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Reliability-based Design Optimization applying Polynomial Chaos Expansion: Theory and Applications

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1. Abstract
Reliability-based Design Optimization is achieving more and more agreement in the industrial design community. In fact, most of the industrial processes are permeated by uncertainties: the manufactured product is generally different, from a geometric point of view, from the product design because of the dimensional tolerances, and, more frequently, the working point is not fixed, but is characterized by some fluctuations in the operating variables.
This uncertainty is commonly transferred to the performance of the system, which cannot be determined with an exact and single value, but which is better described by a statistical distribution of results.
In this environment, a frequent industrial requirement is the satisfaction of constraints or limits, for which the percentage of solutions not satisfying these (failure probability) must be minimized as much as possible, to improve the reliability and quality of the product.
We propose in this paper a new methodology to deal efficiently with a reliability-based design optimization problem of industrial relevance, which conjugates accuracy and short number of needed evaluations.
The methodology is derived from Robust Design Optimization implementing Polynomial Chaos, which estimates analytically therefore with high accuracy the main moments of the performance distribution. The polynomial coefficients can then be used to evaluate the complete cumulative distribution function of the performances of the design, and then to retrieve accurately the failure probability for the prescribed limits/constraints of the problem.
This paper reports the theoretical details of the methodology proposed, and includes a validation test and an application case. An analythical problem (deflection of a cantilever beam) will be first proposed in order to describe the problem and to compare the results obtained by one of the traditional approaches to solve this kind of problems (FORM) and by the new approach proposed here, highlighting the accurate convergence of the results in the two cases and the difference in terms of number of simulations required, in advantage of the new approach.
The new approach is therefore applied to a more challenging and not conventional application problem, which is the reliability-based optimization of a boomerang throw.
The boomerang trajectory is computed through the analytical integration of motion equations, using the boomerang aerodynamic coefficients varying at each time step and computed by a CFD code on a sampling set of different flying conditions and extended on the whole domain through the usage of a Response Surface in modeFRONTIER. The optimization problem consists in the reliability optimization of the throw, subjected to many uncertainties like throw angle, velocity and spin, with the purpose of minimizing the percentage of throws which do not return accurately.
Besides presenting a not conventional and challeging application case, the main purpose of the work is to stress the efficiency of the methodology that could be used in a wide range of industrial application cases.

2. Keywords: Reliability, Optimization, Polynomial Chaos.

3. Background: Need for Reliability-based Design Optimization
Reliability-based Design Optimization (RBDO) is achieving more and more agreement in the industrial design community. In fact, most of the industrial processes are permeated by uncertainties: the manufactured product is generally different, from a geometric point of view, from the product design because of the dimensional tolerances, and, more frequently, the working point is not fixed, but is characterized by some fluctuations in the operating variables.
This uncertainty is commonly transferred to the performance of the system, which cannot be determined with an exact and single value, but which is better described by a statistical distribution of results. In this environment, a frequent industrial requirement is the satisfaction of constraints or limits, which should be achieved for a specified percentage of the performance distribution, or for which the percentage of solutions not satisfying the limits (failure probability) must be minimized as much as possible, to improve the reliability and quality of the product.

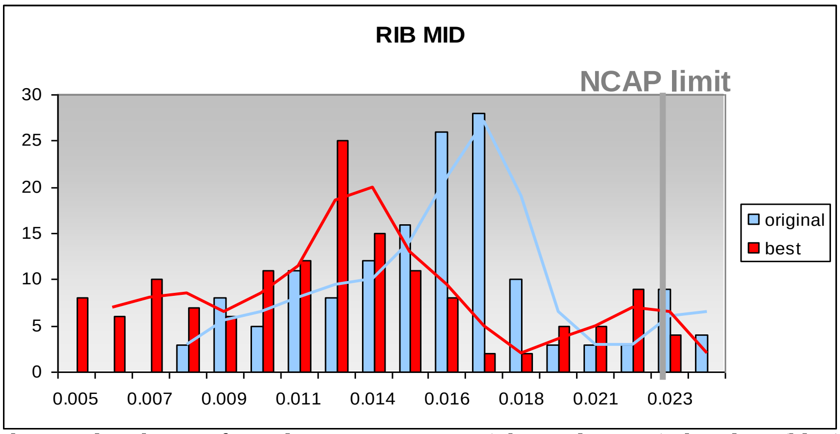

Figure 1. Crash Test distribution for a damage parameter (rib acceleration): baseline (blue) and optimized (red) restraint system

An example to better understand the problem, could be given by the aeronautical or automotive industry. In the first case, a common problem [1] could be represented by the drag minimization of an airfoil, subjected to uncertainties of relative thickness (manufacturing tolerances), Mach number and angle of attack (fluctuations during cruise), and the main requirement is to guarantee a minimum value of lift and momentum for the highest possible percentage of the stochastic distribution: if for instance this is 99%, it means that in 1% of the unpredictable flight conditions, the minimum lift coefficient is not respected, causing instability effects, commonly called “turbulences”.

