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A study on symmetry properties in structural optimization

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1. Abstract
In the present paper, some symmetry results for optimal solutions in structural optimization have been proposed and proven. It is found that under some invariant assumptions, for many structural optimization problems that can be formulated as quasi-convex programs, there exists at least one symmetric global optimal solution if the prescribed loading and support conditions are symmetric. Meanwhile, for some specific non-convex cases, a weaker result is also presented. Furthermore, some new results about the symmetry properties of robust optimization, which has seldom been toughed in the literature, are also presented. Numerous concrete examples illustrate the claims made in the present work explicitly.

2. Keywords: Symmetry; Structural optimization; Quasi-convex programming; Robust optimization.

3. Introduction
Symmetry is a very important property for optimal solutions of structural optimization problems. It is not only of crucial theoretical importance such that from which deep insights of optimal solutions can be obtained, but also has great value from computational point of view since a large amount of computational efforts can be saved in finding optimal designs if the symmetric properties of the considered problem (if they really exist) can be utilized appropriately. Recently, discussions about the symmetry properties of optimal solutions have received considerable attentions in structural optimization community. Kanno et al. (2002) studied the symmetry properties of the solution of semi-definite programming (SDP) and pointed out that for the optimization problem of symmetric trusses under fundamental frequency constraint, which can be formulated as a linear SDP (LSDP), at least one of the optimal solutions is symmetric and can be obtained by the primal-dual interior-point method without any additional constraints. In an educational article, Stolpe (2010) showed by a concrete example that the optimal solution of a symmetric discrete topology optimization problem may be non-symmetric! This is somewhat beyond one's intuition. More recently, in a review article, Rozvany (2011a) presented a number of conjectures concerning the fundamental properties of exact optimal topologies. Among them, the author concluded that for a particular class of problems that satisfies some specific restrictions, at least one optimal topology is symmetric if the external loads, the boundary of the design domain and the support conditions are all symmetric. These results provided a deep insight into the fundament properties of optimal solutions. Cheng and Liu (2011) demonstrated through benchmark examples that for fundament frequency-related topology optimization of frame structures, the corresponding optimal solution may not be symmetric even though the structural node geometry, the concentrated mass distribution and the ground structure are symmetric. They also pointed out that this may be due to the non-linear and non-convex natures of the considered problem.
Inspired by the seminal work of Rozvany (2011a), in the present paper, the symmetry properties of a class of optimization problems are revisited from a more general point of view. The main finding is that quasi-convexity is a key point to guarantee the existence of symmetric solutions for symmetric optimization problems. Our analysis shows that under some invariant assumptions, for a large class of structural optimization problems that can be formulated as quasi-convex programs, there is at least one symmetric global optimal solution if the prescribed loading and support conditions are symmetric. In this sense, some of the conjectures proposed by Rozvany (2011a) can be seen as the special cases of our claims. Besides, for some specific non-convex problems, based on the results of sequential convexification, it is also proved that at least one of the K-K-T points of the underlying problem is symmetric. It is worth noting that in order to illustrate the basic ideas more explicitly, the theoretical results are only developed for the two dimensional discrete structures (truss structures, frame structure and/or structures composed by them). Furthermore, the symmetry analysis of robust optimization problems, which is seldom touched in the literature, is also discussed. We also illustrate through a practical structural optimization example that the global optimum of a symmetric optimization problem which is not quasi-convex may be highly asymmetric even when continuous design variables are considered. Heuristic symmetry reductions without serious theoretical analysis may result in strongly suboptimal solutions.

4. Main theoretical results
In this section, we proposed a basic theorem which indicates a sufficient condition that guarantees the existence of

symmetry global optimal solutions. Firstly, some basic mathematical definitions will be presented briefly (Boyd and Vandenberghe, 2004).

