Adaptive Reduction of Design Variables Using Global Sensitivity in Reliability-Based Optimization

Nam H. Kim[*] and Haoyu Wang[†]

Dept. of Mechanical & Aerospace Engineering, University of Florida, Gainesville, Florida, 32611

Nestor V. Queipo[‡]

Dept. of Mechanical & Aerospace Engineering, University of Florida, Gainesville, Florida, 32611
Applied Computing Institute, Faculty of Engineering, University of Zulia, Venezuela

This paper presents an efficient shape optimization technique based on stochastic response surfaces (SRS) and adaptive reduction of random variables using global sensitivity information. Each SRS is a polynomial chaos expansion that uses Hermite polynomial bases and provides a closed form solution of the model output from a significant lower number of model simulations than those required by conventional methods such as modified Monte Carlo Methods and Latin Hypercube Sampling. Random variables are adaptively fixed before constructing the SRS if their corresponding global sensitivity indices calculated using low-order SRS are below a certain threshold. Using SRS and adaptive reduction of random variables, reliability-based optimization problems can be solved with a reasonable amount of computational cost. The efficiency and convergence of the proposed approach are demonstrated using a benchmark case and an industrial reliability-based design optimization problem (automotive part).

Nomenclature

$\mathbf{u}$	=	vector of standard random variables
$\mathbf{x}$	=	vector of random variables
$\mathbf{d}$	=	vector of design parameters
$\Gamma_p(u_i,\ldots,u_p)$	=	multidimensional Hermite polynomials of degree p
$[\mathbf{K}]$	=	structural stiffness matrix
$\{\mathbf{F}\}$	=	structural load vector
$\{\mathbf{D}\}$	=	nodal solution vector (displacement)
S_i	=	global sensitivity index of i-th random variable
S_i^{total}	=	total sensitivity index of i-th random variable
$E(\cdot)$	=	expected value
$V(\cdot)$	=	variance
G^p	=	approximation of the performance function with p-th order Hermite polynomials
$\sigma_{\max}$	=	maximum allowable equivalent stress
$c(\mathbf{d})$	=	cost function
P_f	=	failure probability
β_t	=	target reliability
Φ	=	cumulative distribution function of the standard random variable

[*] Assistant Professor, Member AIAA
[†] Graduate Assistant
[‡] Visiting Professor

I. Introduction

UNCERTAINTY in the design parameters makes shape optimization of structural systems a computationally expensive task due to the significant number of structural analyses required by traditional methods. Critical issues for overcoming these difficulties are those related to uncertainty characterization, uncertainty propagation, ranking of design variables, and efficient optimization algorithms. Traditional approaches for these tasks often fail to meet constraints (computational resources, cost, time, etc.) typically present in industrial environments.

In particular, reliability-based design optimization (RBDO) involving a computationally demanding model has been limited by the relatively high number of required analyses for uncertainty propagation during the optimization process. While there has been progress addressing this issue, such as more efficient moment-based optimization algorithms (e.g. RIA[1], PMA[1]), and the construction of stochastic response surfaces (SRS) for uncertainty propagation,[2] the possibility of reducing the number of analyses by systematically fixing unessential design variables throughout the optimization process has not been fully explored.

In this paper, in order to avoid the shortcomings of the conventional moment-based methods (FORM or SORM) and modified Monte Carlo Methods and Latin Hypercube Sampling, and those associated with the use of SRS: i) local sensitivity information at sampling points is also used, and ii) global sensitivity indices are calculated to decide whether to fix random variables whose contribution to the output variability is less than a certain threshold.

With reference to Figure 1, the proposed approach for RBDO initially constructs a low-order SRS using all variables, and adaptively reduces them depending on the values of their corresponding Global Sensitivity Indices (GSI). GSI are calculated using a variance-based method[3,4,5] – a rigorous and theoretically sound approach for global sensitivity. Using the reduced number of random variables, a high-order SRS is constructed from which the reliability of the performance function is evaluated.

The paper is structured as follows: Section 2 describes the uncertainty characterization of model inputs, and the uncertainty propagation to the output using the SRS. Section 3 presents the procedure to compute global sensitivity indices in order to fix unessential random variables during the construction of the SRS. An RBDO problem is formulated and the results obtained using the proposed approach are the subject of Section 4, followed by numerical examples in Section 5