In the automotive industry, a typical problem could be for instance represented by crash safety analysis: the safety restraint system should guarantee in case of impact an acceleration value on different parts of the human body less than a critical limit (NCAP normative), and of course the lower is the percentage of the crashes for which the accelerations violate this limit (considering a stochastic distribution due to the uncertainties of the system), the better is the reliability of the system (fig.1).

In literature, there are basically two types of approaches to solve this kind of problems.

One approach is the Robust Design Optimization [1,2], commonly followed by many industries employing software like modeFRONTIER [5]. RDO basically consists in evaluating, for each candidate design proposed by the optimization algorithm, the stochastic distribution of its performances, and in defining objectives based on mean and standard deviations of the same (for instance, maximize mean performances and minimize their standard deviations, in order to optimize the stability at the fluctuations). The strategy is particularly efficient, also because it may take now advantage of Polynomial Chaos Expansion (PCE) [3], an efficient methodology which exploits proper ortho-normal Polynomials to estimate analytically therefore with high accuracy the main moments of the performance distribution, i.e. mean and standard deviation, through a reduced number of sampling evaluations.

The application limit of this methodology for the reliability-based problems as described above is indeed the fact that mean and standard deviation are not enough to compute the complete stochastic distribution of the performance, therefore it is not possible to find the exact failure probability corresponding to any prescribed limit.

The other approach followed in literature is the reliability analysis which implements methodologies like FORM or SORM [4], that evaluate the failure probability of any candidate design on the basis of its uncertainties distribution and of the given limits to be respected. One limit of this methodology can be represented by the high number of evaluations that may be required by the algorithm to compute the failure probability with accuracy, which makes often practically unfeasible its application to optimization problems of industrial relevance, in particular when the computational time required for each evaluation is very expensive.

For these reasons, we propose in this paper a new methodology to deal efficiently with a reliability-based design optimization problem of industrial relevance, which conjugates accuracy and short number of needed evaluations.

The methodology is derived from the first approach described above, i.e. Robust Design Optimization applying Polynomial Chaos, which means that for each candidate design proposed by the optimization algorithm, Polynomial Chaos is applied to a small sampling set. The polynomial coefficients are at this point used to evaluate the complete cumulative distribution function of the performances of the design, from which it is possible to retrieve accurately the failure probability for the prescribed limits/constraints of the problem, on which the optimization objectives are defined.

In the following chapters all the theoretical details of the proposed methodology will be reported, and a validation and an application case will follow.

4. Classical Reliability formulation: FORM and SORM

Approximation methods, such as the First Order Reliability Method (FORM) and Second Order Reliability Method (SORM) [4] have been developed in the area of structural reliability. These methodologies have been described in detail for instance by Dr. Xiaoping Du in ref. [6]. Below, we provide a short overview.

Reliability analysis aims to estimate the probability that any structure or mechanical component will fail to meet a pre-defined criterion. A reliability problem is often formulated in terms of a vector of random input variables **X**= (X_1, X_2,…, X_N) representing uncertain quantities, such as loads, material properties, structure dimensions, environmental factors, etc. and a performance function $g(\mathbf{X})$, which describes the limiting state of the structure in terms of **X**. By convention, $g(\mathbf{X}) \le 0$ denotes the failure domain, whereas $g(\mathbf{X}) > 0$ denotes the safe set. Since the boundary between the safe and failure set is given by $g(\mathbf{X}) = 0$, the performance function is also referred to as the *limit state function* (LSF).

Instead of computing the reliability $R = P\{g(X) > 0\}$, one usually computes its complement, i.e. the *probability of failure*, which reads

$$p_f = P\{g(X) \le 0\} = \int_{g(X)\le 0} f_X(X)dX \tag{1}$$

with $f_X(\mathbf{X})$ denoting the joint probability density function (PDF) of the random variable vector **X**.

In order to overcome problems which might arise in the calculation of the failure probability integral in real-world applications, approximation methods like FORM and SORM have been developed over the years.

Two principal steps are involved in these approximation methods. The first step consists in simplifying the integrand $f_X(\mathbf{X})$ so that its contours become more regular and symmetric, the second consists in simplifying the integration domain. After the execution of these two steps, an analytical solution to the probability integration will be easily found.

