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# The Effect of Ignoring Dependence between Failure Modes on Evaluating Structural Reliability

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**Abstract**
Statistical dependence between failure modes induces a difficulty in evaluating structural reliability. Instead, the dependence is often ignored and the sum of individual reliabilities has been used as a conservative approximation. However, it is not well known that the error of ignoring dependence between failure modes depends on the level of reliability. In many cases, engineering structural systems require a high level of reliability in which a structural failure is rarely seen. In such a rare failure event, the dependence between failure modes becomes loose, and therefore, the error of ignoring dependence also becomes small. This paper focuses on quantifying the effect of ignoring dependence in multiple failures modes with various copula models. It is found that the effect is ignorable even with the presence of high statistical dependence, $\rho$ =0.8, between failure modes when the reliability index of interest is greater than 3.6.
**Keywords:** Multiple failure modes; Statistical dependence; Reliability evaluation

## Introduction

Calculating reliability has been recognized as an important part in structural design due to various sources of uncertainty. Due to natural variability in loads, materials and manufacturing processes, identically manufactured structures from the same design have randomness in their failure events. Hence, structural reliability has to be calculated.

A structural failure is often associated with multiple failure modes, and statistical dependence occurs when different failure modes interact. To calculate the probability of failure accurately, the effect of statistical dependence has to be properly taken into account.

Many reliability analysis methods, such as surrogate-based methods, sampling-based methods (e.g. Monte Carlo simulation and importance sampling) and MPP-based methods (e.g. FORM and SORM), have been developed. However, evaluating structural reliability including dependence in failure modes is still challenging. Surrogate-based methods and sampling-based methods easily account for dependence, however the methods suffer the computational burden with high dimensions, often called curse of dimensionality. MPP-based methods are computationally effective, but they cannot take into account dependence in failure modes accurately. Consequently, approximate approaches, such as the PNET method and the lower-upper bound method, have been developed. However they inevitably introduce error in ignoring statistical dependence between failure modes.

It is not well known that error due to ignoring dependence is smaller at the high reliability levels. Often structures are required to be highly reliable. For example, the U.S. Army’s introduction of a structural fatigue reliability criterion for rotorcraft has been interpreted as a requirement for component lifetime reliability of 0.999999 [13]. With such a high level of reliability, component failures are extremely rare events. For such rare events, dependence in component failure modes can be negligible. In other words, taking into account dependence in failure modes may no longer an issue for designs of highly reliable components.

The objective of this paper is to study the effect of ignoring dependence on reliability using various examples, in terms of error in probability of failure. In addition, the factors that affect the error in probability of failure are discussed. A structural failure is defined with a limit state function, and the variability in limit state is modeled as a distribution. For multiple failure modes, the concept of copula is used to model dependence between failure modes. Probabilities of failure considering dependence and ignoring it are compared to see the effect of ignoring dependence.

It is observed that for high reliabilities neglecting dependence between failure modes gives only 10% error in calculating probability of failure even with a strong dependence between two failure modes (i.e. $\rho$ = 0.8). The error is sensitive to the ratio of the standard deviation of limit states and the difference between the means of limit states.

## Error from neglecting the dependence between component failures

A structural failure is commonly defined with multiple failure modes. The general formulation of structural failure is defined in terms of failure modes. Let $\mathbf{X}=\{X_1,X_2,...,X_n\}^T$ be the vector of input model uncertainties for a structure, such as variability in loads, variability in strength of materials. Uncertainty in input variables is defined with cumulative distribution functions, $F_{X_i}\left(x_i\right)$. Failure modes can be defined with limit state functions with the input model uncertainties. The limit state function is defined here in such a way that the failure event occurs when

$$G_i\left(\mathbf{X}\right)\le 0 \tag{1}$$

while the structure is intact when

$$G_i\left(\mathbf{X}\right)> 0 \tag{2}$$

where the subscript $i$ represents $i^{\text{th}}$ failure mode of system, and $\mathbf{X}$ is a vector of random inputs. If $G_i\left(\mathbf{X}\right)\triangleq G_i$ The failure event of $i^{\text{th}}$ failure mode is defined by $F_i=\left\{G_i\left(\mathbf{X}\right)\le 0\right\}$, and the probabilities of failures for potential failure modes can be defined as

$$P_{f\,i}=\Pr\left(G_i<0\right) \tag{3}$$

In general, determining structural failure in terms of multiple failure modes is not easy since they commonly have statistical dependence. Two commonly used concepts for multiple failure modes are a series failure model where the structure fails if any of its failure mode is activated, and a parallel model where the structural fails if all of its failure modes are activated [8]. The series failure model takes account of union of failures, and the parallel model takes account of intersection of failures. Both models are affected by dependence of failure modes. In this study, the series model is used to show the effect of ignoring dependence, since it is more common in structural design.