4.1. Quasi-convex program

Definition 1: A function $f(\boldsymbol{x})\colon \mathbf{R}^{\mathrm{n}} \to \mathbf{R}$ is said to be quasi-convex if its domain and all its sublevel sets

$$S_\alpha = \{\boldsymbol{x} \in \mathbf{dom}\, f \,|\, f(\boldsymbol{x}) \le \alpha\}, \tag{1}$$

for any $\alpha \in \mathbf{R}$, are convex. A function is quasi-concave if $-f$ is quasi-convex.

Definition 2: A mathematical programming problem

$\mathbb{QC}\wp$

$$\begin{aligned} &\text{Find} && \boldsymbol{x} = (x_1, \dots, x_n)^\top \in \mathbf{R}^{\mathrm{n}} \\ &\text{Minimize} && f(\boldsymbol{x}) \\ &\text{Subject to} && \\ & && g_i(\boldsymbol{x}) - \bar{g}_i \le 0, \quad i = 1, \dots, m \end{aligned} \tag{2}$$

is said to be a quasi-convex program if f and g_i, $i = 1, \dots, m$ involved in Eq. (2) are all quasi-convex.

Continuity of the objective/constraint functions

Assume the design variables of the is the cross sectional areas of some structures (Fig.1), The continuity of f means that

$$\lim_{\epsilon \to 0^+} f\left(A_1^{S_1}, A_2^{S_1}, A_3^{S_1}, A_4^{S_1}, \epsilon, \epsilon\right) = f'\left(A_1^{S_1}, A_2^{S_1}, A_3^{S_1}, A_4^{S_1}\right). \tag{3}$$

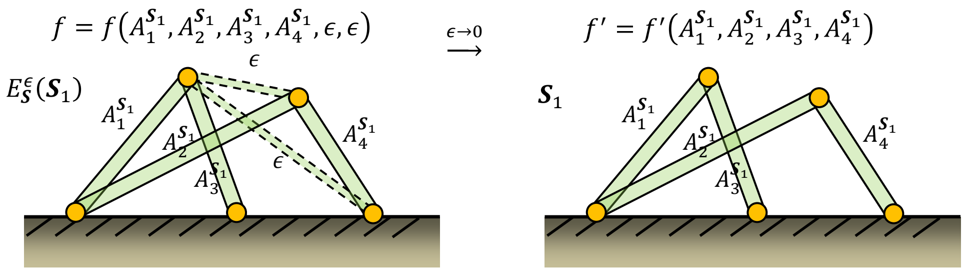


Fig.1 Schematic illustration of the continuity of the structural performance functions.

Remark 1: In the above definition, it is emphasized that f and f' are two different functions defined on two structures with different topologies. Under this circumstance, f and f' should be associated with the same structural response that reflect the *overall* or *global* performance of the structure such as the structural weight/compliance, the critical buckling load, the fundamental frequency of the structure, etc. It is worth noting, however, that the above requirement is only necessary for topology optimization problems. For size optimization problems, local structural responses such as the stresses and internal forces of the bars can be considered in the objective or constraint functions.

Remark 2: The continuity of the objective/constraint functions when structural topology degenerates from one form into another, first initiated in Cheng and Jiang (1992), is a subtle issue in structural topology optimization especially when ground structure approach is adopted for finding the optimal topologies. As shown in Cheng and Guo (1997), for some types of structural topology optimization problems (e.g., topology optimization with local stress constraints, topology optimization with local buckling constraints), if gradient-based optimization algorithms are employed, the global optimal solutions are expected to be found only if the corresponding problem formulations have some required continuities. The continuity of some specific objective/constraint functions were discussed in Cheng and Guo (1997), Petersson (2001), Svanberg and Werme (2009), Cheng and Liu (2011) respectively. It was found that whether the continuity exists or not is strongly dependent on the forms adopted to describe the objective/constraint functions. Furthermore, it was also proved that under some regularity assumptions, the structural compliance and the internal forces of the bars in truss structures are continuous structural responses when structural topology degenerates. It is worth noting, however, that the fundament frequency of truss and frame structures may be discontinuous when structural topology degenerates. This is due to the fact that local vibration resulting from very slender bars in the structure may become the fundamental vibration mode, which, however, will be disappeared if all these slender bars are removed. For more detailed discussions about the issue of continuity in structural topology optimization, we refer the readers to the above mentioned references.