The first simplification is achieved by transforming the original random variables **X** into independent random variables **U**= (U_1, U_2,…, U_N) that follow a standard normal distribution (i.e. a Gaussian distribution with zero mean and unit standard deviation). In the literature, the original variable space is called **X**-space, while the standard normal space is called **U**-space.

After the variable transformation , the failure probability becomes

$$p_f = P\{g(U) \le 0\} = \int_{g(U)<0} \phi_U(u)du \tag{2}$$

where $\phi_U(u)$ is the joint PDF of **U**.

The second simplification is achieved by approximating the limit state function (constraint satisfaction). The name First Order Reliability Method comes from the fact that the LSF is linearized with a first order Taylor expansion. The linearization is performed at a convenient point, i.e. the point that has the highest probability density on the frontier $g(\mathbf{U}) = 0$ between safe and failure set. This point is named *Most Probable Point* (MPP) and it is the LSF shortest distance point from the **U**-space origin, as shown in fig. 2. The distance between the MPP and the origin represents the so-called *reliability index* β.

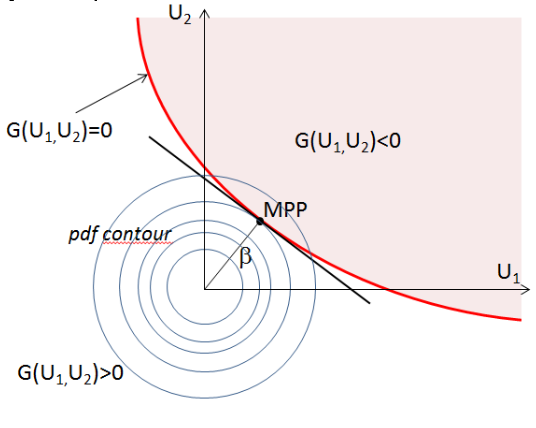

Figure 2. Reliability problem definition

The model for the MPP search can be rewritten as a constrained single-objective optimization problem:

$$\begin{cases} \min\limits_{u} \|u\| \\ subject \quad to \qquad g(u) = 0 \end{cases} \tag{3}$$

where $\|u\|$ stands for the norm (length or magnitude) of a vector. The solution to the model given in Eq.3 is the MPP.

After the two steps described above, the failure probability is easily computed as

$$p_f = \Phi(-\beta)$$

with Φ denoting the standard normal *cumulative distribution function* (CDF)

Second Order Reliability Methods account for second-order non-linearities of the performance function. Further details about the MPP search in FORM and SORM approaches are available in the literature ([4], [6]).

Here, we only stress the following facts. In industrial applications, the calculations of derivatives might be very expensive, assuming that the performance function is differentiable at all. Besides, for particular problems, FORM might not even converge, or locating the MPP might be very difficult.

5. Reliability quantification through Polynomial Chaos

In the first paragraph of this chapter we introduce the basic theory of Polynomial Chaos (PC) [3]. PC is one of the most efficient methodologies employed for uncertainty quantification. As noted in chapter 1, it is commonly used to solve RDO problems [1,7]. The main advantage of this methodology relies in the possibility of computing the mean and standard deviation of a given distribution with high accuracy by means of a reduced number of sampling points. In particular, it can be proved that the convergence to exact statistical moments follows an exponential rate with respect to the number of sampling points [8], whereas sampling techniques, such as Monte Carlo (MC) [9] or Latin-Hypercube sampling (LHS), converge much more slowly (as the inverse square root of the sampling size and as the inverse sampling size respectively).

In the second paragraph, we describe how to use PC for the purpose of this paper, i.e. reliability quantification in the presence of any constraints, as an alternative to FORM/SORM procedures.

5.1. The generalized Polynomial Chaos

Under specific conditions [10], a stochastic process can be expressed as a spectral expansion based on suitable orthogonal polynomials, with weights associated to a particular probability density function.