A series model composed of $N$ failure modes can be defined using $N$ limit functions. Input model uncertainties are propagated to uncertainties in multiple limit states $G_{\text{i}}$. Statistical dependence between failure modes appears as dependence between limit states. If $F_i$ represents the failure event of $i^{\text{th}}$ failure mode, the probability of failure for dependent multiple failure modes (or exact probability of failure) is obtained as

$$\begin{aligned} P_{f\,sys} &= \Pr\left(\bigcup_{i=1}^{N} F_i\right) \\ &= \Pr\left[\bigcup_{i=1}^{N}\left\{G_i\le 0\right\}\right] \end{aligned} \tag{4}$$

Using the well-known expansion theorem for the probability of the union of events,

$$\Pr\left(\bigcup_{i=1}^{N} F_i\right)=\sum_{k=1}^{n}\left(-1\right)^{k-1}S_k \tag{5}$$

$$S_k=\sum_{1\le i_1\le i_2\le\ldots\le i_k\le n}^{k}\Pr\left(F_{i_1}\cap F_{i_2}\cap\ldots\cap F_{i_k}\right) \tag{6}$$

Since evaluating the probability of the union of events is difficult, assuming independence would be preferable if the assumption does not lead to large error in the probability of failure evaluation. With the independence assumption, the probability of failure can be calculated based on the marginal probabilities of failure. The system probability can be simply calculated by replacing the Eq. (6) with the marginal probabilities of failure. The probability of the intersection region is approximated as

$$S_k=\sum_{1\le i_1\le i_2\le\ldots\le i_k\le n}^{k}\Pr\left(F_{i_1}\right)\Pr\left(F_{i_2}\right)\ldots\Pr\left(F_{i_k}\right) \tag{7}$$

Figure 1 shows the difference in system probability of failure with and without considering dependence for two failure modes. When the independent assumption is used, the joint PDF is equal to the product of two marginal PDFs of limit states. The probability of intersection is equal to the product of the probabilities of failure of each mode.

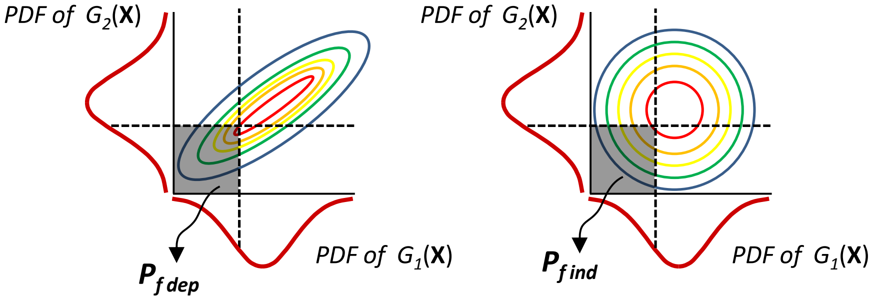
The effect of ignoring dependence appears as error in the probability of intersection and the error is propagated to error in probability of failure. The error by ignoring dependence is calculated as

$$\text{Error}=\left|\frac{P_{f\,ind}}{P_{f\,sys}}-1\right|\times 100\ \left(\%\right) \tag{8}$$

where Pfsys is the system probability of failure with considering dependence, while Pfind is the one without considering dependence. The error can also be expressed in terms of the reliability index as

$$\text{Error}=\left|\frac{\beta_{ind}}{\beta_{sys}}-1\right|\times 100=\left|\frac{-\Phi^{-1}\left(P_{f\,ind}\right)}{-\Phi^{-1}\left(P_{f\,sys}\right)}-1\right|\times 100\ \left(\%\right) \tag{9}$$

where $\beta_{ind}$ is reliability index ignoring dependence and $\beta_{sys}$ is reliability index considering dependence (exact reliability index). $\Phi^{-1}(\bullet)$ is the inverse CDF of the standard normal distribution. Reliability index is defined with probability of failure as $\beta = -\Phi^{-1}(P_f)$. The errors in reliability index are shown in the numerical example section.



(a) Joint PDF considering dependence (b) Joint PDF ignoring dependence

**Figure 1:** Difference between system probabilities with considering dependence and ignoring dependence

**Copula models**

The main focus of this section is to review how to model a joint PDF using the theory of copulas. The word 'copula' is a Latin noun which means "a link". The word was employed in a statistical term by Sklar (1959) in the theorem describing the functions which join together marginal cumulative distribution functions (CDF) to form multivariate CDF [2]. In this context, copula is a function that links multivariate CDFs to their marginal CDFs [3,4,5]. This implies that copulas are multivariate CDF whose one-dimensional margins are uniform on the interval (0,1). Copulas and their application are important concepts for modeling a joint CDF.

Let $\mathbf{X} = \{X_1, X_2, ..., X_n\}^T$ be a $n$-dimensional random vector. And the random variables are defined by CDFs, $F_{X_i}(x_i)$.

$$F_{X_1,...X_n}(x_1,...,x_n) = \Pr(X_1 \le x_1,..., X_n \le x_n) \tag{10}$$

Then the multivariate CDF of the random vector is defined using copula function as

$$F_{X_1,...X_n}(x_1,...,x_n) = C\left(F_{X_1}(x_1),..., F_{X_n}(x_n)\right) \tag{11}$$

where C is a copula function. All arguments of the copula function have a domain of [0, 1], because CDFs have a domain of [0, 1]. Also, due to the property of multivariate CDF, the output of the copula function also has a domain of [0, 1].