4.2. Main results

Theorem 1: If for the considered structural optimization problem,

(1) the set of the invariant rigid motions is the same for both the external loads and the support conditions and has at least one generator;

(2) it is or can be formulated as a quasi-convex program if the cross sectional areas of the structural components are used as design variables;

(3) the concerned objective and the constraint functions are invariant under the corresponding invariant rigid motions and have the required continuity properties;

(4) its feasible set is not empty,

then there must exist a global optimal solution of the considered structural optimization problem such that the corresponding optimal structure has the same symmetry as the external loads and the support conditions.

Remark 3: In Rozvany (2011a), the author discussed the uniqueness and symmetry properties of optimal solutions in the light of exact optimal layout theory. It is found that for a large class of optimization problems that can be formulated as linear programming (LP) program such as elastic stress or compliance design of trusses and grillages under single load case and plastic design of both single and multiple loaded trusses and grillages, at least one of the corresponding optimal solutions must be symmetric. This is quite consistent with Theorem 1 presented above since LP is a quasi-convex program. In Rozvany (2011a, Page 310-311) and Rozvany (2011b), the author also pointed out some cases that will yield non-symmetric optimal solutions. These anomalous phenomena can also be explained by the proposed theorem since for the case of 12.1 (Topology optimization with piece-wise concave cost functions), case 12.2 (Compliance minimization for axisymmetric perforated plates) and case 12.3 (Topology optimization with non-symmetric specific cost functions) in Rozvany (2011a) and the counter example presented in Rozvany (2011b, example 1), the quasi-convexity assumption of the considered problems is all violated. Therefore it seems that some of the conjectures made in Rozvany (2011a) for the finite dimension case can be seen as the special cases of our claims. Furthermore, the counter example presented by Stolpe (2010) can also be explained by the proposed theorem since all the discrete optimization problems are obviously non-convex in nature.

Non-convex case

The second theoretical result of this paper, which is concerned about the non-convex case is presented as follows.

Theorem 2: If for the considered non-convex structural optimization problem,

(1) the set of the invariant rigid motions is the same for both the external loads and the support conditions;
(2) the set of the invariant rigid motions has at least one generator;
(3) its feasible set is not empty and the cross sectional areas of the structural components are used as design variables;
(4) at least one of the K-K-T points of the considered non-convex optimization can be approximated by the K-K-T points of a series of convex optimization problems;
(5) the objective and the constraint functions of the approximated convex optimization problems are invariant under the corresponding invariant rigid motions and have the required continuity properties,

then there must exist at least one K-K-T point of the considered problem has the same symmetry as the external loads and support conditions. Furthermore, if all the K-K-T points of the considered non-convex optimization can be approximated by the K-K-T points of some series of convex optimization problems, then the global optimal solution of the considered problem is symmetric.

To get more details about the prove of the two theorems above please see (Guo et al. 2012) for reference.

4.3. Results about robust optimization

In recent years, a lot of research interest has been devoted to the study of structural optimization problems considering uncertainties. As a more recent approach to structural optimization under uncertainty, robust structural optimization has received ever increasing attention in recent years since it is computational tractable if formulated appropriately and not dependent too much on the probabilistic descriptions of uncertain parameters. (Sigmund 2009, Takezawa et al., 2011 and Guo et al., 2009, 2013).

In general, a robust structural optimization problem can be formulated as follows:

$$\begin{aligned}
&\text{Find} && \boldsymbol{x} = (x_1, \dots, x_n)^\top \in \mathbf{R}^\mathrm{n} \\
&\text{Minimize} && \mathcal{F}(\boldsymbol{x}) = \max_{\boldsymbol{p}\in U_{\boldsymbol{p}}} f(\boldsymbol{x};\boldsymbol{p}) \\
&\text{Subject to} && \\
& && \mathcal{H}(\boldsymbol{x}) = \max_{\boldsymbol{p}\in U_{\boldsymbol{p}}} h_j(\boldsymbol{x};\boldsymbol{p}) \le 0\,, \ \ j = 1, \dots, m, \\
& && x_i^l \le x_i \le x_i^u, \ \ i = 1, \dots, n,
\end{aligned} \tag{20}$$

