Taking Advantage of Separable Limit States in Sampling Procedures

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Sampling procedures are commonly used to estimate probability of failure in reliability-based structural design. For estimating very low probabilities the number of required samples can be high, and the present paper suggests that this number can be reduced when the limit state (or failure criterion) is expressed as the difference of two functions of independent sets of random variables (e.g., capacity minus response). For this case of separable limit states, the sampling may be performed in two stages. First the cumulative distribution function (CDF) of one of the function created by sampling one set of random variables, and then the probability of failure is obtained by sampling the other set of variables using the CDF constructed in the first phase. The paper first considers simple Monte Carlo sampling, then incorporates tail-modeling for constructing the CDF. A simple example of two uniformly distributed variables is used for illustrating the method, and a beam problem is used to demonstrate its usefulness.

Nomenclature

a	=	lower bound of a uniform distribution
b	=	upper bound of a uniform distribution
C_{Vp}	=	coefficient of variation of probability of failure simulation
c	=	capacity of a system
F_c	=	cumulative distribution function of the capacity
F_r	=	cumulative distribution function of the response
F_x	=	random horizontal load on beam
F_y	=	random vertical load on beam
f_c	=	probability density function of the capacity
f_G	=	probability density function of the limit state function
f_r	=	probability density function of the response
f_x	=	probability density function of the random variable, x
G	=	limit-state function for probability of failure
g	=	threshold value of tail model
I	=	indicator function
i	=	random variable index
N	=	sample size
N_g	=	number of tail data above the threshold
N_{mc}	=	sample size of crude Monte Carlo simulation
N_{smc}	=	sample size of separable Monte Carlo simulation
p_f	=	actual probability of failure
R	=	range of a distribution
r	=	response of random input variables
S	=	random applied stress on beam
t	=	thickness of beam
w	=	width of beam
x	=	generic random variable
$\mathbf{x}_1$, $\mathbf{x}_2$	=	mutually independent random variable vectors
Y	=	random yield strength of beam
y	=	generic function
z	=	distribution of random limit state values, G, above the threshold, g

α	=	inequality test of indicator function
γ_c	=	ratio of failure region to the range of the capacity, for uniform distributions
γ_r	=	ratio of failure region to the range of the response, for uniform distributions
μ	=	mean of random variable
σ	=	scale parameter of tail model
σ_p	=	standard deviation of the average of N probability of failure simulations
ζ	=	shape parameter of tail model

I. Introduction

PROBABILITY of failure in reliability-based structural design are often estimated using sampling procedures. When estimating very low probabilities, the number of required samples can be high, thus Monte Carlo simulation (MCS) becomes a costly process. Several methods for predicting the probability of failure have been developed to relieve the simulation time burden. Importance sampling is a method that concentrates the random sample values to the region where failure occurs (Kalos and Whitlock, 1986). Though proven to be an effective alternative, importance sampling requires prior knowledge of the conditions for failure, which is not always available. As done by Qu *et al.* (2000 and 2004) and Kale *et al.* (2005), response surface approximation is a different method used to reduce computation time. Kim *et al*. (2006) used a technique called tail modeling to accurately predict the extreme tail of the limit state function using generalized Pareto distributions. Tail modeling required fewer samples since only the tail of the cumulative distribution function (CDF) was being fit.

This paper considers a method that exploits the fact that in most structural problems, the failure condition may be written as response exceeding capacity, which are both functions of independent sets of random variables. For very safe structures, failure usually occurs when very high response happens for a structure with unusually low capacity (e.g., due to damage). The small probability of failure corresponds to a small portion, or extreme tail, of the limit state's distribution. Each of the probabilities of high response and low capacity may not be extremely small, but when these occurrences are independent, the probability of both occurring simultaneously is the product of the two. Hence, when the response and capacity are independent, it may be possible to analyze them separately with a moderate sample size, and still be able to estimate very low probabilities of failure. Therefore, to bypass the requirement of sampling the extreme tail of the limit-state function, the variables could be considered independently, by separating the response and the capacity, as discussed by Melchers (1999).

The objective of the present paper is to explore this possibility for the case where the response and capacity are controlled by two different sets of random variables, which we call a separable limit state. First, the a background on limit states, Monte Carlo sampling, and tail modeling is presented. Then separable Monte Carlo method (SMC) is explained and the formula for the probability of failure is derived from the definition of a mean of a function. The probability of failure equations for SMC was analyzed and profitable scenarios for implementation of SMC on limit states are discussed. An efficiency comparison of limit state Monte Carlo and separable Monte Carlo was performed using uniform distributions. Finally, a beam problem example was analyzed using variations on limit state and separable simulation methods to demonstrate the usefulness of SMC. Tail modeling was also considered for the beam problem to determine its effectiveness with both sampling methods.