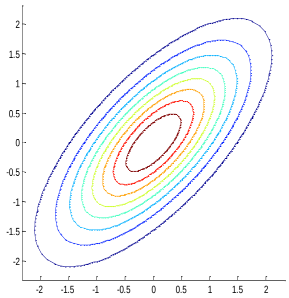
As shown in Eq. (10), a copula makes a connection between marginal CDFs and multivariate CDFs. The joint PDF of the random vector **X** is obtained as

$$f_{X_1,...X_n}(x_1,...,x_n) = c\left(F_{X_1}(x_1),..., F_{X_n}(x_n)\right)\prod_{i=1}^{n} f_{X_i}(x_i) \tag{12}$$

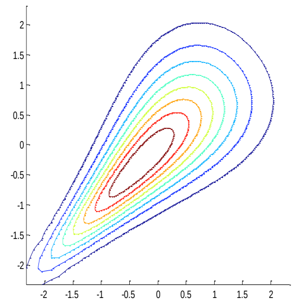
$$c(u_1,...,u_n) = \frac{\partial^n C(u_1,...,u_n)}{\partial u_1 ... \partial u_n} \tag{13}$$

Two commonly used simple copulas are elliptical copula and Archimedean copula. Elliptical copulas belong to a class of symmetric copulas because the horizontal cross-sections of their joint PDFs take the shape of ellipse; a simple linear transformation can transform the elliptical cross-section to circular one. Two widely used elliptical copulas are the Gaussian and the t copula. Note that the multivariate Gaussian distribution is a multivariate CDF that is modeled with the Gaussian copula and the normal distributions as its marginal distributions. Archimedean copulas are an associative class of copulas. While the Gaussian copula expresses the implicit formula of $C$ with standard normal distributions, most common Archimedean copulas admit an explicit formula for the copula function $C$.

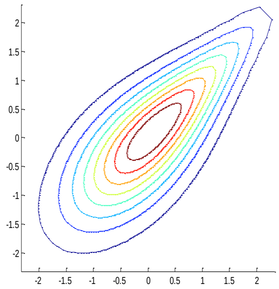
For modeling dependence in failure modes, this paper uses the Gaussian, Claton, Gumbel, Frank copulas. Note that the Gaussian copula is elliptical, while all others are Archimedean. All the copulas are defined with a single parameter that defines the strength of dependence; the linear correlation coefficient for the Gaussian copula and Kendall's tau for other copulas. The linear correlation coefficient is not scale invariant, whereas the Kendall's tau is scale invariant. For the same linear correlation coefficient, Kendall's tau can give different values for different copulas. Figure 2 shows a contour of each copula with two standard normal marginal distributions. Kendall's tau of each copula is determined in such a way that the corresponding linear correlation coefficient becomes 0.7.



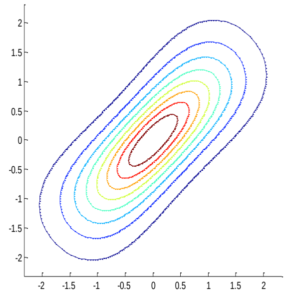

(a) Gaussian copula




(b) Claton copula




(c) Gumbel copula




(d) Frank copula

**Figure 2:** Various joint PDF shapes with different copulas with two standard normal marginal distributions with the linear correlation coefficient 0.7

**Numerical example 1: The effect of ignoring dependence with two failure modes**

In this section, the effect of ignoring dependence is shown with the four bivariate copulas and two normally distributed marginal distributions. Since the effect of the dependence is most important when the two marginal probabilities of failure are comparable, it is assumed that limit state distributions are the same normal distribution as $G_1 \sim N(z,1)$ and $G_2 \sim N(z,1)$. By changing the mean values of the marginal distributions, various levels of probability are investigated. The copulas represent various types of dependence between failure modes. For each copula model, system probability of failure considering dependence is calculated with Eqs. (5) and (6), while the system probability of failure ignoring dependence is calculated with Eqs. (5) and (7). Then, the error in the system probability of failure is calculated in terms of reliability index.

For modeling dependence between two failure modes, marginal probability of failure in defined. Limit states follow the normal distribution with the mean of z and the standard deviation of 1.

$$\begin{aligned} \Pr(F_1) &= \Pr(G_1 < 0) \\ &= F_{G_1}(0; z, 1) \\ \Pr(F_2) &= \Pr(G_2 < 0) \\ &= F_{G_2}(0; z, 1) \end{aligned} \tag{14}$$

The intersection probability of failure in Eq. (6) is defined with copulas. Since the failure is associated with negative values of limit states, the intersection probability of the two failures can be expressed as

$$\begin{aligned} \Pr(F_1 \cap F_2) &= \Pr(G_1 < 0, G_2 < 0) \\ &= C\left(F_{G_1}(0; z, 1), F_{G_2}(0; z, 1), \theta\right) \end{aligned} \tag{15}$$

where $F_{G_1}$ and $F_{G_2}$ are marginal CDFs of limit states with mean of *z* and standard deviation of 1. *C* is a bivariate copula with a parameter $\theta$ that represents the strength of dependence. The magnitude of probability of failure and the strength of dependence are controlled by changing *z* and $\theta$, respectively. The system probability of failure considering dependence is rewritten as

$$
\begin{aligned}
P_{f\,sys} &= \Pr\left(G_1^* < 0\right) + \Pr\left(G_2^* < 0\right) - \Pr\left(G_1^* < 0 \cap G_2^* < 0\right) \\
&= F_{G_1}\left(0; z, 1\right) + F_{G_2}\left(0; z, 1\right) - C\left(F_{G_1}\left(0; z, 1\right), F_{G_2}\left(0; z, 1\right), \theta\right) \qquad (16) \\
\beta_{sys} &= -\Phi^{-1}\left(P_{f\,sys}\right)
\end{aligned}
$$