where $f\colon \mathbf{R}^\mathrm{n} \times \mathbf{R}^\mathrm{k} \to \mathbf{R}$ and $h_j : \mathbf{R}^\mathrm{n} \times \mathbf{R}^\mathrm{k} \to \mathbf{R}$ are assumed to be continuous differentiable functions in their domains. $\boldsymbol{x} \in \mathbf{R}^{\mathrm{n}}$ is the vector of design variables and $\boldsymbol{p} \in \mathbf{R}^{\mathrm{k}}$ is the vector of uncertain parameters. The uncertainty set is denoted by $U_{\boldsymbol{p}} \subset \mathbf{R}^{\mathrm{k}}$, which is supposed to be closed in this paper. x_i^l and x_i^u denote the lower and upper bounds of x_i, respectively. Although some general results about the symmetry properties of optimal solutions have been obtained for deterministic structural optimization problems (Rozvany, 2011a; Guo et al., 2012), the corresponding results for robust structural optimization problems remains unexplored. In this section, we shall make a preliminary study on this topic.

With the main theorems in section 2.2, it is obvious that the symmetric property of optimal solutions of following problems could be guaranteed(see Fig. 2 for reference).

(1) Robust optimization of truss structures considering the uncertainty of node locations
(2) Robust optimization of truss structures considering the uncertainty of elastic boundary support stiffness
(3) Robust optimization of truss structures considering the uncertainties of node locations and external load

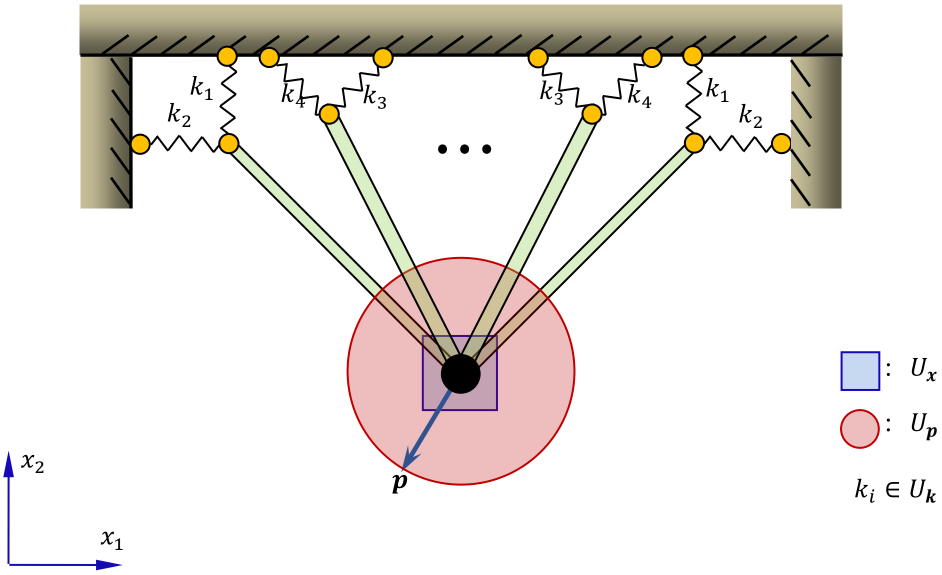


Fig. 2 Robust structural optimization of a truss structure considering the uncertainties of node locations, external load and the stiffness of the elastic boundary supports.

More discussions about the symmetric properties of robust structural optimization is proposed in (Guo et al. 2013). It was abbreviated in this paper for the conciseness.

5. Concrete examples

In this section, three concrete structural optimization problems with cross sectional areas as design variables are proposed to the main theorem and results for robust optimizations. Besides, a counter example is also provided which announced that the importance of quasi-convexity to assure the symmetric property of optima.