II. Limit States and Probability of Failure Calculations

The limit state is the criterion evaluated to determine if failure occurs. We consider the special case when the limit state can be expressed as the difference between capacity, c, and a response, r. Furthermore, we assume that uncertainty in the response is due to one set of random variables, $\mathbf{x}_1$, and the uncertainty in the capacity is due to a second set of random variables, $\mathbf{x}_2$. The general limit state function for probability of failure calculations is shown in Eq. (1).

$$G(\mathbf{x}_1, \mathbf{x}_2) = c(\mathbf{x}_1) - r(\mathbf{x}_2) \tag{1}$$

Therefore, the system fails when $G < 0$ and safe when $G \geq 0$. The general form of the probability of failure is observed in Eq.(2), for when the limit state function is less than zero.

$$p_f = \int_{G<0} f_G(G)\, dG \tag{2}$$

Where, $f_G(G)$ is the probability density function (PDF) of the limit state function, G, which is integrated for values less than zero. Furthermore, for a small probability of failure, the domain of failure according to the limit state is very small and in the extreme tail of the PDF. Accurate modeling of tails requires either many Monte Carlo samples to obtain it analytically, or a more advanced technique, such as tail modeling. To relieve some of the computational costs, this paper investigates separable methodology on the limit state of two procedures for calculating the probability of failure: Monte Carlo method and tail modeling of the limit state function.

A. Monte Carlo Sampling

Monte Carlo simulation (MCS) is a random sampling technique used to determine information about functions of random variables, including mean, distributions, and, in this case, probability of failure. The standard Monte Carlo approach involves designating 0 or 1 for each run in the simulation, corresponding to pass or fail of the limit-state function, respectively. Denoting the terms of Eq. (1) as $c_i = c(x_{1i})$ and $r_i = r(x_{2i})$, the probability of failure according using the limit state is

$$p_f \approx \frac{1}{N}\sum_{i=1}^{N} I\left[G(c_i, r_i) < 0\right] \tag{3}$$

where, $I[\alpha]$ is the indicator function, which equals 1 if α is true and 0 if α is false. The capacity and response have a subscript, i, to represent the independent random sample value generated for each simulation, according to their respective distributions. Obviously, since only the sum of the number of failures is being averaged, a very large sample size is required to accurately predict a very small probability of failure.

B. Tail Modeling

Accurately modeling the extreme tail of a distribution is often difficult for complex distributions, or without using a very large number of samples. However, it is also very critical to estimate accurately since the tail is where failure occurs for low probability of failures. Kim *et al.* used a generalized Pareto distribution (GPD) to approximate the tail region of the limit-state function. Figure 1 displays the upper tail region of the cumulative distribution function (CDF) modeled by generalized Pareto distribution. Tail modeling approximates the CDF of limit-state function, G, above a designated threshold value, g, written as $z = G - g$.

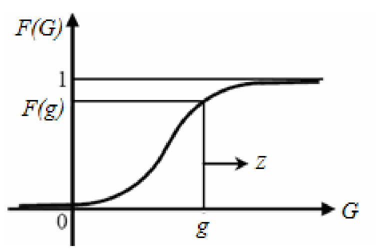

Figure 1. Tail modeling of *F(G)* using the threshold value of *g*

The expression for the tail model using generalized Pareto distributions is

$$F(G)_{\xi,\sigma} = 1 - \frac{N_g}{N}\left\langle 1 + \frac{\xi}{\sigma}(G - g)\right\rangle_+^{-\frac{1}{\xi}} \tag{4}$$

where, N is the total number of points sampled, N_g are the number of sample points above the threshold, and ξ and σ are shape and scaling parameters, respectively. The GPD parameters are determined using maximum likelihood, based on the N samples. The failure is defined by the limit state function when $G < 0$ and safe when $G \geq 0$, therefore the probability of failure can be determined at $G = 0$. Rewriting Eq. (4), the probability of failure can be written as

$$p_f = \frac{N_g}{N}\left(1 - \frac{\xi}{\sigma} g\right)^{-\frac{1}{\xi}} \tag{5}$$

III. Calculations for Separable Limit States

The calculation for the probability of failure using separable Monte Carlo (SMC) is similar to finding the mean of a function, *y(x)*, where *x* is a random variable with a probability density function of $f_x(x)$.

$$\mu_y = \int y(x) f_x(x) dx \tag{6}$$

This equation forms the basis for SMC, because the probability of failure can be expressed (Melchers, pp. 31-49, 1987) as

$$p_f = \int_{Rr} F_c(r) f_r(r) dr \tag{7}$$

where, F_c is the cumulative distribution function (CDF) of the capacity and f_r is the PDF of the response; which are integrated over the range of the response, R_r.