The basic idea is to project the variables of the problem onto a stochastic space spanned by a set of complete orthogonal polynomials ψ that are functions of random variables $\xi(\theta)$, where θ is a random event [7]. For example, the variable ϕ has the following spectral infinite dimensional representation:

$$\phi(\mathbf{x},t,\theta) = \sum_{i=0}^{\infty} \phi_i(\mathbf{x},t)\psi_i(\xi(\theta)) \tag{4}$$

Eq.4 divides the random variable $\phi(\mathbf{x},t,\theta)$ into a deterministic part, i.e. the coefficient $\phi_i(\mathbf{x},t)$, and a stochastic part, i.e. the Polynomial Chaos $\psi_i(\xi(\theta))$. The set of polynomials $\{\psi_i\}$ forms a complete orthogonal basis in the Hilbert space determined by their support. The orthogonality relation takes the form

$$\langle \psi_i \psi_j \rangle = \langle \psi_i^2 \rangle \delta_{ij} \tag{5}$$

where δ_{ij} is the Kronecker delta and <.,.> denotes the ensemble average, i.e. the inner product in the Hilbert space of the variables ξ, which reads

$$\langle f(\xi), g(\xi) \rangle = \int f(\xi) g(\xi) w(\xi) d\xi \tag{6}$$

with $w(\xi)$ a weighting function. According to the *Askey-chaos* theory [11], to any probability density function a different weighting functions $w(\xi)$ is to be selected. For example, the weight for Gaussian distribution corresponds to *Hermite* polynomials, Exponential and Gamma distributions to *Laguerre* and *generalized Laguerre* polynomials respectively, Beta distribution to *Jacobi* polynomials, and Uniform distribution to *Legendre* ones (*Wiener-Askey scheme*). In Ref. [10] it has been proven that, by choosing a polynomial chaos expansion with weights corresponding to the input-variable distribution, the expansion convergence rate is optimal (exponential). Thus, in the case of random inputs with Gaussian distribution, we represent the variable $\phi(\mathbf{x},t,\theta)$ in terms of an *Hermite* spectral representation (or *Wiener* chaos) for which the weighting function reads

$$w(\xi) = e^{-\frac{1}{2}\xi^T \xi} \tag{7}$$

apart from normalization factors.

In numerical applications, the series in Eq.4 has to be truncated to a finite number of terms (here denoted with *N)* or, equivalently, to a finite order. Hence, for a Wiener chaos, Eq.4 becomes:

$$\phi(\mathbf{x},t,\theta)=\sum_{i=0}^{N}\phi_i(\mathbf{x},t)\psi_i(\xi)=\sum_{p_1=0}^{P_1}\sum_{p_2=0}^{P_2}\ldots\sum_{p_n=0}^{P_n}\phi_{p_1p_2..p_n}(\mathbf{x},t)H_{p_1}(\xi_1)H_{p_2}(\xi_2)\ldots H_{p_n}(\xi_n) \tag{8}$$

where $H_{p_k}(\xi_k)$ is the *Hermite* polynomial of order p_k in terms of the *k*th random variable ξ_k with Gaussian distribution *N*(0,1). The number of total terms of the series in eq.4 where a Tensorial-expanded representation has been adopted, is determined by:

$$N+1=\frac{(n+p)!}{n!\,p!} \tag{9}$$

with *n* the dimension of the uncertain variables ξ and *p* the highest-order of the polynomials $\{\psi_i\}$.
By applying the orthogonality condition to the truncated spectral expansion, the expectation value and variance of $\phi(\mathbf{x},t,\theta)$ are straightforwardly found to be respectively given by

$$\begin{aligned}E_{PC}(\phi)&=\mu_\phi=\phi_0(\mathbf{x},t)\\ Var_{PC}(\phi)&=\sigma_\phi{}^2=\sum_{i=1}^{N}\left[\phi_i{}^2(\mathbf{x},t)\left\langle\psi_i{}^2\right\rangle\right]\end{aligned} \tag{10}$$

with $\left\langle\psi_i{}^2\right\rangle$ the polynomial normalization.

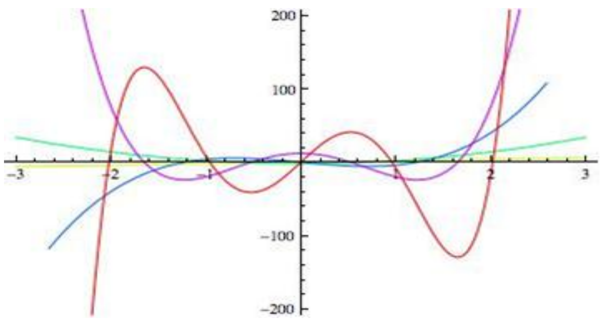

Figure 3. Example of Polynomial Chaos expansion (Hermite polynomials)

Thus, the problem of uncertainty quantification is shifted to the determination of the polynomial expansion coefficients $\phi_i(\mathbf{x},t)$. Various techniques have been developed for the solution of this problem (see ref. [3]). We follow the one described in ref. [15], using a *regression* or *collocation method* based on the least square minimization of the discrepancy between $\phi(\mathbf{x},t,\theta)$ and its truncated expansion:

$$\min\sum_{j=1}^{N_S}\left|\phi(X,t,\theta)-\sum_{J=0}^{N}\phi_j(X,t)\psi_j(\xi)\right|^2 \tag{11}$$

with N_S the number of sampling points used to evaluate the discrepancy.
The sample can be arbitrarily chosen, except for its size which has to be equal to or greater than the number of points reported in Eq.9. In this paper, we generate the sample randomly, by means of a Latin Hypercube sampling. To solve the least square problem, we use the standard Levenberg-Marquardt method.
For problems with a low-dimensional uncertainty space, like the ones considered in this paper, the computational effort required by this method is small.