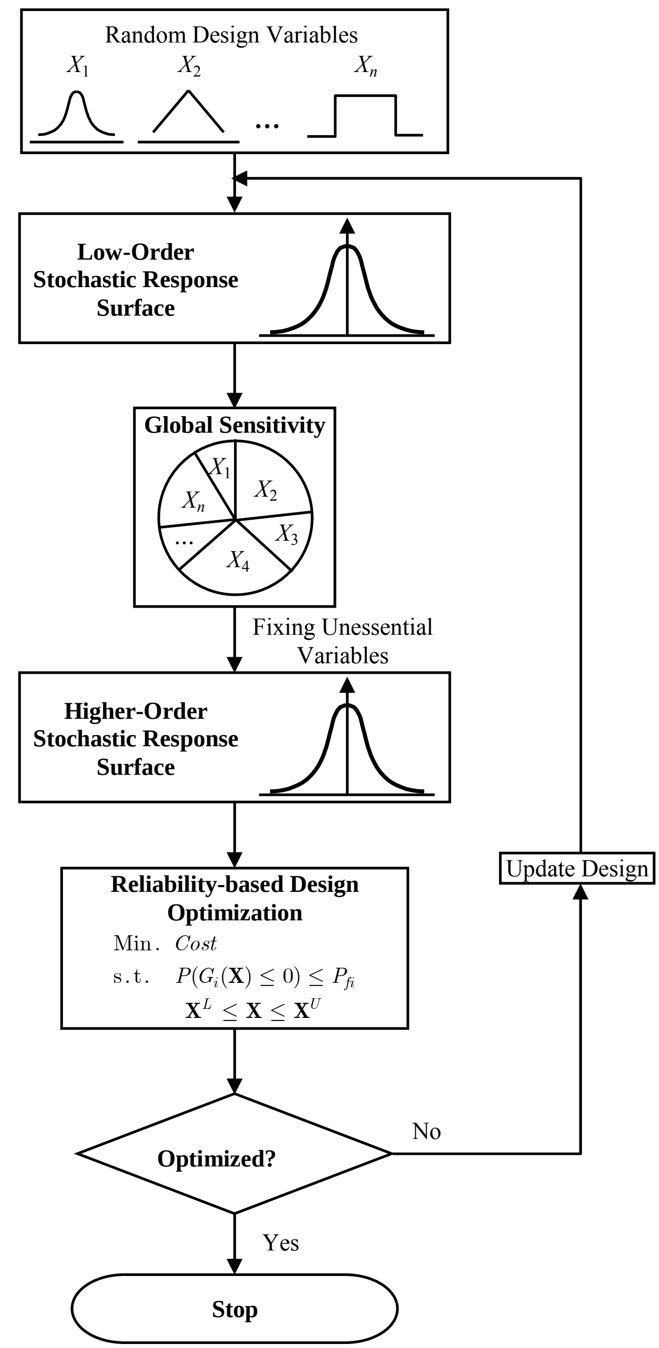


Figure 1. Adaptive reduction of unessential random design variables using global sensitivity indices in RBDO. Low-order SRS is used for global sensitivity analysis, while a higher-order SRS is used to evaluate the reliability of the

II. Uncertainty Quantification

Uncertainty quantification can be decomposed in three fundamental steps: i) uncertainty characterization of model inputs, ii) propagation of uncertainty, and iii) uncertainty management/decision making. To assist the latter step, global sensitivity of model input to outputs is also incorporated.

The uncertainty in model inputs is represented in terms of standardized random variables (SRV) with mean zero and variance equal to one. The selection is supported by the fact that they are widely used and well-behaved. For other types of random variables, an appropriate transformation must be employed. We will assume that the model inputs are independent so each one is expressed directly as a function of SRV through a proper transformation. Devroye[6] presents the required transformation techniques and approximations for a variety of probability distributions. More arbitrary probability distributions can be approximated using algebraic manipulations or by series expansions.

The uncertainty propagation is based on constructing stochastic response surfaces (polynomial chaos expansion). Stochastic response surfaces[7] can be view as an extension of classical deterministic response surfaces for model outputs constructed using uncertain inputs and performance data collected at heuristically selected collocation points. The polynomial expansion uses Hermite polynomial bases for the space of square-integrable probability density functions (PDF) and provides a closed form solution of model outputs from a significant lower number of model simulations than those required by conventional methods such as modified Monte Carlo Methods and Latin Hypercube Sampling.

Let n be the number of random variables and p be the order of polynomial. The model output can then be expressed in terms of SRV $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}^{\mathrm{T}}$ as:

$$G^p = a_0^p + \sum_{i=1}^{n} a_i^p \Gamma_1(u_i) + \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij}^p \Gamma_2(u_i, u_j) + \sum_{i_1=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} a_{ijk}^p \Gamma_3(u_i, u_j, u_k) + \cdots \tag{1}$$

where G^p is the approximated model output, the $a_i^p, a_{ij}^p, \ldots$ are deterministic coefficients to be estimated, and the $\Gamma_p(u_i, \ldots, u_p)$ are multidimensional Hermite polynomials of degree p given by:

$$\Gamma_p(u_i, \ldots, u_p) = (-1)^p e^{1/2\mathbf{u}^T\mathbf{u}} \frac{\partial^p}{\partial u_i \ldots \partial u_p} e^{-1/2\mathbf{u}^T\mathbf{u}} \tag{2}$$

where $\mathbf{u}$ is the vector of p independent and identically distributed normal random variables $\{u_k\}_{k=1}^{p}$ that represent the model input uncertainties. In general, the approximation accuracy increases with the order of the polynomial and should be selected reflecting the accuracy needs and computational constraints. In addition, the approximation in Eq. (1) includes robust coefficients hence exhibiting relatively small changes from low to high-order approximations.