5.1. Quasi-convex case

Let us consider a truss structure shown in Fig.3. The Young's modulus, material density and length of bar 1 and bar 2 are $E_i = 2\sqrt{2},\ \rho_i = 1.0e-5, l_i = \sqrt{2},\ \ i = 1{,}2,$ respectively. Bar 3 is a non-designable support bar, whose cross sectional area, Young's modulus, material density and length are $A_3 = 1.0, E_3 = 1.0, \rho_3 = 1.0e-5, l_3 = 1,$ respectively. The concentrated mass at the free node is $M = 1$. The optimization problem is to find the optimal values of A_1 and A_2 such that the total volume of the structure is minimized while the constraint imposed on the fundamental frequency of the structure $\omega_1^2 \geq \alpha = 1$ should be satisfied.

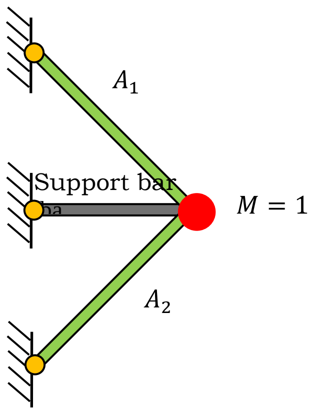


Fig.3 A three-bar truss example.

This is a *nonlinear convex* optimization problem. The corresponding feasible domain and the iso-line of the objective function are plotted schematically in Fig.4. From Fig.13, it can be observed that the global optimal solution is $A_1 = 1/2, A_2 = 1/2$, which represents a symmetric structure. All unsymmetrical structures have the same volume cannot satisfy the prescribed frequency constraint. This is also consistent with our theoretical result.

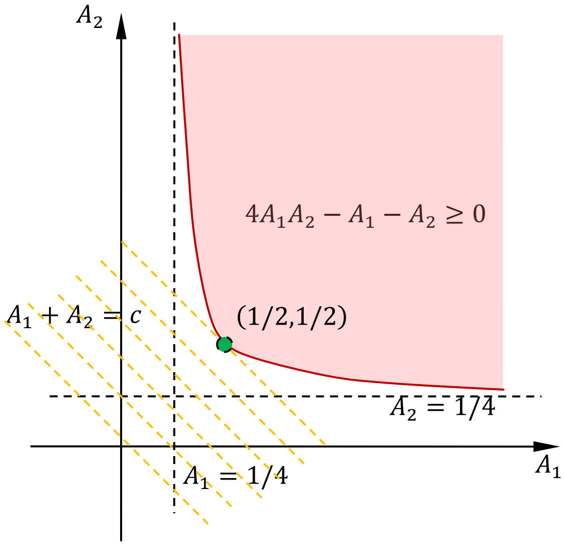


Fig.4 Schematic illustration of the feasible domain of the three-bar truss problem.

5.2. Non-convex case

Let us analyze the frame structure in Fig.5a. A rigid block under a unit moment is supported by two cantilever beams and a rigid bar restricting the vertical displacement. The objective function is also the compliance of the structure with the section areas A_1 and A_2 of the beams as design variables satisfying that the total volume of the two beams $V_1 + V_2 \leq 1$ and $0 \leq A_i,\ i = 1, 2$, respectively. For simplicity, we assume that the Young's modulus of the two beams is 3 and the cross sections are square, the lengths and the density of the beams are all 1.0 simultaneously.

This optimization problem is non-convex in nature. It can be verified that the global optimal solution of this problem is a non-symmetric one ($A_1 = 1, A_2 = 0$ or $A_1 = 0, A_2 = 1$) as shown in Fig.5b. On the contrary, the symmetric solution $A_1 = 0.5,\ A_2 = 0.5$ illustrated in Fig.5c, which can be verified as one of the K-K-T points of the optimization problem, corresponds to a solution with the *maximum* value of the objective function!

Actually, this optimization problem is so-called DC optimization. From the theoretical results provided in (Hong et al., 2011), the K-K-T points of a DC optimization problems can be obtained by sequential convex optimizations. Therefore the above result for the optimal solution of the frame structure is also quite consistent with the claims made in Theorem 2.