In contrast, probability of failure assuming independence is expressed as

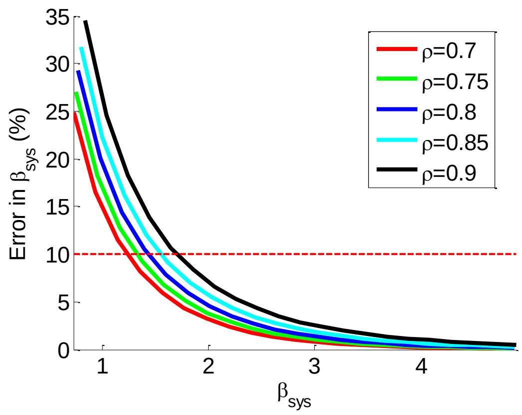
$$
P_{f\,ind} = F_{G_1}\left(0; z, 1\right) + F_{G_2}\left(0; z, 1\right) - F_{G_1}\left(0; z, 1\right) F_{G_2}\left(0; z, 1\right) \qquad (17)
$$

The Error is calculated with Eqs. (8) and (9). The strength of dependence is usually categorized from very weak to very strong in terms of the linear correlation coefficient, $\rho$, as shown in Table 1 [11]. For this example, the error due to ignoring dependence is calculated for strong to very strong correlation (0.7-0.9) and the system probability of failure level of $10^{-1}$ to $10^{-6}$ with the four copula models.

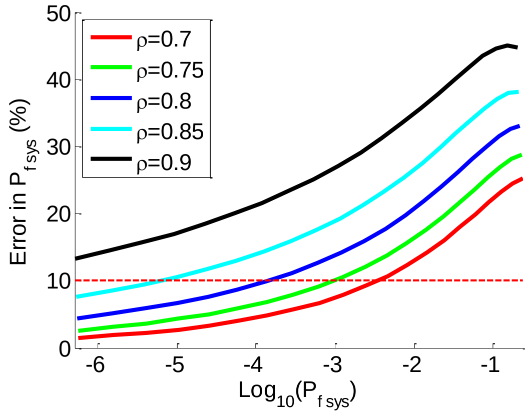
**Table 1.** The strength of dependence

| Range of $\rho$ | [0,0.2] | [0.2,0.4] | [0.4,0.7] | [0.7,0.9] | [0.9,1.0] |
|---|---|---|---|---|---|
| Strength | Very weak | Weak | Moderate | Strong | Very strong |

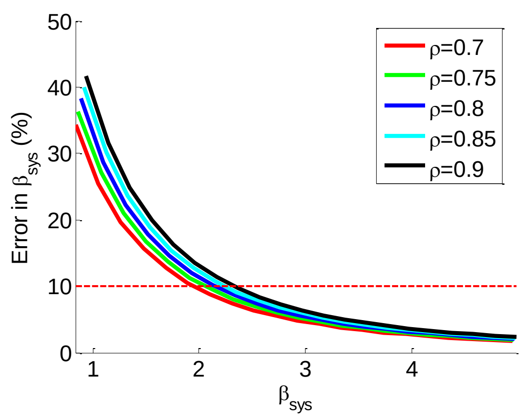
The error in reliability index and the probability of failure by ignoring dependence is shown in Figure 3 with respect to the different strengths of linear correlation coefficient. To change the level of probability of failure and reliability index, different z values are applied to Eqs. (16) and (17). As mentioned before, the error depends on the level of probability of failure. In most cases, the error in reliability index converges to zero as the reliability index increases. That is, the error in no considering dependence is ignorable when the level of reliability is high. In Table 2, the magnitudes of reliability index that has 10% and 1% errors are shown. In Table 3, the corresponding magnitudes of probability of failures for 10% and 1% errors are shown. Note that the relative error in reliability index is smaller than that of the probability of failure.



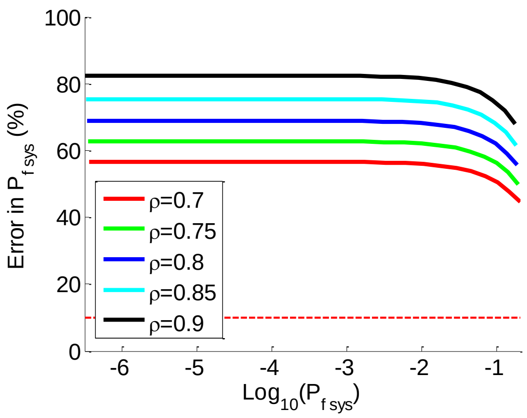
(a) Error vs reliability index (Gaussian)