The number of model simulations required to construct the SRS could be reduced when local sensitivity information is available. The issue is how efficiently the local sensitivity information can be calculated. If the global finite difference method is employed, there is no advantage in using sensitivity information because each sensitivity information requires additional analyses. Recently, Isukapalli *et al.*[8] used an automatic differentiation program (ADIFOR) to calculate the local sensitivity of the model output with respect to random variables and used them to construct a stochastic response surface. Their results showed that local sensitivity information can significantly reduce the number of sampling points required. However, the computational cost of the automatic differentiation is often higher than that of direct analysis[9].

In contrast, when the finite element method is used, as discussed by van Keulen *et al.*[10], design sensitivity analysis can provide a very efficient tool for calculating gradient information because the sensitivity equation uses the same coefficient matrix that is already factorized from the original analysis. In this paper, the continuum-based sensitivity analysis is utilized to calculate the gradient of the model output with respect to random variables. In many finite element-based structural analyses, the discrete system is often represented using a matrix equation as

$$[\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{F}\} \tag{3}$$

where $[\mathbf{K}]$ is the stiffness matrix, $\{\mathbf{F}\}$ is the load vector, and $\{\mathbf{D}\}$ is the nodal solution. The model output in Eq. (1) can be expressed as a function of the nodal solution. Thus, the local sensitivity of the model output can be easily calculated if the local sensitivity of the nodal solution is available. When design parameters are defined, the matrix equation (3) can be differentiated with respect to the design parameter d_i to obtain the following design sensitivity equation:

$$[\mathbf{K}]\left\{\frac{\partial \mathbf{D}}{\partial d_i}\right\} = \left\{\frac{\partial \mathbf{F}}{\partial d_i}\right\} - \left[\frac{\partial \mathbf{K}}{\partial d_i}\right]\{\mathbf{D}\} \tag{4}$$

The above equation can be solved inexpensively because the matrix [**K**] is already factorized when solving Eq. (3). The computational cost of sensitivity analysis is less than 20% of the original analysis cost. The efficiency of the uncertainty propagation approach is critical to RBDO (uncertainty management) since at each design cycle an updated version of the PDF for the constraint function (related to model outputs) is required.

III. Global Sensitivity Indices and an Adaptive Approach for Fixing Unessential Variables

To reduce the number of simulations required to construct the SRS even further, unessential random variables are fixed during the construction of the SRS. A random variable is considered unessential (and hence it is fixed) if its contribution to the variance of the model output is below a given threshold. Global sensitivity indices are calculated to quantify the model input contributions to the output variability hence establishing which factors influence the model prediction the most so that: i) resources can be focused to reduce or account for uncertainty where it is most appropriate, or ii) unessential variables can be fixed without significantly affecting the output variability. The latter application is the one of interest in the context of this work.

Variance-based methods are the most rigorous and theoretically sound approaches for total sensitivity calculations.[3,4,5] Variance based methods decompose the output variance into partial variances of increasing dimensionality as

$$f(\mathbf{x}) = f_0 + \sum_i f_i(x_i) + \sum_{i<j} f_{ij}(x_i, x_j) + \ldots + f_{12\ldots n}(x_1, x_2, \ldots, x_n) \tag{5}$$

subject to the restriction that:

$$\int f_{i_1 \ldots i_s} dx_k = 0 \;\text{ for }\; k = i_1, \ldots, i_s \tag{6}$$

Specifically, the global sensitivity index S_i (main factor) and total sensitivity index S_i^{total} associated with x_i are represented by Eqs. (7) and (8), respectively:

$$S_i = \frac{V(E(f \mid x_i = x_i^*))}{V(f)} \tag{7}$$

$$S_i^{total} = \frac{E(V(f \mid x_{-i} = x_{-i}^*))}{V(f)} \tag{8}$$

where E and V denote expected value and variance, respectively. The symbol $-i$ refers to all input variables except x_i. During the design cycles variables will be fixed based on the values of the global sensitivity indices (main factors) hence reducing the number of function evaluations required for the construction of the SRS.