5.2. Failure probability computed through Polynomial Chaos Expansion
In the previous paragraph, we described how to get the mean and variance of a function of some uncertain variables by using a Polynomial Chaos Expansion.
We have already pointed out in chapter 1 that the main moments of the distribution of the performance of any design can be used to quantify its robustness, i.e. they can be used as criteria for a RDO problem (for instance, one could maximize the mean performance and minimize its standard deviation). By the way, a RBDO problem needs,

for the optimization criteria, the definition of a reliability index and a *failure probability*, (see chapter 2 and fig. 2, which basically expresses the probability for any design to violate the given constraints).To determine the failure probability, as alternative to FORM/SORM methodologies briefly introduced in chapter 2, we propose here a different methodology, based indeed on the Polynomial Chaos Expansion.
Eq.4 or its explicit formulation for Gaussian distribution uncertainties, i.e. Eq.8, can in fact be used as a *meta-model* for the $\phi(\mathbf{x}, t, \theta)$ function, which represents basically the output performance function of the input variables $\boldsymbol{X}$ and t and of the random event θ or uncertain parameters $\xi(\theta)$. The PCE can be used to determine, by means of a MC or LHS, an accurate CDF of the function ϕ. The evaluation of the performance function in industrial use-cases can be very demanding, since it often involves expensive CFD or structural numerical simulations. In our strategy, these expensive evaluations are required only to determine the coefficients of the PCE. Once found, the evaluation of the meta-model on any sample is practically free in terms of CPU. Once the CDF is accurately obtained, from the given constraint value we can easily retrieve the corresponding percentage in the distribution, i.e. the *failure probability* (fig.4).

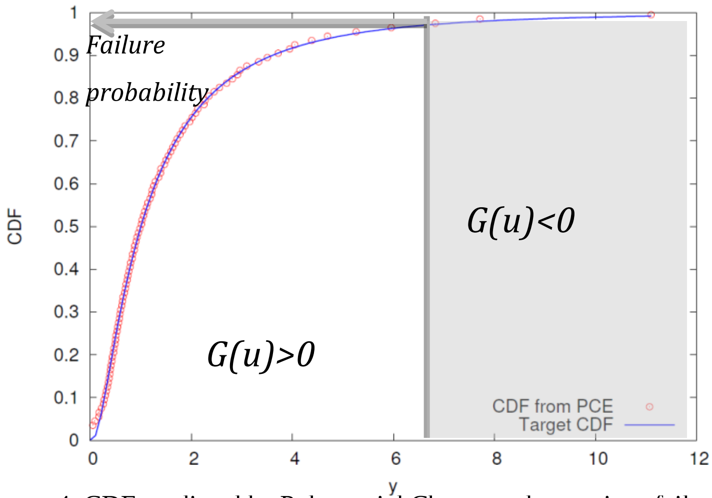

Figure 4. CDF predicted by Polynomial Chaos used to retrieve failure probability

In this way, a Robust Design Optimization problem can be defined, using as criteria for the optimization the minimization of the *failure probability*: in other words, we search for a new design for which the failure probability for the given uncertainties distribution is minimum. The big advantage of this approach with respect to using FORM/SORM methodologies is the reduced number of sampling points needed to obtain the Polynomial Chaos based meta-model (Eq.4), if compared to the iterations needed to compute the reliability index for each design required by FORM/SORM methodologies.
In the following chapter, we illustrate a comparison of the two procedures for a given test case.