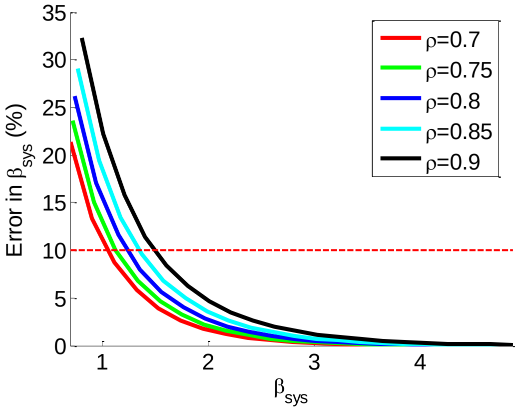
(b) Error vs log probability of failure (Gaussian)



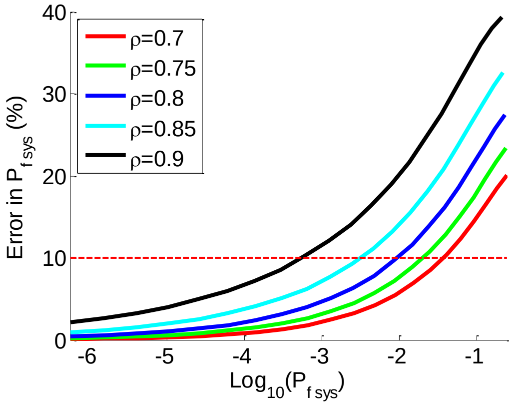
(c) Error vs reliability index (Clayton)



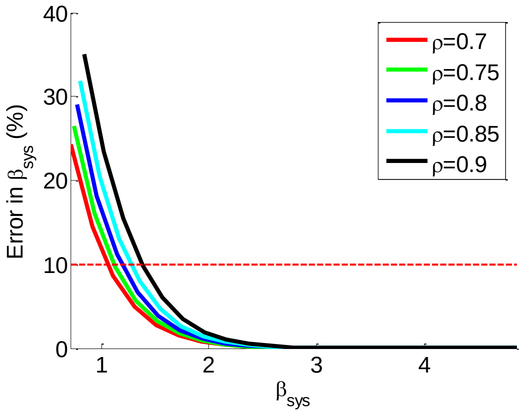
(d) Error vs log probability of failure (Clayton)



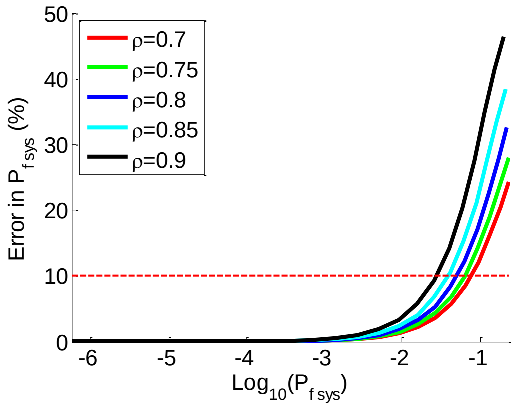

(e) Error vs reliability index (Gumbel)




(f) Error vs log probability of failure (Gumbel)




(g) Error vs reliability index (Frank)




(h) Error vs log probability of failure (Frank)

**Figure 3:** The relative error with respect to the strength of the magnitude of system probability of failure; the red dashed line represents 10% error. The ordinate of error vs reliability index plots is error in reliability index and the ordinate of error vs log probability of failure is error in probability of failure (not error in logarithmic probability of failure)

For dependence between failure modes that are modeled with the Gaussian, Gumbel and Frank copula, the effect of ignorance decreases as reliability index increases. Only the Clayton needs a high beta for 1% error. For the Clayton copula model, error in reliability index decreases as reliability increases but error in probability of failure increases as reliability increases because error in probability of failure is much sensitive than error in reliability index. For example, system probability of failure considering dependence is 0.1 and corresponding probability ignoring dependence is 0.18, there is 80% error in probability of failure and 22% error in reliability index. 100 % error in the probability of failure $10^{-5}$ is equivalent to 4% error in terms of reliability index. In Table 3, for the Clayton copula, probability of failure values corresponding to 10% and 1% errors are not available since error increases as probability of failure decreases.

**Table 2.** Reliability index for 10%~~,~~ and 1% errors (in reliability index) with respect to the strength of dependence measured by the linear correlation coefficient

| Copula | Error \ ρ | ρ =0.7 | ρ =0.75 | ρ =0.8 | ρ =0.85 | ρ =0.9 |
|---|---|---|---|---|---|---|
| Gaussian | 10% | 1.23 | 1.32 | 1.43 | 1.55 | 1.70 |
| | 1% | 2.82 | 3.03 | 3.28 | 3.60 | 4.04 |
| Clayton | 10% | 1.95 | 2.05 | 2.15 | 2.24 | 2.33 |
| | 1% | 6.50 | 6.72 | 6.92 | 7.10 | 7.27 |
| Gumbell | 10% | 1.04 | 1.13 | 1.23 | 1.35 | 1.49 |
| | 1% | 2.27 | 2.42 | 2.60 | 2.82 | 3.12 |
| Frank | 10% | 1.05 | 1.11 | 1.19 | 1.27 | 1.38 |
| | 1% | 1.86 | 1.92 | 1.98 | 2.06 | 2.17 |