IV. Reliability-Based Optimization

In order to illustrate and evaluate the proposed approach, a simple formulation of the more general RBDO problem[11-14] is discussed. The cost function is assumed to be easily evaluated using the design variables and the constraints are defined using probabilistic distributions of the performance functions. Specifically, consider the following form of the RBDO problem:

$$\begin{aligned} &\text{minimize} && c(\mathbf{d}) \\ &\text{subject to} && P(G_j(\mathbf{x}) < 0) \le P_{f,j}, \quad j = 1, 2, \ldots, np \\ & && \mathbf{d}_L \le \mathbf{d} \le \mathbf{d}_U \end{aligned} \tag{9}$$

where $\mathbf{x} = [x_i]^T$ (i = 1, 2,…, n) denotes the vector of random parameters, $\mathbf{d} = [d_i]^T = [\mu_i]^T$ represent the design variables chosen as the mean values of **x**, and $c(\mathbf{d})$ identifies the cost function. The system performance criteria are described by the performance functions $G_j(\mathbf{x})$ such that the system fails if $G_j(\mathbf{x}) < 0$. Each $G_j(\mathbf{x})$ is characterized by its cumulative distribution function $F_G(g)$:

$$F_{G_j}(g) = P(G_j(\mathbf{x}) < g) = \int_{G_j(\mathbf{x})<g} \cdots \int f_{\mathbf{X}}(\mathbf{x})\, dx_1 \cdots dx_n \tag{10}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint PDF of all random system parameters and g is the probabilistic performance measure. The reliability analysis of the performance function requires evaluating the non-decreasing $F_G(g) \sim g$ relationship,[1] which is performed in the probability integration domain bounded by the system parameter tolerance limits. Since the probability integration domain is in general complicated, many approximation methods (FORM or SORM) are often used. In this paper, the PDF estimated using the proposed uncertainty propagation scheme is used for evaluating reliability constraints hence providing better approximations than traditional linearization and thus significantly improving the rate of convergence of RBDO. Once the cost and constraint functions are evaluated, the optimization problem in Eq. (9) can be solved using conventional mathematical programming techniques.

V. Numerical Examples

A. Stochastic Response Surface for the Torque-Arm Model

Consider the torque arm model depicted in Figure 2.[15] The locations of boundary curves have uncertainties due to manufacturing processes. Thus, the relative locations of corner points of the boundary curves are defined as random variables.

For simplicity, we assumed that all random variables exhibit a normal distribution with mean zero and standard deviation equal to 0.1; i.e., $\mathbf{x} \sim N(0, 0.1)$. The mean values of these random variables are chosen as design parameters, while, without loss of any generality, the standard deviation remains constant during the design process.

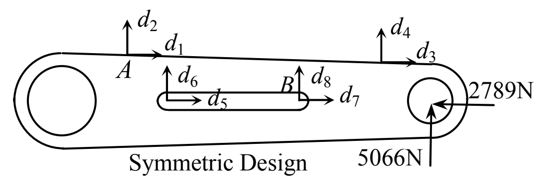


Figure 2. Shape design parameters for the torque arm model. Design parameters are the mean values of corner coordinates of boundary curves. Due to manufacturing processes, the

As illustrated in Figure 2, the initial model consists of eight design parameters. For example, design parameter d_1 is the mean of the relative location of point A in the x-direction. In order to show how the SRS is constructed and the PDF of the model output is calculated, we choose the three design parameters (d_2, d_6, and d_8) that most significantly contribute to the stress performance at points A and B.

A meshfree method[15] is employed to solve the structural response. In the initial design, the maximum stress of 305 MPa occurs at location A. For reliability analysis, the stress limit is established to be 800MPa. In the reliability analysis the performance function is defined such that $G \leq 0$ is considered failure. Thus, in the case of stress constraints, the following performance function is defined:

$$G(\mathbf{x}) := \sigma_{\max} - \sigma_A(\mathbf{x}) \tag{11}$$

where $\sigma_{\max}$ is the maximum allowed equivalent stress and σ_A is the stress at location A.

Before constructing the stochastic response surface, it is important to transform the random variables $\{x_i(d_i, 0.1)\}$ into standard normal distributions $\{u_i\}$. After transforming to the standard normal distributions, the stochastic response surface can be defined using the polynomial chaos expansion. The 2nd- and 3rd-order Hermite polynomial chaos expansions can be written as, respectively,

$$G^2 = a_0^2 + \sum_{i=1}^{n} a_i^2 u_i + \sum_{i=1}^{n} a_{ii}^2 (u_i^2 - 1) + \sum_{i=1}^{n-1} \sum_{j>i}^{n} a_{ij}^2 u_i u_j \tag{12}$$

and

$$
\begin{aligned}
G^3 = a_0^3 &+ \sum_{i=1}^{n} a_i^3 u_i + \sum_{i=1}^{n} a_{ii}^3 (u_i{}^2 - 1) \\
&+ \sum_{i=1}^{n} a_{iii}^3 (u_i{}^3 - 3u_i) + \sum_{i=1}^{n-1} \sum_{j>i}^{n} a_{ij}^3 u_i u_j \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ijj}^3 (u_i u_j{}^2 - u_i) + \sum_{i=1}^{n-2} \sum_{j>i}^{n-1} \sum_{k>j}^{n} a_{ijk}^3 u_i u_j u_k
\end{aligned}
\tag{13}
$$