6. Reliability optimization test by two approaches: FORM and Polynomial Chaos
In ref.[12] a classical reliability-based optimization problem is reported, which basically consists in:

$$\text{Maximize } y$$
$$\text{subject to } \begin{cases} x^2 - 1000y \geq 0 \\ y - x + 200 \geq 0 \\ x - 3y + 400 \geq 0 \\ -400 \leq x, \quad y \leq 300 \end{cases} \tag{12}$$

In this problem, $\mathbf{x}$ = (x,y) are uncertain variables, and we assume independent and normally distributed uncertainties with $\sigma_x = \sigma_y = 10$ and a desired reliability index of $\beta=4$ (which corresponds to about 1.4 10^{-4} % of failure probability).
Fig.5-left illustrates the variable space with the feasible region highlighted in grey. The solution A is the ideal global optimum, since the circle represents a reliability index β equal to 4 for the given uncertainties, and it is the solution which maximize y respecting the given constraint.
In [12] different algorithms to solve the reliability-based optimization problem are proposed, most of them based on GA, EA or other multi-objective optimization algorithms, which basically aim to optimize the objective function and maximize the reliability index at the same time, computing the index with FORM or SORM approximation techniques as introduced in chapter 2. Results of the application of this approach will be reported in

table 1 moreover.
To apply instead our approach defined in chapter 3, we first need to set up the Polynomial Chaos Expansion. In particular, fig.5-right reports a validation test for the Polynomial Chaos approximation. The red curve represents the real CDF distribution curve of the quadratic constraint for the optimal design point A. The green curve is instead obtained by Monte Carlo approximating the quadratic constraint function by a meta-model based on the Polynomial Chaos Expansion as described in chapter 3.1, in particular using an order of expansion equal to 3 and 12 sampling points. As it is possible to observe (the blue line reports the differences between the two curves), the accuracy is particularly high, with a mean squared error of about 1.55E-4, which means that we can compute the CDF of any design practically without significant errors just using a sampling of 12 points.

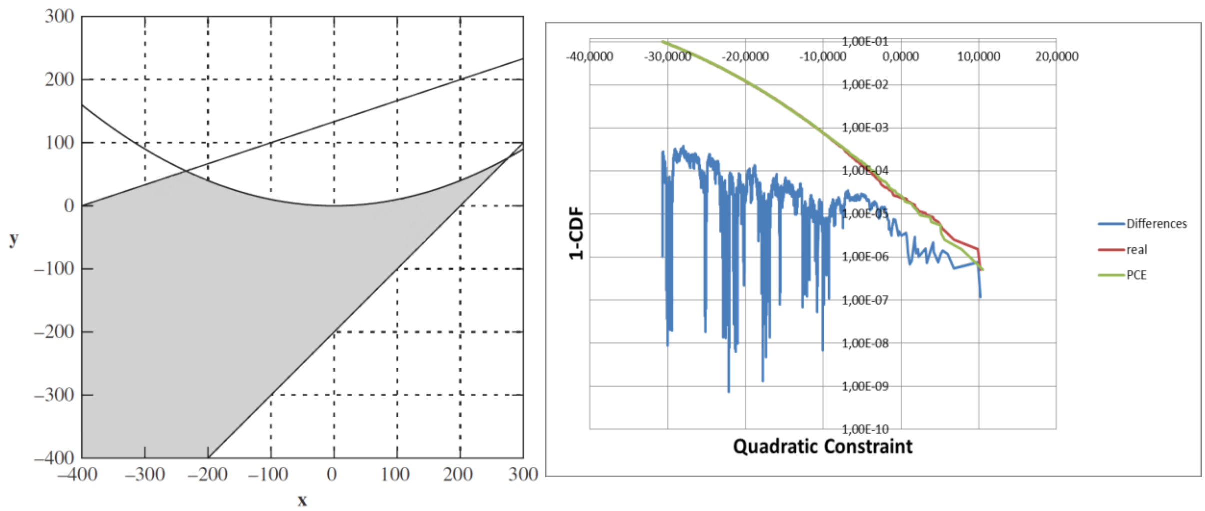

Figure 5. Reliability design space (left); CDF of quadratic constraint predicted by PCE vs real (right)

At this point, the optimization problem can be solved by applying a specific algorithm, Simplex, available in modeFRONTIER multi-objective design environment software [5], in which the complete methodology of CDF approximation by means of Polynomial Chaos is implemented as well, obtaining a value close to the optimal solution of the optimization problem in very few iterations.
This number of iterations (50), combined with the number of samples needed to compute the CDF for the design defined at each iteration (12), gives an overall number of needed samples, 600, particularly small, if compared to the other approaches.
Table1 reports in fact the comparison of the ideal (analytical) solution with the results obtained by the two approaches: both methodologies reach the same ideal global optimum, but the approach based on Polynomial Chaos requires a much lower overall number of design simulations. The results from FORM are obtained as indicated in the reference [12], while the reported range of results with the PC technique corresponds to the repetition of the procedure 50 times, starting from different initial points in order to validate the robustness of methodology.
After having validated the efficiency of this methodology, in the next section we will apply it to solve a problem of industrial relevance, even if not conventional.