**Table 3.** Logarithmic system probability of failure for 10%, and 1% errors in probability of failure (not error in logarithmic probability of failure) with respect to the strength of dependence measured by the linear correlation coefficient

| Copula | Error | ρ =0.7 | ρ =0.75 | ρ =0.8 | ρ =0.85 | ρ =0.9 |
|---|---|---|---|---|---|---|
| Gaussian | 10% | -2.46 | -3.02 | -3.85 | -5.19 | -7.83 |
| | 1% | -7.08 | -8.35 | -9.86 | -11.89 | -15.48 |
| Clayton | | N/A | N/A | N/A | N/A | N/A |
| Gumbell | 10% | -1.41 | -1.68 | -2.02 | -2.49 | -3.25 |
| | 1% | -3.81 | -4.32 | -5.02 | -6.07 | -7.57 |
| Frank | 10% | -1.21 | -1.22 | -1.31 | -1.41 | -1.56 |
| | 1% | -2.11 | -2.21 | -2.35 | -2.48 | -2.57 |

It is observed that the shape of copula affects error in probability of failure. Joint PDF contour of the Clayton copulas has pointed contour shape in the failure region and error in probability of failure increases as probability of failure decreases. The Gaussian copula and the Frank copula have similar joint PDF contours, but the Frank copula has more rounded contour in the failure region, the effect of dependence is smaller for the Frank copula then the Gaussian copula.

From the results, the error in probability of failure calculation is 10% even with the strong dependence level of $\rho = 0.8$ at the probability of failure level $10^{-4}$. Even at the very strong dependence level of $\rho = 0.9$, the Frank copula model has error less than 1% at the level of $10^{-4}$. In terms of reliability index, error is very small at the strong correlation of $\rho = 0.9$.

**Numerical example2: The effect of the ratio and difference between limit states**

In the previous section, we used the same distributions for both limit states $G_1$ and $G_2$. In this section the effect of the ratio and difference between limit states are shown. The Gaussian and Clayton copulas are considered since they have relatively larger error than the other copula models.

Definitions of $G_1$ and $G_2$ are the same as the previous section. The ratio and difference are taken into account as

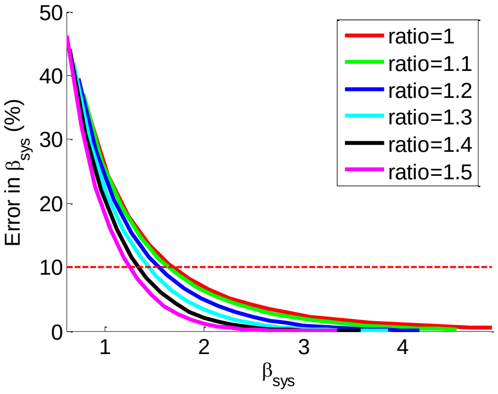
$$\begin{aligned} G_1^* &= G_1 \\ G_2^* &= r\left(G_2 - z\right) + d \end{aligned} \tag{18}$$

Corresponding probability of failure is calculated as

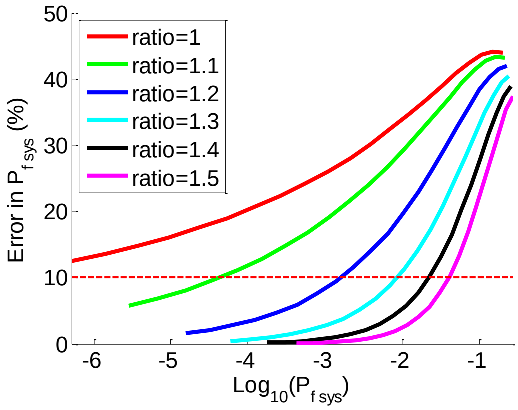
$$\begin{aligned} P_{f\,sys} &= \Pr\left(G_1^* < 0\right) + \Pr\left(G_2^* < 0\right) - \Pr\left(G_1^* < 0 \cap G_2^* < 0\right) \\ &= F_{G_1^*}\left(0; z, 1\right) + F_{G_2^*}\left(0; z+d, r\right) - C\left(F_{G_1^*}\left(0; z, 1\right), F_{G_2^*}\left(0; z+d, r\right), \theta\right) \\ \beta_{sys} &= -\Phi^{-1}\left(P_{f\,sys}\right) \end{aligned} \tag{19}$$

where $z$ is a regulator of the magnitude of probability of failure, and $r$ is the ratio and $d$ is the difference between limit states, $G_1^*$ and $G_2^*$.

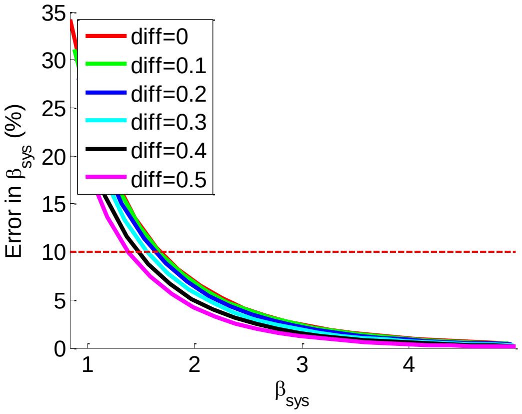
Figure 4 shows the error with respect to the magnitude of reliability index and logarithmic probability of failure while the strength of dependence is kept as $\rho = 0.8$. Table 4 and 5 show the error in terms of reliability index and probability of failure regarding of the ratio and the difference.