Note that the polynomials are constructed in the standard Gaussian space rather than the original design space. For 2nd- and 3rd-order expansion, the numbers of unknown coefficients, denoted by N_2 and N_3, are defined as N_2 = 10 and N_3 = 20.

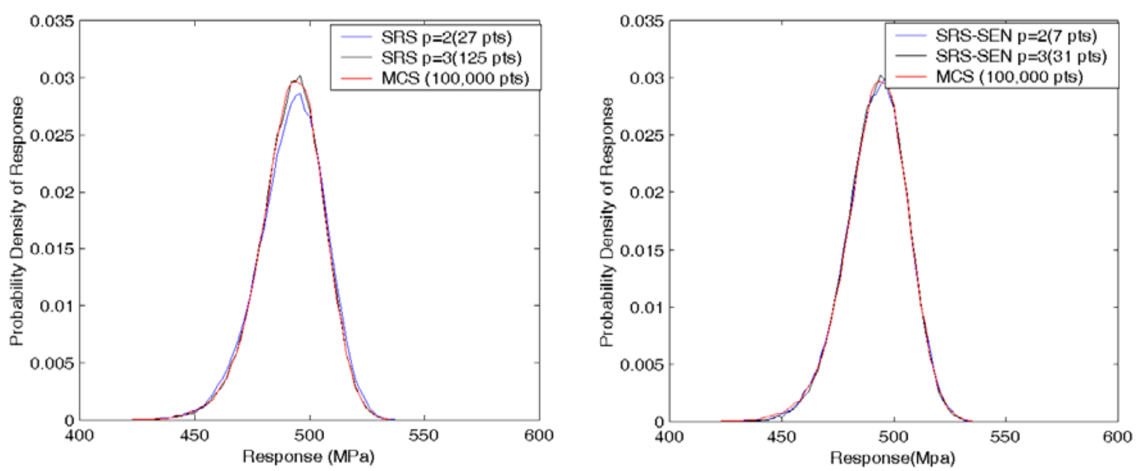
The coefficients of the polynomial chaos expansion are obtained using the model outputs at selected collocation points. The collocation points are selected from the roots of the polynomial that is one order higher than the polynomial chaos expansion.[16] For example, to solve for a three-dimensional second order polynomial chaos expansion, the roots of the third order Hermite polynomial, $-\sqrt{3}$, 0, and $\sqrt{3}$ are used, thus the possible collocation points are (0, 0, 0), ($-\sqrt{3}$, $-\sqrt{3}$, $-\sqrt{3}$), ($-\sqrt{3}$, 0, $\sqrt{3}$), etc. There are 27 possible collocation points and 10 unknown coefficients in the case of second-order expansion. For robust estimation of the regression coefficient, the number of collocation points in general should be twice the number of unknown coefficients.

After choosing collocation points in the standard normal space, a transformation is applied from standard Gaussian space to design space according to the PDF associated with the design variables. In the torque arm model, the PDF of the performance function is plotted in Figure 3(a) for polynomials of different orders. The accuracy and the convergence of the stochastic response surfaces are compared with the PDF obtained using Monte Carlo simulation with 100,000 sample points. As expected, the root mean square error is reduced for higher-order polynomials.

In order to reduce the required number of sampling points to construct the SRS, local sensitivity information is also used. At each sampling point, n+1 data are available (function value + gradients of n random variables). In order to account for the local sensitivity information, the expressions in Eqs. (12) and (13) are differentiated with respect to the random variables. However, the stochastic response surfaces are defined in the standard Gaussian space. As a result, it is necessary to transform the local sensitivity in the design space into standard Gaussian space using the following equation:

$$
\nabla G(\mathbf{u}) = \nabla G(\mathbf{x}) \frac{\partial \mathbf{T}^{-1}(\mathbf{u})}{\partial \mathbf{u}} \tag{14}
$$

where $\mathbf{T}: \mathbf{x} \rightarrow \mathbf{u}$ is the transformation between the design and standard Gaussian spaces.