Rewriting the integral in terms of simulations yields

$$p_f \approx \frac{1}{N} \sum_{i=1}^{N} F_c(r_i) \tag{8}$$

Considering Eq. (8), it can be said for this method that sample values of the response are inserted into the CDF of the capacity, thus evaluating each response at the corresponding level of failure according to the capacity. Separable Monte Carlo method takes the average of the capacity's CDF at randomly generated, different response sample values. Figure 2 graphically shows the process for an SMC simulation from Eq. (8).

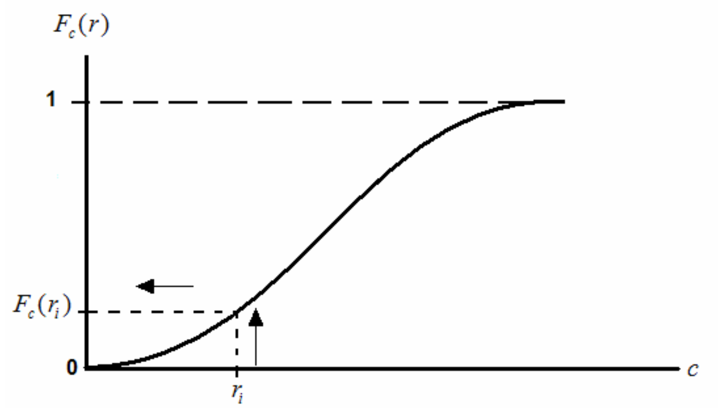

Figure 2. Transformation of the random response sample value in the capacity's CDF

Figure 2 is one run in the simulation, which is repeated *N* times using a new random number each run. The appealing aspect of this method of simulation is that a single run yields a much better estimate of the probability of failure, than does limit-state Monte Carlo. Separable Monte Carlo method can provide more than just a binary result of 0 or 1 for each run. Each run of SMC determines the probability of failure of a random response in the capacity. Alternately, the probability of failure can be expressed as evaluating the capacity in the cumulative distribution of the response, as in Eq. (9).

$$p_f = \int_{Rc} \left(1 - F_r(c)\right) f_c(c) dc \tag{9}$$

Or for simulations,

$$p_f \approx \frac{1}{N}\sum_{i=1}^{N}\left(1 - F_r(c_i)\right) \tag{10}$$

This representation is the opposite of Eq. (8), in which the capacity is sampled at values of the CDF of the response. Depending on the scenario of the system, either Eq. (8) or Eq. (10) might be more effective when performing simulations to estimate the probability of failure. If the response has a small standard deviation or is deterministic, then the Eq. (8) form of SMC would require fewer simulations for the same level of accuracy than if Eq. (10) were applied. On the other hand, if the capacity has a smaller standard deviation, then Eq. (10) will more efficiently estimate the probability of failure.

Of course, the key to taking advantage of Eq. (7) or Eq. (9) is the easy availability, or ease of generation, of the cumulative distribution. Fortunately, this is the case for many problems in structural design; where the CDF is determined analytically, which is often the case for the capacity. If that information is not directly known, then Monte Carlo sampling could be used to estimate the CDF. However this leads to a two-stage MCS, where sampling is done to obtain the CDF, then more sampling is required to determine the probability of failure. Another option is to us a combination of MCS and tail modeling to acquire the tail modeling parameters which estimate the CDF, as in Eq. (4). This paper will consider each of these scenarios in a beam problem example, presented in a later section.

IV. Efficiency Comparison via Analytical Example

To explore the efficiency of the two methods, expressions for the probability of failure and required sample size were derived for uniform distributions. For simplicity, assume that the random determinants of response and capacity produced a uniform distribution for each. Uniform distributions were chosen for convenience of calculations. Figure 3 shows a general scenario of probability of failure for two uniform distributions.

