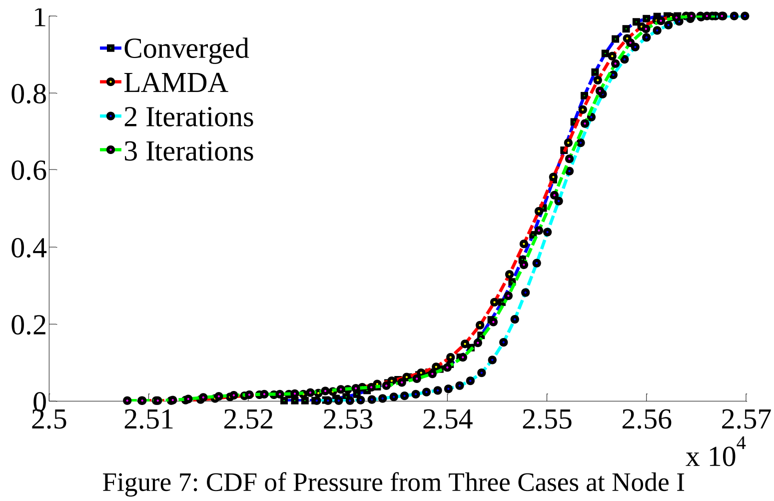


Figure 7: CDF of Pressure from Three Cases at Node I

Fig. 7 shows the CDF of Cases III, IV and V at node I. Both case III and V are closer to the converged distribution compare with case IV at Node I. For further comparison, the Kullback-Leiber divergence (also called K-L distance) is adopted to quantify the differences between the converged PDF and the approximated result by case I - V at each node. The K-L distance of distribution Q from P, denoted by $D_{KL}(P||Q)$, is a measurement of information lost when P is approximated by Q. Details about the methodology can be found in [9]. For *n*-variable distribution, the K-L distance for marginal distributions are computed, and a total average is estimated in formula (20). In the MDA problem, D_{KL}^{i} denotes the K-L distance at the i^{th} node.

$$\text{Total Average} = \frac{\sqrt{\sum_{i=1}^{n} \left(D_{KL}^{i}(P||Q)\right)^2}}{n} \tag{7}$$

Tables 1 summarizes the results of the MDA. The K-L distance at the first 10 nodes are listed:

Table 1: Kullback-Leiber divergence for different scenarios

Node	LAMDA			Case 4	Case 5
	Case 1	Case 2	Case 3		
1	4.06	3.33	2.52	1.81	2.76
2	-1.54	-2.14	-2.04	-2.83	-1.69
3	1.71	0.74	0.97	-2.96	-0.09
4	-1.8	-2.6	-1.93	-4.68	-3.02
5	-2.79	-3.31	-2.89	-4.31	-3.13
6	-3.99	-3.8	-3.77	6.08	-0.34
7	-0.62	-0.87	-1.02	3.79	0.68
8	-1.1	-0.36	-0.55	10.96	3.01
9	-2.81	-2.46	-2.47	12.02	2.05
10	-0.64	-0.05	1.2	-5.24	-1.13
Total Average	0.19	0.14	0.14	0.44	0.19

The following observations are made:

1. Case 1, Case 2 and Case 3: as the utilized principal components increases, the K-L distance tend to be smaller. This is due to the reason that the more principal components are taken, the more variances of the original samples will be caught. Therefore, the results become more accurate.
2. Case 4 and Case 5: Case 5 approaches the converged results much better than Case 4. As iteration increases, the stability of system will be enhanced, and the differences between the results from contiguous iterations will become smaller.
3. Case 2, Case 3 and Case 5: The proposed LAMDA method together with the model reduction technique for data uncertainty propagation could overall well quantify the converged distribution of coupling variable just using the result from the first 3 iterations of the analysis instead of performing the converging analysis.
4. The computational time for 3-iteration analysis on average is 3 min/input on desktop, and it takes 120min/input on average for the analysis to converge. Therefore, the proposed approach as a great superiority in speed.

Therefore, the proposed LAMDA method together with model reduction technique is evidently computationally economical. It is an efficient tool for dealing with MDA system with large amount of exchanging variables and long converging time. The proposed method can be easily extended to the system with data uncertainty and model error c Bayesian network

7. Conclusion

In this paper, we proposed approaches for data uncertainty and model error quantification and propagation through multidisciplinary system with field quantity as coupling variable. The likelihood-based approach for MDA is improved by using a Bayesian network together with copula sampling and principal component analysis, such that the high-dimensional coupled problem could be decoupled and the dimension could be decreased. The use of the proposed method is explained by an aero-elasticity problem of a 3-D aircraft wing, which requires fluid-structure analysis. The proposed method has obvious advantage in computational efficiency, while still providing satisfactory results.

9. Acknowledgments

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10. References

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