(a) Only function values are used (b) Function values and local sensitivities are used

Figure 3. PDF of performance function $G(\mathbf{x})$ – Torque Arm Problem

Using local sensitivity information increases by a factor of n the number of data obtained from each sampling point (from one to four for the case of three design variables), hence the number of sampling points can be reduced n+1 times. In Figure 3(b), the PDFs of the performance function are plotted for alternative polynomials expansions and that obtained using Monte Carlo simulation. Note that a stochastic response surface with the same level of accuracy to that showed in Figure 3(a) can be obtained with four times less number of sampling points.

B. Reliability-Based Design Optimization

The reliability optimization problem under consideration requires to minimize the mass of the torque arm while satisfying stress reliability constraints. Let the model output G_i be defined as

$$G_i(\mathbf{x}) = 1 - \frac{\sigma_i}{\sigma_{\max}} \tag{15}$$

Using Eq. (9), the design optimization problem can be defined as

$$\begin{aligned} &\text{Minimize} \quad \text{Mass}(\mathbf{d}) \\ &\text{subject to} \quad P(G_i(\mathbf{x}) \le 0) \le \Phi(-\beta_t), \qquad i = 1, \ldots, NC \\ &\qquad\qquad\quad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \end{aligned} \tag{16}$$

where β_t is the target reliability indices and Φ is the cumulative distribution function of the standard normal distribution. For the reliability analysis, a target reliability index of 3.0 is used, which is equivalent to 99.87% reliability. The stress values at four (i.e., NC = 4) different locations are monitored. Table 1 shows the lower and upper bounds of the mean values associated with the design variables (modeled as random variables). Note that the design parameters are the relative movement of the corner points, the initial values for all design parameters is zero. The lower and upper bounds are chosen such that the topology of the boundary is preserved throughout the whole design process.

Table 1. Definition of random design parameters and mean value bounds

Random Variables	$\mathbf{d}^L$	$\mathbf{d}$ (Initial)	$\mathbf{d}^*$ (Optimum)	$\mathbf{d}^U$	Standard Deviation	Distribution type
d_1	–3.0	0.0	–0.7532	1.0	0.1	Normal
d_2	–0.5	0.0	–0.5000	1.0	0.1	Normal
d_3	–1.0	0.0	–0.1346	1.0	0.1	Normal
d_4	–2.7	0.0	–2.5443	1.0	0.1	Normal
d_5	–5.5	0.0	–0.8508	1.0	0.1	Normal
d_6	–0.5	0.0	1.9998	2.0	0.1	Normal
d_7	–1.0	0.0	0.8319	7.0	0.1	Normal
d_8	–0.5	0.0	0.0000	0.0	0.1	Normal

For comparison purposes, this RBDO problem is solved using all random variables without any adaptive reduction. At each design point, the eight random variables are used to construct the SRS. In order to generate the third-order SRS, a total of 89 sampling points are used; at each sampling point stress and local sensitivity information is gathered. The optimization problem converges at the 21-th iteration. The design variables at the optimum design are listed in the fourth column of Table 1, and the optimum geometry is plotted in Figure 4(a). Figure 4(b) shows the stress distribution of the torque arm model at the optimum design. The maximum stress occurs at Point A with a value of 704 MPa. Considering the maximum allowable stress limit is 800 MPa, the mean value of the optimum design has about 96 MPa margin. Figure 5 shows the design history of the cost function. The initial mass of 0.878 kg is reduced to 0.522 kg (about 59.4%) at the optimum design. Most reduction has been achieved in the first five design cycles, and after that the optimization slowly converged by adjusting design parameters.

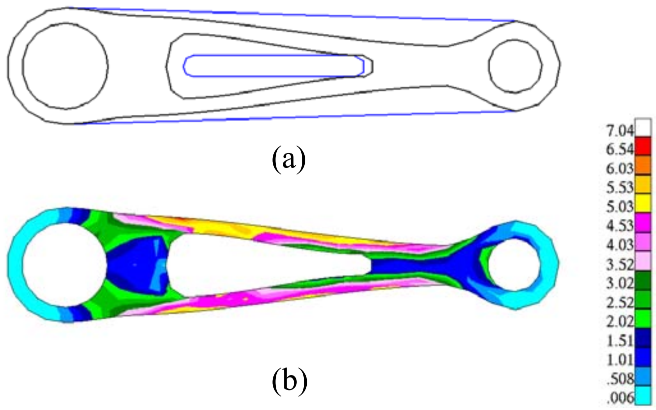


Figure 4. Optimum design and stress distribution of the torque arm model with 8 random variables. (a) Blue color = initial design, black color = optimum design (b) Max. equivalent stress = 704 MPa at Point *A*.