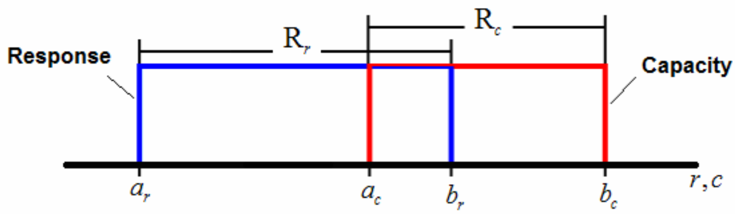

Figure 3. General uniform probability density functions for response and capacity

From Fig. 3, R is the range of the distribution, a is the lower bound, and b is the upper bound. Since the probability of failure will not be changed by translation or inflation of r and c, it is convenient to represent the distributions and their overlap by ratios γ_r and γ_c, as shown in Eq. (11).

$$\gamma_r = \frac{b_r - a_c}{R_r} \qquad \gamma_c = \frac{b_r - a_c}{R_c} \tag{11}$$

The definition of these ratios only applies when the bounds of the two distributions remain in the same sequence shown in Fig. 3 ($a_r \leq a_c$ and $b_r \leq b_c$) This implies that $0 \leq p_f \leq 0.5$, which is most often the case in engineering situations. Through some derivation, which is attached in the appendix, the probability of failure was found in terms of γ_r and γ_c.

$$p_f = \frac{1}{2}\gamma_r\gamma_c \tag{12}$$

A measure of the accuracy of a limit state Monte Carlo simulation can be quantified with the variance of the estimate.

$$\sigma_p^2 = \frac{p_f(1 - p_f)}{N} \tag{13}$$

As a way to express the cost of a simulation, Eq. (13) was rewritten in terms of the number of samples required, N_{mc}, to achieve a specified level of accuracy or coefficient of variation, $C_{Vp} = \sigma_p / p_f$.

$$N_{mc} = \frac{(1-p_f)}{p_f C_{Vp}^2} = \frac{1}{C_{Vp}^2}\left[\frac{2}{\gamma_r \gamma_c} - 1\right] \tag{14}$$

This form better displays the inverse nature of sample size to probability of failure and the overlap ratios. Analyzing Eq. (14), it is apparent that for very small probability of failures, or very small γ_r and γ_c, a very large sample size is required.

Similarly, the variance of the probability of failure calculation by SMC using Eq. (7) is given in Eq. (15).

$$\sigma_p^2 = \int_{Rr} \left(F_c(r) - p_f\right)^2 f_r(r)dr \tag{15}$$

The expression was again written in terms of required sample size and simplified by substituting Eqs. (11) and (12), resulting in Eq. (16).

$$N_{smc} = \frac{1}{C_{Vp}^2}\left[\frac{4}{3\gamma_r} - 1\right] \tag{16}$$

For completeness, the required sample size was also derived for the alternate separable Monte Carlo form from Eq. (9).

$$N_{smc} = \frac{1}{C_{Vp}^2}\left[\frac{4}{3\gamma_c} - 1\right] \tag{17}$$

Analyzing Eqs. (16) and (17) reveal that depending on the SMC method chosen, depends only on the variable being sampled. Additionally, both expressions for sample size of SMC are identical when the ranges of the response and capacity are equal. The efficiency analysis considers Eq. (16), where the response is sampled in the CDF of the capacity. The effect of γ_r and γ_c would just be reversed if Eq. (17) was used instead.

Considering small probabilities, γ_r and γ_c are typically small, so the second term in Eqs. (14) and (16) may be neglected. Then after some manipulation of the equations, another descriptive formula for efficiency analysis is shown in Eq. (18).

$$\frac{N_{smc}}{N_{mc}} = \frac{2\sqrt{2}}{3}\sqrt{p_f}\sqrt{\frac{\gamma_c}{\gamma_r}} \tag{18}$$

Dissecting the components of the equation shows when SMC is more efficient than standard Monte Carlo. First, the smaller the probability of failure gets, the smaller the number of samples are required for SMC with respect to limit state MC. Also, the method is advantageous when more of the uncertainty comes from the capacity rather than the response ($\gamma_r > \gamma_c$). For example, when the response is deterministic, $\gamma_r = 1$.

To graphically illustrate the effect of p_f on both simulation methods, a plot of sample size vs. p_f was generated in Fig. 4.