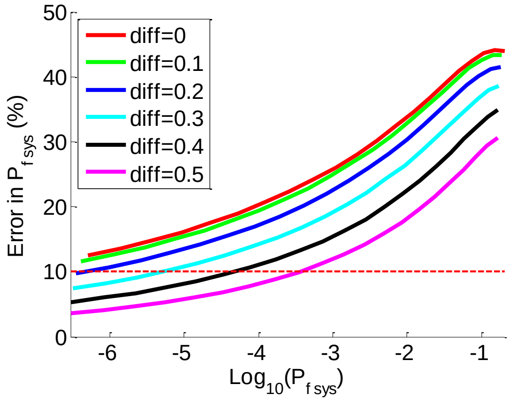

(a) Error vs reliability index for the ratio of the standard deviations of two limit state variables (Gaussian)



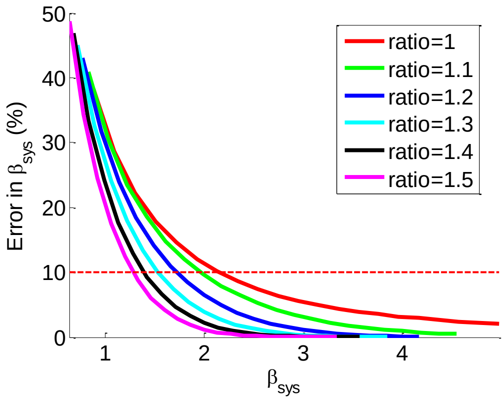

(b) Error vs log probability of failure for the ratio of the standard deviations of two limit state variables (Gaussian)



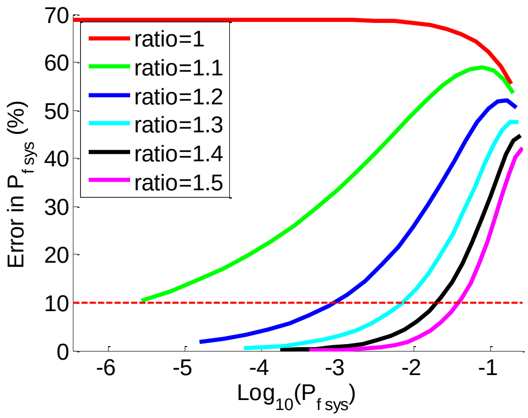

(c) Error vs reliability index for the difference between the means of two limit state variables (Gussian)



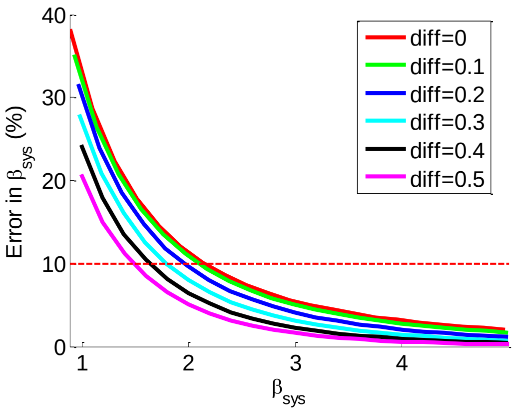

(d) Error vs log probability of failure for the difference between the means of two limit state variables (Gussian)



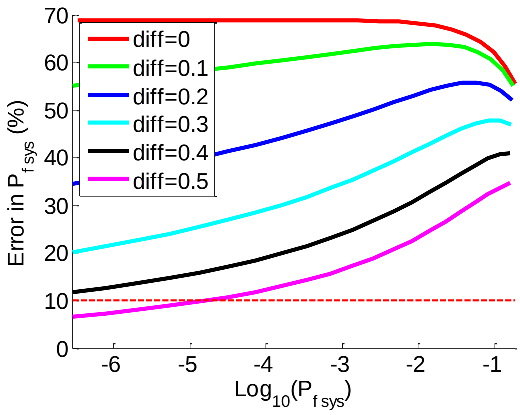

(e) Error vs reliability index for the ratio of the standard deviations of two limit state variables (Clayton)




(f) Error vs log probability of failure for the ratio of the standard deviations of two limit state variables (Clayton)




(g) Error vs reliability index for the difference between the means of two limit state variables (Clayton)




(h) Error vs log probability of failure for the difference between the means of two limit state variables (Clayton)

**Figure 4:** The magnitude of error with respect to the ratio and difference between limit states, $G_1^*$ and $G_2^*$, with $\rho = 0.8$ ("ratio" and "diff" are $r$ and $d$ in Eq. (18), respectively)

In contrast, probability of failure with the independence assumption is calculated as

$$\begin{aligned} P_{f\,dep} &= \Pr\left(G_1^* < 0\right) + \Pr\left(G_2^* < 0\right) - \Pr\left(G_1^* < 0\right)\Pr\left(G_2^* < 0\right) \\ &= F_{G_1^*}\left(0; z, 1\right) + F_{G_2^*}\left(0; z+d, r\right) - F_{G_1^*}\left(0; z+d, 1\right) F_{G_2^*}\left(0; z+d, r\right) \end{aligned} \tag{20}$$

Errors are calculated with Eqs. (8-9) as the previous section. The effect of ignorance decreases as the magnitude of probability of failure decreases. Table 4 shows reliability index for errors of 10% and 1% in reliability index calculation with respect to the ratio and difference between limit states. Table 5 shows logarithmic system probability of failure for errors in probabilities of failure of 10% and 1% in reliability index calculation with respect to the ratio and difference.