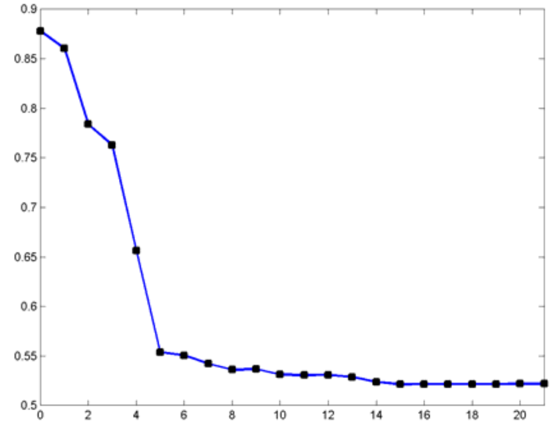


Figure 5. Optimization history of cost function (mass) for the torque arm model with 8 random variables.

C. Adaptive Reduction of Random Variables

The RBDO problem in the previous section was solved with all random variables. However, some random variables did not significantly contribute to the stress function variance. Thus, a significant amount of computational cost can be saved if the random variables whose contribution to the variance of the output is small are considered as deterministic variables at their mean values. This section describes how the global sensitivity indices (main factors) can be used for deciding whether to fix unessential random variables during the construction of stochastic response surfaces.

At the initial design stage, a lower-order stochastic response surface is constructed using all random variables. In this particular example the first-order SRS is constructed using 17 sampling points. At the initial design, the first-order SRS with eight random variables can be expressed as,

$$
\begin{aligned}
G^1 &= a_0 + a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 + a_6u_6 + a_7u_7 + a_8u_8 \\
&= 4.95 + 0.0063u_1 + 0.117u_2 + 0.00008u_3 - 0.0019u_4 + 0.0026u_5 - 0.052u_6 - 0.0002u_7 - 0.016u_8
\end{aligned}
\tag{17}
$$

One useful aspect of the polynomial chaos expansion is that the coefficients in Eq. (17) are a measure of the contribution of the corresponding random variable to the variation of the output, and these coefficients will not change significantly in higher-order SRS. On the other hand, typically the main factor associated with a particular variable is responsible for most of its contribution to the output variance. Thus, evaluating the global sensitivity indices (main factors) using the first-order SRS can be justified. Note that all random variables are transformed into SRV, the variance of G^1 can be evaluated analytically. Using Eqs. (7) and (8), the global sensitivity index of each random variable is calculated. Using Eq. (17) and assuming the design variables are independent, the global sensitivity index can be calculated as:

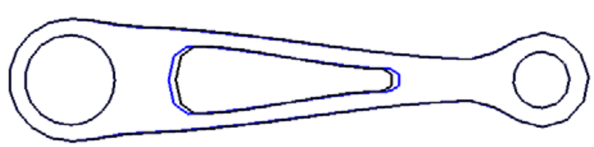


Figure 6. Optimum designs for the full SRS (black color) and adaptively reduced SRS (blue color). Because some variables are fixed, the interior cutout of the design from the adaptively reduced SRS is larger than that from the full SRS.

$$
S_i = \frac{a_i^2}{\sum_{j=1}^{n} a_j^2}
\tag{18}
$$

If the global sensitivity index of a specific variable is less than a threshold value, the variable is considered as deterministic and fixed at its mean value.

In order to show the advantage of the adaptive reduction of random variables, the torque arm problem is solved using a threshold value of 1.0%. Table 2 shows the first-order SRS of the torque arm model at the initial design. The total variance of stress function is 1.670×10^{-2}. Based on the global sensitivity indices, there are only three random variables whose GSI is greater than 1.0%; i.e., u_2, u_6, and u_8. Thus, in the reliability analysis only these three random variables are used in constructing the third-order SRS, which now requires only 19 sampling points. All other random variables are considered as deterministic variables at their mean values. If the total number of sampling points for both low (17) and higher-order (19) polynomial expansions are compared with the higher-order SRS using all random variables (89), a significant reduction of the number of sampling points was achieved.

Table 2. Global sensitivity indices (main factors) for the torque arm model at the initial design. Only three random variables (u_2, u_6, and u_8) are preserved when a threshold value of 1.0% is in place.

SRV	Variance	GSI (%)
u_1	3.916×10^{-5}	0.235
u_2	1.369×10^{-2}	**82.0**
u_3	6.403×10^{-9}	0.00003834
u_4	3.667×10^{-6}	0.02197
u_5	6.864×10^{-6}	0.04109
u_6	2.702×10^{-3}	**16.179**
u_7	4.818×10^{-8}	0.0002885
u_8	2.538×10^{-4}	**1.519**

The RBDO problem, defined in Eq. (16) is now solved using the proposed adaptive reduction of random variables. The optimization algorithm converges after the 17-th iteration. As seen in Figure 6, the optimum design using the adaptively reduced SRS is slightly different from that obtained in the previous section (without adaptive reduction). The former has a longer interior cutout than the latter. This can be explained from the fact that some variables were considered deterministic throughout the design process. Furthermore, the optimum value achieved using the adaptively reduced SRS converges to a lower value than the one without adaptive reduction). The total mass of the torque arm is reduced in 57.6%. The difference between the two approaches is approximately 1.8%.