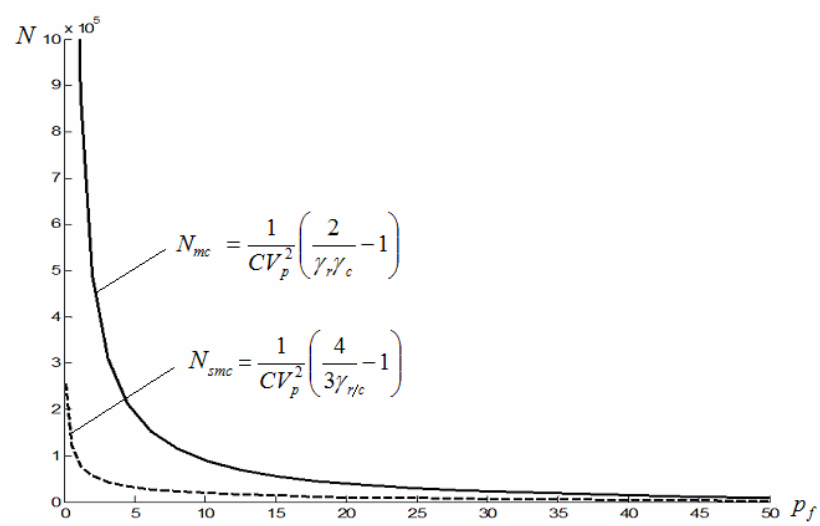

Figure 4. Sample size of crude and separable MC for uniform distributions with C_{Vp} = 1% and $\gamma_r = \gamma_c$

For a better representation of the advantage of SMC at small probability of failures, the ratio of sample sizes were compared in Fig. 5 versus the two terms of interest, p_f and γ_c / γ_r. The actual plots use the ratio of Eqs. (19) and (20), whereas the estimated plots are from Eq. (21).

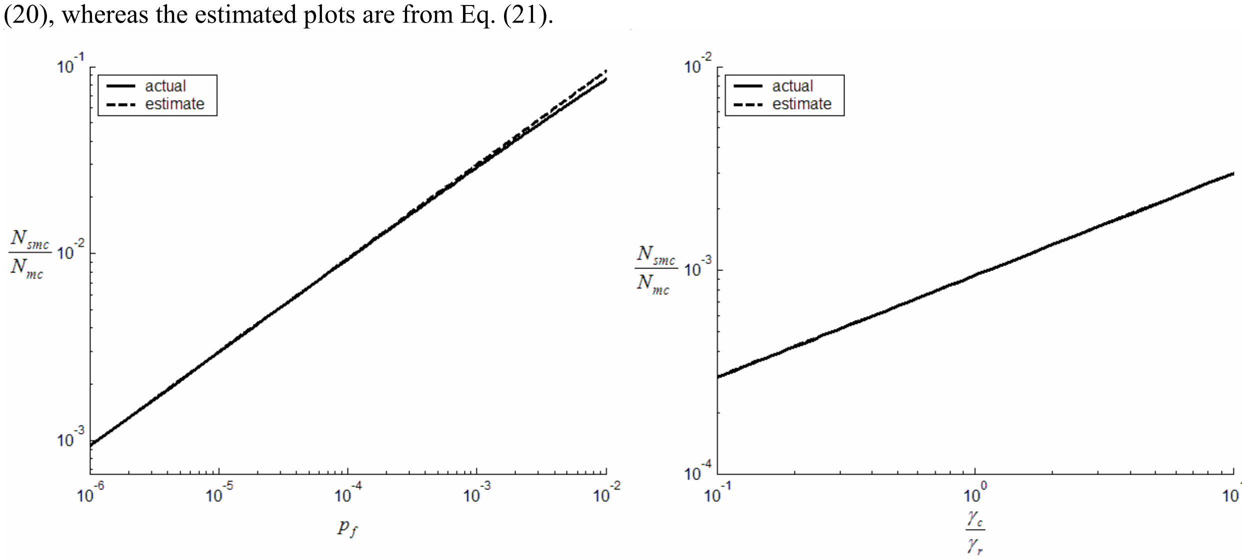

Figure 5. Actual and estimated ratio of separable to limit state MC sample sizes for uniform distributions with C_{Vp} = 1% a) versus p_f with γ_c / γ_r = 1 and b) versus γ_c / γ_r with $p_f = 10^{-6}$

V. Beam Problem Example

The cantilever beam problem presented by Kim *et al.* was considered to demonstrate separable Monte Carlo application. As shown in Figure 6, a beam with length, L, width, w, and thickness, t, is subjected to random horizontal and vertical loads, F_x and F_y, respectively. The width and thickness were taken as deterministic values of 2.453in and t = 3.884in, respectively.