**Table 4.** Reliability index for 10% and 1% errors with respect to the ratio and difference between limit states , $G_1^*$ and $G_2^*$ , with $\rho$ = 0.8

| Copula | Error | ratio =1 | ratio =1.1 | ratio =1.2 | ratio =1.3 | ratio =1.4 | ratio =1.5 |
|---|---|---|---|---|---|---|---|
| Gaussian | 10% | 1.68 | 1.64 | 1.54 | 1.44 | 1.33 | 1.24 |
| | 1% | 3.98 | 3.50 | 2.95 | 2.55 | 2.26 | 2.04 |
| Clayton | 10% | 2.15 | 1.96 | 1.72 | 1.54 | 1.39 | 1.28 |
| | 1% | 6.92 | 3.95 | 3.07 | 2.60 | 2.28 | 2.05 |
| | | diff.=0 | diff.=0.1 | diff.=0.2 | diff.=0.3 | diff.=0.4 | diff.=0.5 |
| Gaussian | 10% | 1.68 | 1.67 | 1.62 | 1.56 | 1.47 | 1.38 |
| | 1% | 3.98 | 3.94 | 3.81 | 3.62 | 3.39 | 3.16 |
| Clayton | 10% | 2.15 | 2.09 | 1.96 | 1.79 | 1.63 | 1.48 |
| | 1% | 6.92 | 6.22 | 5.25 | 4.49 | 3.92 | 3.48 |

**Table 5.** Logarithmic system probability of failure for 10% and 1% errors with respect to the ratio and difference between limit states, $G_1^*$ and $G_2^*$ , with $\rho$ = 0.8

| Copula | Error | ratio =1 | ratio =1.1 | ratio =1.2 | ratio =1.3 | ratio =1.4 | ratio =1.5 |
|---|---|---|---|---|---|---|---|
| Gaussian | 10% | -7.42 | -4.36 | -2.79 | -2.06 | -1.65 | -1.39 |
| | 1% | -14.86 | -8.11 | -5.24 | -3.7 | -2.85 | -2.34 |
| Clayton | 10% | N/A | -5.66 | -3.04 | -2.15 | -1.69 | -1.41 |
| | 1% | N/A | -8.56 | -5.38 | -3.73 | -2.86 | -2.34 |
| | | diff.=0 | diff.=0.1 | diff.=0.2 | diff.=0.3 | diff.=0.4 | diff.=0.5 |
| Gaussian | 10% | -7.42 | -7.11 | -6.30 | -5.30 | -4.31 | -3.44 |
| | 1% | -14.86 | -14.58 | -13.72 | -12.61 | -11.43 | -10.22 |
| Clayton | 10% | N/A | N/A | N/A | N/A | N/A | -4.77 |
| | 1% | N/A | N/A | N/A | N/A | N/A | N/A |

From the results, it is observed that the error is more sensitive to the ratio than the difference. From the Table 5, for the Gaussian copula model, the logarithmic probability of failure level for 10% error is decreased by 82% as the ratio changes from 1 to 1.5. The logarithmic probability of failure for 10% is decreased by 46% as the difference changes from 0 to 0.5.

For the Clayton copula model, error in calculated probability of failure is significantly increased as the level of probability of failure decreases unlike that other copula models. However, error is decreased by 75% as the ratio changes from 1.1 to 1.5. For the ratio less than 1.3, for both Gaussian and Clayton copula models, logarithmic probability of failures for both 10% and 1% errors are almost same. The difference does not effect on error as much as the ratio does.

## Conclusion

In this paper, the effect of ignoring dependence in multiple failure modes is studied. Various dependence types between failure modes are considered using copulas. Errors in probability of failure and corresponding reliability index due to ignoring dependence are compared with different strengths of dependence and different levels of reliability. It is found that the effect of ignoring dependence significantly decreases as the reliability increases, except for the Clayton copula. For the Gaussian copula, for example, even with the presence of strong dependence between failure modes, such as $\rho$ = 0.8, ignoring dependence can give 10% error in the order of probability of failure of $10^{-4}$. For the case of the Gumbel or Frank copulas, the 10% error occurs in the order of probability of failure of $10^{-2}$ ~$10^{-3}$.

It is also found that the ratio between standard deviations and the difference between means affect the error. From the numerical example, it is observed that the error is sensitive to the ratio of the standard deviations than the difference between the means. Therefore, when a probability of failure is expected to be lower than $10^{-5}$, unless the ratio of the standard deviations is very close to 1, dependence in failure modes is ignorable even though the strength of dependence is strong.

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