Table 1: Results and convergence of FORM-based algorithm and Polynomial Chaos method

Method	Optimal solution (y)	Number of evaluations
Ideal solution	**11.820**	**-**
FORM (GA)	**close to 11.820**	**1200**
PC	**[11.4-12.2]**	**600**

7. Application: Reliability Optimization of a Boomerang throw by Polynomial Chaos

The realistic application problem we propose in this section is the RBDO of a boomerang throw (figure 6 below). The boomerang trajectory is computed through the Runge-Kutta integration of motion equations, taking in account the boomerang aerodynamic coefficients in function of angle of incidence and speed which vary at each time step.

To evaluate these coefficients, a CFD code (STAR-CCM+) is used on a sampling set for each boomerang design, and the results are extended on the whole domain through the usage of a Response Surface in modeFRONTIER. The complete loop to define an automatic process flow and to optimize the shape of the boomerang for given objectives (minimum energy for launch, maximum range and maximum returning accuracy) is described in one of our previous works [13].
In this paper, however, we extend the optimization problem to a reliability-based optimization concept, since we want to optimize the boomerang throw, subjected to many uncertainties like throw angles, velocity and spin, with the purpose of minimizing the percentage of throws which do not return accurately, i.e. the *failure probability*.
For simplicity reasons, we assume that the geometry of the boomerang is given, as the optimal one found by the deterministic optimization described in ref. [13], and corresponding to the one reported in fig.6-right.

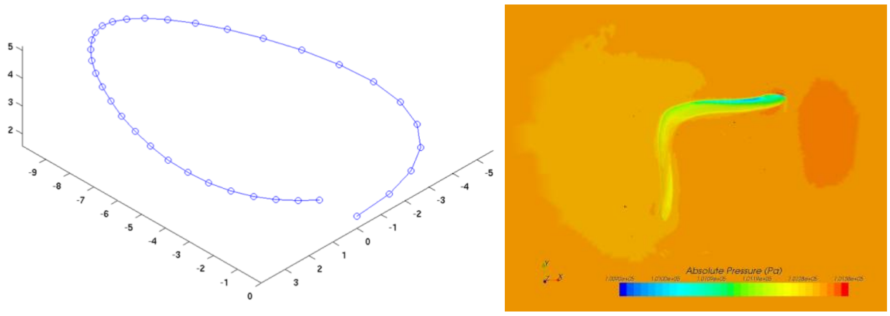

Figure 6. Boomerang trajectory (left); CFD analysis of boomerang flight (right)

The input variables for the reliability optimization problem are therefore just the ones related to the boomerang throw, as expressed in table 2 below with their range of variation and uncertainty.
More in particular, *V* is the initial boomerang translational velocity, *Spin* is the initial boomerang spin, *Aim* is the angle between the initial boomerang translational velocity and the horizontal plane, and *Tilt* is the angle between the initial boomerang rotational plane and the vertical axis (0° tilt corresponds to a vertical boomerang plane of rotation).
The range of variation includes all feasible values that can be obtained by a common launch, and the uncertainties (to be quantified just by their standard deviation considering a Gaussian distribution) take into account any possible random variation that a generic thrower (not a professional one of course) might produce.
The optimal set of launching parameters will therefore give the lowest failure probability taking into account the random perturbation due to the thrower behavior or other random events. As optimization objective, we arbitrarily consider the minimization of the 99-th percentile of the return distance: this means that the 99% of throws will return to a distance lower or equal to this value, and the purpose is to minimize it.
For a more efficient result, we add a second objective, which is the maximization of the range of the throw; in this case, since there is not any particular constraint to achieve, we just consider the average value of range.
Table 2 below reports a summary of the input variables range of variation, their uncertainty parameters (considering a Normal distribution for each one), and the definition of the objectives.
To solve the RBDO problem as described above, modeFRONTIER software has been applied. Figure 7 below reports the workflow that has been created in order to have an automatic execution of the simualtion codes for each design proposed in the optimization.

Table 2. Input variables and objectives considered for the reliability-based optimization

Input variables	**Range of variation**	**Uncertainty (standard deviation)**
Velocity (V)	[5-30]m/s	2m/s
Spin	[0-10] Hz	1Hz
Aim angle	[0-30]°	2°
Tilt Angle	[0-50]°	2°
Objectives	**Goal**	
Returning distance RD	Minimize 99-ile of RD	
Range	Maximize average value	

In particular, looking at the workflow from top to bottom, the input variables with their ranges and properties are first defined by the corresponding nodes, and they are then connected to the simulation software, in this case the Runge-Kutta script coded in Matlab and needed to calculate the trajectory, on the basis of the aerodynamic coefficients previously computed by another modeFRONTIER workflow for the designed geometry. The performances needed to compute the objectives (in particular range and arrival distances) are then extracted by

Matlab, and the objectives are automatically computed using the Polynomial Chaos formulation .