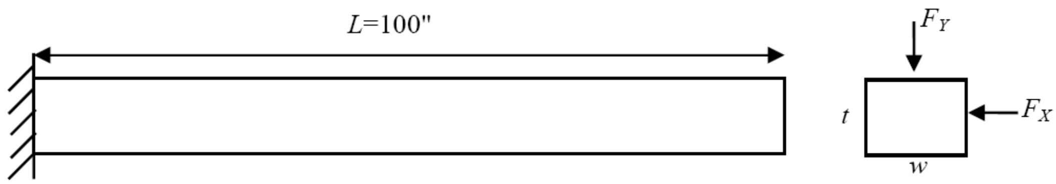

Figure 6. Cantilever beam subjected to random horizontal and vertical random loads

The probability of failure calculations are based on the strength limit-state function given in Eq. (19) along with the random variables' distributions in Table 1.

$$G(Y,S) = S - Y = \left(\frac{600}{w^2 t} F_x + \frac{600}{wt^2} F_y \right) - Y \tag{19}$$

Table 1. Distributions of random variables in cantilever beam problem

Random Variable	F_X	F_Y	Y
Distribution	Normal (500,100)lb	Normal (1000,100)lb	Normal (40000,2000) psi

In this problem, the capacity is yield strength, *Y*, and the response is the applied stress *S*. The probability of failure of the cantilever beam system was found in the previous work and verified to be nominally 0.00134. The next two sections evaluate the accuracy of predicting the probability of failure through five methods. First, the probability of failure was determined through standard Monte Carlo simulations using the limit state method from Eq. (3). Then separable Monte Carlo was used in both directions, response into capacity and capacity into response, as discussed in Eqs. (8) and (10). The final two methods of determining the probability of failure was based on tail modeling. The tail model was first considered for the limit state of the beam stress, as performed by Kim *et al*. (2006) (Eq. (5)). The other use of tail modeling involved separable Monte Carlo methods to separate the response and capacity. Since the tail model only estimates the CDF above the threshold, the value of *g* was fixed at about the 95% level, to ensure equal analysis of both methods with different sample sizes. In other words, the tail model estimated the top five percent of the CDF of the limit state and response.

For separable Monte Carlo, one of the random variables must be chosen for the CDF and the other chosen for the PDF. Since the probability density function of the capacity is easily obtained from the given information, a tail model was generated for the CDF of the response. Given this choice of CDF and PDF, Eq. (9) was used to calculate the probability of failure as

$$p_f = \int_{-\infty}^{g} \left[1 - F_r(c)_A\right] f_c(c)dc + \int_{g}^{\infty} \left[1 - F_r(c)_{\xi,\sigma}\right] f_c(c)dc \tag{20}$$

Since more than just the upper five percent estimate of the CDF from tail modeling was required, Arena® software was used to estimate the CDF below *g*, labeled as $F_r(c)_A$. This is a safe estimate that still emphasizes tail modeling, because Arena® can model the middle part of the CDF well, however the tail is more difficult. Therefore tail modeling was used for the part of the integral above the threshold

Table 2 presents the coefficient of variation of the five probability of failure calculation methods, for different sample sizes.

Table 2. Coefficient of variation (%) of probability of failure from beam example

N	Limit state MCS	SMC simulation (R into C)	SMC simulation (C into R)	Tail model of limit-state	Tail model of response (C into R)
10^3	87.65	33.30	13.49	-	26.92
10^4	27.72	11.37	2.59	19.55	10.47
10^5	8.76	4.44	1.12	7.30	2.19
10^6	2.77	1.64	0.44	1.72	0.95

Referring to the standard limit state MCS results in the table, to obtain one failure, at least 10^3 simulations must be performed for $p_f = 0.00134$, nominally. The separable Monte Carlo, where the capacity was sampled in the CDF of the response was the most efficient method in this example, since the capacity had a smaller standard deviation, or was closer to deterministic. This form of SMC obtained about the same lever of accuracy as standard Monte Carlo with two orders of magnitude fewer samples (10^4 vs. 10^6). Using separable Monte Carlo in conjunction with tail modeling was slightly less accurate than without tail modeling. This result is likely due to the uncertainty involved with estimating the tail modeling parameters, ξ and σ, and from using two different CDFs (one from Arena® and one from tail modeling. The tail modeling of the limit state however, was more accurate than just limit state MCS.

There are two factors that could have been considered to better isolate and emphasize the differences in each method. First, tail modeling is most effective for complicated distributions; however, the response in this case was simply normal. This resulted in the known expression of a normal CDF modeling the distribution better than the tail modeling estimate. If the response was a complicated distribution, then the tail modeling should have increased the effectiveness of SMC. The second factor is the lack of consideration of the variability in sampling for estimating the parameters of the response's normal distribution. Though this factor is expected to be quite small, it would have increased the coefficient of variation of the probabilities of failure calculated with the CDF of the response.