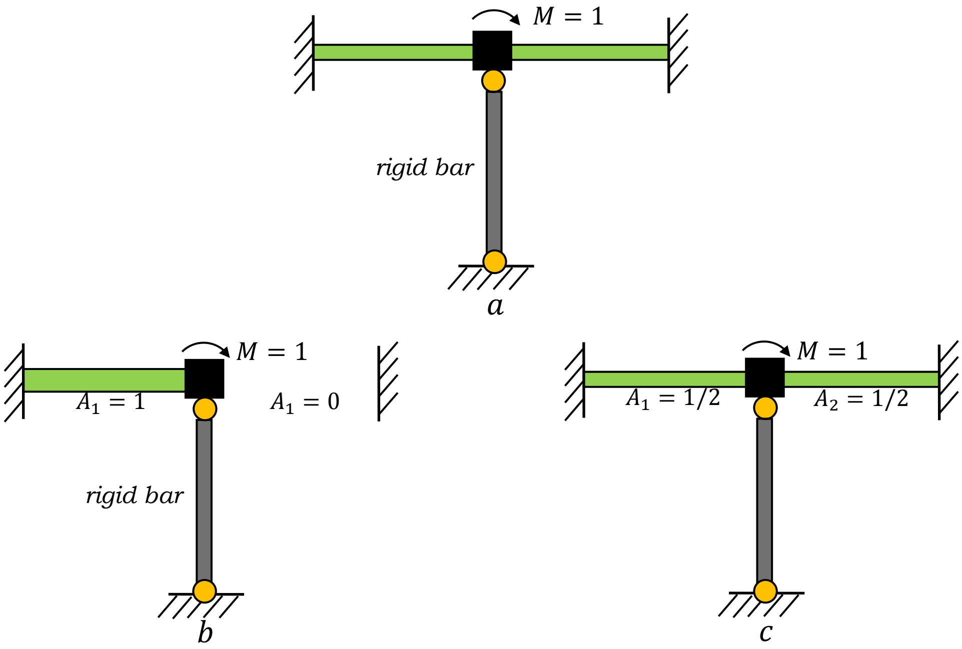


Fig.5 An example illustrating the result in Theorem 1 for non-convex case. (a) Ground structure. (b) Unsymmetric global optimal solution and (c) Symmetric solution maximizes the objective function.

5.3. Robust optimization

Consider a two bar truss example shown in Fig. 6. The robust optimization problem is intended to minimize the worst case structural compliance under total structural weight constraint considering load uncertainty. All quantities (e.g., Young's modulus, mass densities, bar lengths, etc.) involved in this example are assumed to take unit values unless otherwise stated.

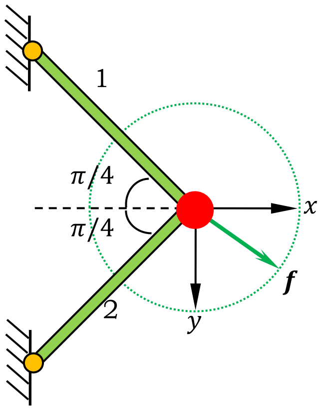


Fig. 6 Robust optimization of a two bar truss structure considering load uncertainty.

As shown schematically in Fig. 7, the global optimal solution is $A_1^* = A_2^* = 1/2$, which represents a symmetric structure.