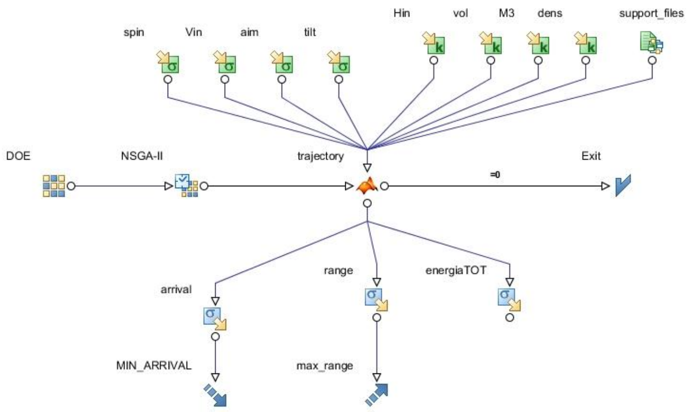

Figure 7. modeFRONTIER workflow for boomerang trajectory simulation

A proper multi-objective optimization algorithm available in modeFRONTIER is at this point selected (NSGA-II [14]), and the optimization is executed automatically, running 10 generations of 16 designs each, for an overall number of proposed designs equal to 160. For each of these designs the Polynomial Chaos Expansion is applied to evaluate the distributions performance, and for this purpose, considering 4 uncertainties and a Polynomial degree equal to 2, 15 sampling points are needed for each design.
As a conclusion, after an overall number of 2400 different evaluations (which is rather a low number for a reliability-based multi-objective optimization problem with 4 uncertainties), the results reported in table 3 are obtained.

Table 3. Optimization results

Input variables	**Original design**	**Optimized design**
Velocity (V)	21.6 m/s	21.7m/s
Spin	4.98 Hz	4.92Hz
Aim angle	4.2°	4.2°
Tilt Angle	20.1°	7.2°
Objectives		
Returning distance RD	8.5m	2.9m
Range	33.4m	21.7m

Two optimal results have been selected and reported in the table: the first one gives optimal performances for the range, while the second guarantees the most accurate throw: in 99% of the cases the returning distance will be less than 2.9m. Finally, fig.8 reports the complete results of the optimization: in abscissa we have the 99-ile of the return distance and in ordinate the average range, while the color represent the nominal value of the return (or arrival) distance.

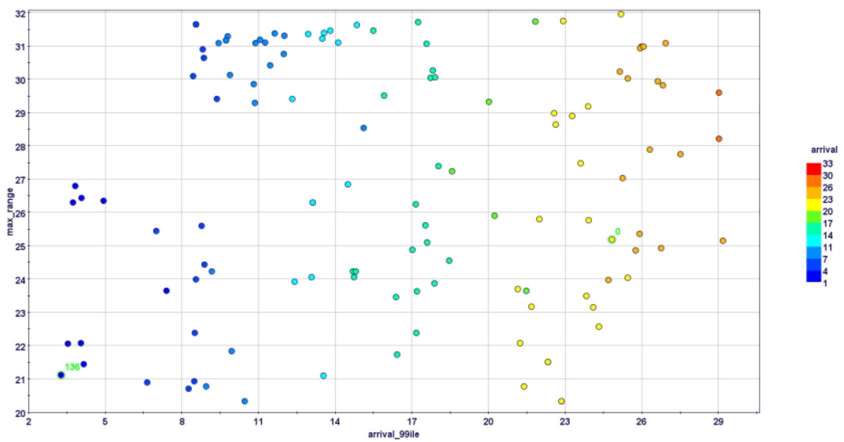

Figure 8. Optimization results: objectives space

8. Conclusion

In this paper we perform a reliability-based multi-objective optimization by computing the distribution parameters (percentiles) needed to define objectives or constraints by means of Polynomial Chaos Expansion, which allow to obtain accurate responses with the lowest possible number of design evaluations, for each configuration proposed by the optimization algorithm. The methodology has been first validated through a mathematical example, proving the rapidity with respect to traditional methodologies, and then applied to the optimization of a boomerang launch, in order to guarantee the 99% of throws returning at a minimum distance.

Besides presenting the not-conventional and challenging application case, the main purpose of the work presented in this paper is to stress the efficiency of the methodology that could be used in a wide range of industrial applications.

9. References

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