VI. Conclusions

A separable method was applied to limit states to separate the response and capacity. Standard limit state Monte Carlo was compared for efficiency to separable Monte Carlo simulation methods using uniform distributions. Simplified forms of the probability of failure and sample size equations were developed using ratios of the overlap region of the uniform distributions to the ranges of response and capacity, respectively. The simplified form of the required sample size using separable Monte Carlo to the standard method was proportional to the probability of failure and a ratio of the standard deviations of the capacity and response. It was observed that the advantage of using separable Monte Carlo is greatest for very small probability of failures. Both simulation methods were then applied to a beam problem. The selection of the appropriate cumulative distribution function is important in separable methods, which depends on the nature of the random variables. The CDF was obtained analytically, through Monte Carlo sampling, and tail modeling. The most effective simulation method for the beam problem was separable Monte Carlo when the capacity was sampled in the CDF of the response. Tail modeling also improved the efficiency of the limit state analysis.

Appendix
Derivation of Sample Size Expressions for Uniform Distributions

Equation Section (Next)

Uniform distributions were chosen for convenience of calculations. Figure A1 shows a general scenario of probability of failure for two uniform distributions.

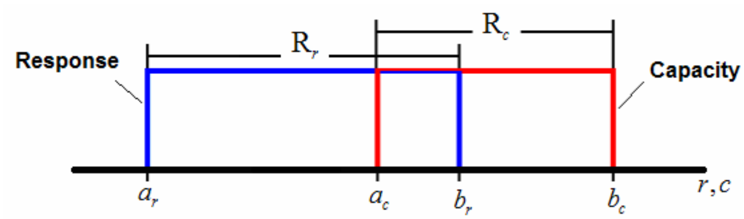

Figure A1. General uniform probability density functions for response and capacity

From Fig. A1, *R* is the range of the distribution, *a* is the lower bound, and *b* is the upper bound. Since the probability of failure will not be changed by translation or inflation of *r* and *c*, it is convenient to represent the distributions and their overlap by ratios as shown on Fig. A2.

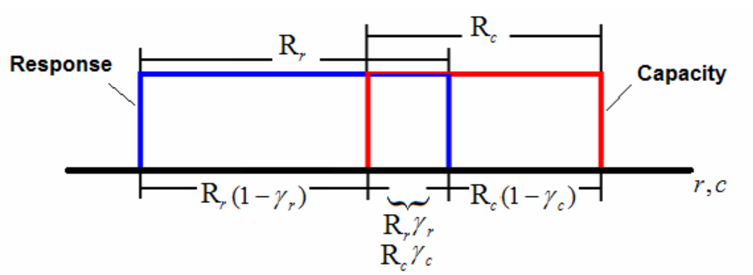

Figure A2. General uniform PDFs for $0 \le p_f \le 0.5$, showing ratios of ranges

The ratios described are labeled γ_r and γ_c, and are given as

$$\gamma_r = \frac{b_r - a_c}{R_r} \qquad\qquad \gamma_c = \frac{b_r - a_c}{R_c} \tag{A1}$$

Figure A2 only applies when the bounds of the two distributions remain in the same sequence shown in Fig. A1 ($a_r \le a_c$ and $b_r \le b_c$). This implies that $0 \le p_f \le 0.5$, which is most often the case in engineering situations. The derivations will show detail for the separable method when the response is sampled in the CDF of the capacity, shown in Eq. (A2).

$$p_f = \int_{Rr} F_c(r) f_r(r) dr \tag{A2}$$

The alternate method has a nearly identical derivation, so only the result is provided. To better understand the scenario in which separable Monte Carlo is being applied, the cumulative distribution function of the capacity for the uniform distribution is shown in Fig. A3.

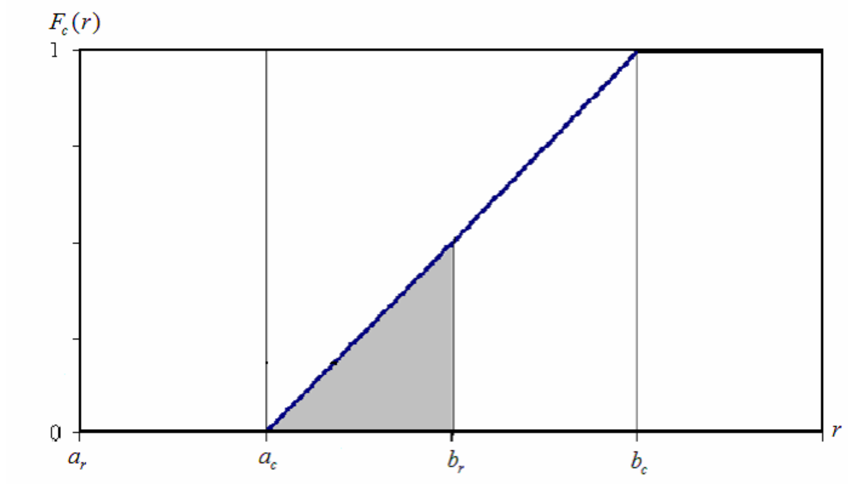

Figure A3. CDF of the capacity for $a_r \le a_c$ and $b_r \le b_c$ ($0 \le p_f \le 0.5$)