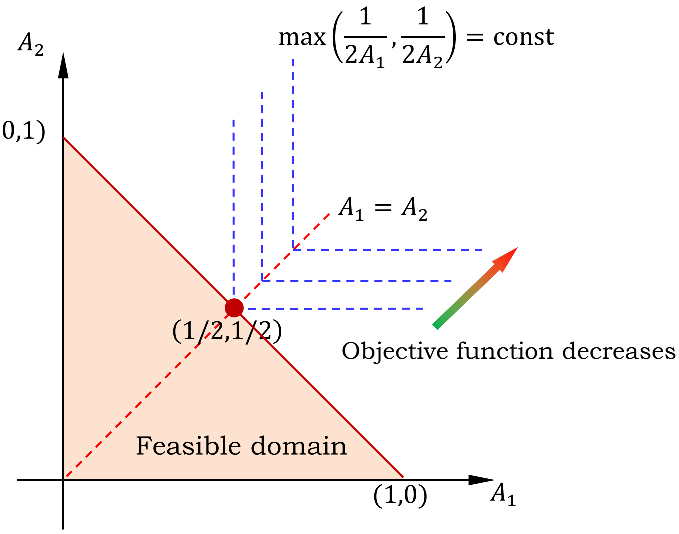


Fig. 7 Feasible domain of the two bar truss example.

5.4 Counter example violate the quasi-convexity

Let us consider the example shown in Fig. 8, which is inspired by Stolpe and Svanberg (2001). The design variables are the cross sectional areas of the bars (i.e., A_1 and A_2, respectively). The optimization problem is to minimize the total weight under the compliance constraint. The lengths, densities and Young's moduli of the bars are all assumed to be unit for simplicity. If the truss bars are made of *porous material*, the stiffness of the bars can be expressed as $k_i = E_i A_i^p / l_i$, $i = 1,2$ with $p > 1$.

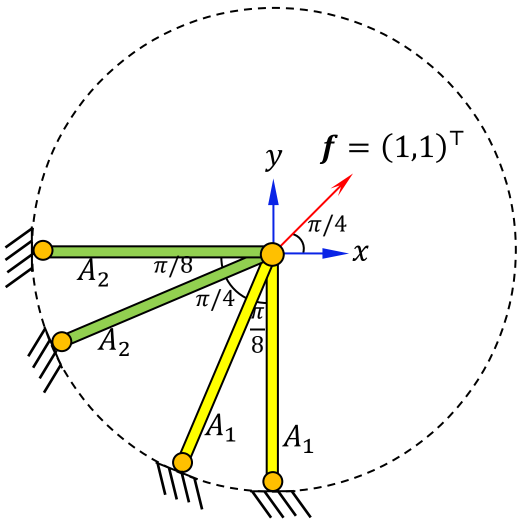


Fig.8 Four-bar truss structure.

When $p = 4$ and $\bar{c} = 8.828$, the feasible domain can be depicted schematically in Fig. 9. It can be observed that the feasible domain is symmetric but not quais-convex with respect to A_1 and A_2. This is due to the fact for this problem c is not a quasi-convex function of A_1 and A_2. Obviously, the *global* optimal solutions of this problem is (1, 0) and (0, 1) and the symmetric feasible solution (0.5378, 0.5378) is only a local optimum!

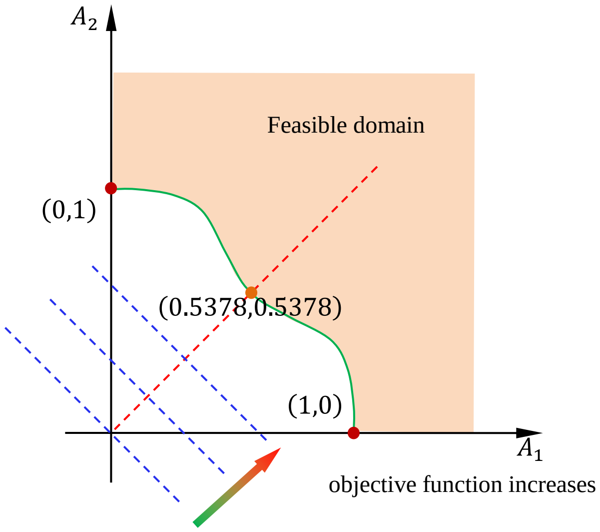
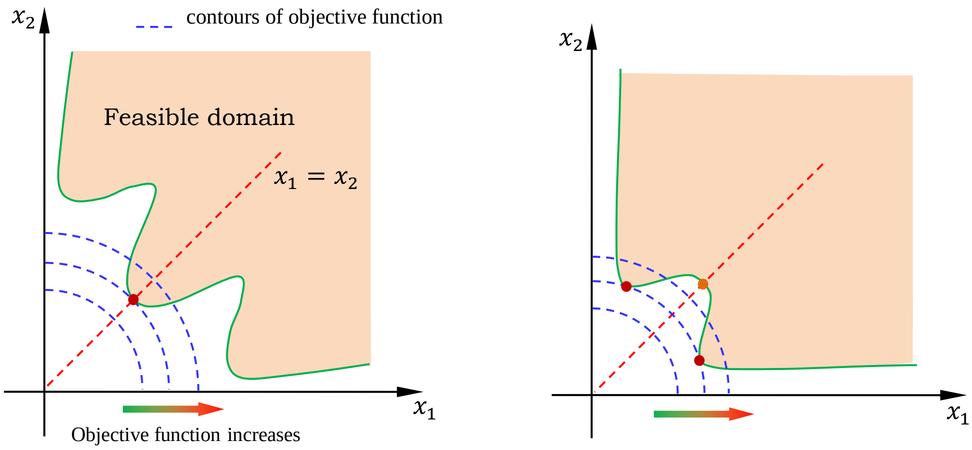