The number of active random variables associated with the modeling of the first constraint during the design iterations are listed in Table 3. On average, four random variables were preserved as such, which implies that only 29 sampling points were required for constructing the SRS. This is three times less than the SRS approach without adaptive reduction (89 sampling points).

Table 3. Comparison of the number of random variables in each design cycle. The threshold of 1.0% is used. The first constraint is listed.

Iter	Full SRS	Reduced SRS
1	8	3
2	8	3
3	8	3
4	8	3
5	8	4
6	8	4
7	8	5
8	8	4
⋮	⋮	⋮
17	8	4
⋮	⋮	-

VI. Conclusions

In this paper, we present an approach for solving RBDO problems involving a computationally demanding model. Key aspects of the approach are: i) the uncertainty propagation of random variables using a polynomial chaos expansion and local sensitivity information, and ii) the use of global sensitivity information to adaptively reduced the number of random variables throughout the design process. The convergence and accuracy of the proposed approach was demonstrated using a benchmark case and an industrial reliability-based design optimization problem (automotive part).

VII. References

1. J. Tu, K. K. Choi, and Y. H. Park, 1999, A New Study on Reliability Based Design Optimization, ASME Journal of Mechanical Design, Vol. 121(4)

2. N. H. Kim, H. Wang, N. V. Queipo, 2004, Efficient shape optimization using polynomial chaos expansion and local sensivities, 9th ASCE Specialty Conference on Probabilistic Mechanics and Structural Reliability, July 25-28, Albuquerque, NM

3. I. Sobol, Sensitivity Analysis for Nonlinear Mathematical Models, Mathematical Modeling & Computational Experiment, 1993, Vol. 1, pp. 407-414.

4. T. Homma and A. Saltelli, Importance Measures in Global Sensitivity Analysis of Nonlinear Models, Reliability Engineering and System Safety, 1996, Vol. 52, pp. 1-17.

5. A. Saltelli, S. Tarantola, and K. Chan, A Quantitative Model-Independent Method for Global Sensitivity Analysis of Model Output, Technometrics, February 1999, Vol. 41, No. 1, pp. 39-56.

6. Devroye, L. (1986) Nonuniform random variate generation. New York: Springer Verlag.

7. S. S. Isukapalli, A. Roy and P. G. Georgopoulos, Stochastic Response Surface Methods (SRSMs) for Uncertainty Propagation: Application to Environ-mental and Biological Systems. Risk Analysis, 1998, Vol. 18, No. 3, pp. 351-363.

8. S. S. Isukapalli, A. Roy and P. G. Georgopoulos, Efficient Sensitivity/Uncertainty Analysis Using the Combined Stochastic Response Surface Method and Automated Differentiation: Application to Environmental and Biological Systems, Risk Analysis, 2000, Vol. 20, No. 5, pp. 591-602.

9. Carle, A., M. Fagan, and L. L. Green (1998). Preliminary results from the application of automated adjoint code generation to cfl3d. In 7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, Volume Part 2, AIAA 98-4807, pp. 807–817. Sept. 2-4.

10. F. van Keulen, R. T. Haftka, and N. H. Kim, Re-view of Options for Structural Design Sensitivity Analysis. Part1: Linear Systems, Computer Methods of Applied Mechanics and Engineering, to appear, 2004

11. Enevoldsen, I. and Sorensen, J.D., 1994, "Reliability-Based Optimization in Structural Engineering," Structural Safety, Vol. 15, pp. 169-196.

12. Chandu, S.V.L. and Grandhi, R.V., 1995, "General Purpose Procedure for Reliability Based Structural Optimization under Parametric Uncertainties," Advances in Engineering Software, Vol. 23, pp. 7-14.

13. Wu, Y.T. and Wang, W., 1996, A New Method for Efficient Reliability-Based Design Optimization, Probabilistic Mechanics & Structural Reliability: Proceedings of the 7th Special Conference, pp. 274-277.

14. Grandhi, R.V. and Wang, L.P., 1998, Reliability-Based Structural Optimization Using Improved Two-Point Adaptive Nonlinear Approximations, Finite Elements in Analysis and Design, Vol. 29, pp. 35-48.

15. N.H.Kim et al. Numerical method for shape optimization using meshfree method. Struct Multidisc Optim 24,418-429, Springer-Verlag, 2003

16. J.Villadsen and M.L.Michelsen. Solution of Differential Equation Models by Polynomial Approximation. Prentice-Hall, Englewood Cliffs, New Jersey, 1978