An expression of the probability of failure from Eq. (A2) is

$$p_f = \left(\frac{1}{b_r - a_r}\right)\int_{ar}^{br} \frac{r - a_c}{b_c - a_c} dr = \frac{(a_c - b_r)^2}{2(b_r - a_r)(b_c - a_c)} = \frac{1}{2}\gamma_r\gamma_c \tag{A3}$$

A measure of the accuracy of a Monte Carlo simulation can be quantified with the variance of the estimate.

$$\sigma_p^2 = \frac{p_f(1 - p_f)}{N} \tag{A4}$$

Therefore, to achieve a given variance, σ_p^2, with CMC requires N_{mc} samples. The relative error is given by the coefficient of variation, $C_{Vp} = \sigma_p / p_f$. Rewriting Eq. (A4),

$$N_{mc} = \frac{(1 - p_f)}{p_f C_{Vp}^2} \tag{A5}$$

Simplifying further, Eq. (A3) can be used for p_f to obtain

$$N_{mc} = \frac{1}{C_{Vp}^2}\left[\frac{2}{\gamma_r\gamma_c} - 1\right] \tag{A6}$$

Next, an expression for the required sample size of separable Monte Carlo will be derived. Based on the variance of the probability of failure calculation for SMC is given as

$$\sigma_p^2 = \int_{Rr} \left(F_c(r) - p_f\right)^2 f_r(r) dr \tag{A7}$$

Referring to Fig. A3, the separable Monte Carlo variance now becomes

$$\sigma_p^2 = \frac{1}{N}\left(\frac{1}{b_r - a_r}\right)\int_{ar}^{br}\left(\frac{r - a_c}{b_c - a_c} - p_f\right)^2 dr \tag{A8}$$

Integrating and solving for the sample size yields

$$N_{smc} = \frac{1}{\sigma_p^2}\left(\frac{1}{b_r - a_r}\right)\left[p_f^{\,2}(a_c - a_r) + \frac{b_c - a_c}{3}\left(\left(\frac{b_r - a_c}{b_c - a_c} - p_f\right)^3 + p_f^{\,3}\right)\right] \tag{A9}$$

The expression can be considerably simplified by substituting in the ratios from Eqs. (A1) and (A3). Additionally, the variance is replaced by the coefficient of variation, resulting in Eq. (A10).

$$N_{smc} = \frac{1}{C_{Vp}^2}\left[\frac{4}{3\gamma_r} - 1\right] \tag{A10}$$

Similarly, for the alternate form of SMC where the capacity is sampled in the CDF of the response, the required sample size is

$$N_{smc} = \frac{1}{C_{Vp}^2}\left[\frac{4}{3\gamma_c} - 1\right] \tag{A11}$$

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References

[1]Kalos, M. H., Whitlock, P. A., *Monte Carlo Methods Volume I: Basics*, John Wiley and Sons, New York, 1986, pp. 92-103.

[2]Qu, X., Venkataraman, S., and Haftka, R.T., "Response Surface Options for Probabilistic Optimization of Stiffened Panels," *Joint Specialty Conference on Probabilistic Mechanics and Structural Reliability*, July 2000.

[3]Qu, X., and Haftka, R. T., "Reliability-Based Design Optimization Using Probability Sufficiency Factor," *Structural and Multidisciplinary Optimization*, Vol. 27, No. 5, 2004, pp. 314-325.

[4]Kale, A. A., Haftka, R.T., and Sankar, B.V., "Reliability Based Design and Inspection of Stiffened Panels Against Fatigue," *AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*, AIAA paper 2005-2145, April, 2005.

[5]Kim, N. H., Ramu, P., and Queipo, N. V., "Tail Modeling in Reliability-Based Design Optimization for Highly Safe Structural Systems," *47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*, May, 2006.

[6]Melchers, R. E., *Structural Reliability: Analysis and Prediction*, 2nd ed., John Wiley and Sons, New York, 1999, Chap. 3.

[7]Arena, Software Package, Ver. 3.01, Systems Modeling Corporation, Pittsburgh, PA, 1997.