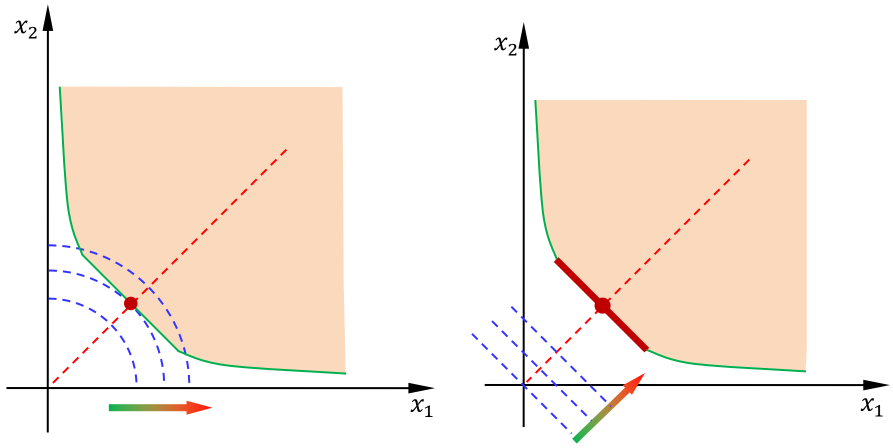
Fig. 9 The feasible domain of the four-bar truss structure.

6. Conclusions

In the present paper, symmetry properties of structural optimization problems are re-examined from a broader point of view. Some extensions of the previous theoretical results about the symmetry properties of structural optimization problems are reported. It is found that generally the condition of convexity can be relaxed to quasi-convexity in order to guarantee the existence of symmetric global optima. This is also true for robust structural optimization problems. Based on these extended theoretical results, deeper insights into the symmetry properties of optimal solutions are obtained. In general, the feasible domain of a symmetric optimization problem under different situations can be illustrated schematically in Fig. 10. As shown in the figure, the existence and uniqueness of the symmetric global optimal solution is strongly dependent on the convex properties of the objective and constraint functions involved in the problem. Cautions should be made when heuristic symmetry reduction procedures are adopted for the solution of a non-convex (non-quasi-convex) structural optimization optimizations since this may result in strongly suboptimal solutions. At last, it is worth noting that, in fact, our results are not only applicable for structural optimization problems, but also applicable for other optimization problems with the same mathematical structures (for example inverse problem and model identifications).



(a) (b)



(c) (d)

Fig. 10 Feasible domain of a symmetric optimization problem under different situations.
(a) non-quasi-convex problem with a unique, symmetric global optimal solution;
(b) non-quasi-convex problem with non-unique, asymmetric global optimal solutions;
(c) quasi-convex problem with unique, symmetric global optimal solution;
(d) quasi-convex problem with both symmetric and asymmetric global optimal solutions.

7. Acknowledgements

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