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Haoyu Wang and Nam H. Kim
Dept. of Mechanical & Aerospace Engineering
University of Florida
Gainesville, Florida, 32611, USA

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Robust Design Using Stochastic Response Surface and Sensitivities

Haoyu Wang[1] and Nam H. Kim[2]

Dept. of Mechanical & Aerospace Engineering, University of Florida, Gainesville, Florida, 32611

A robust design method that can reduce the variance of the output performance as well as the deviation of the mean value is proposed using a stochastic response surface method and an efficient sensitivity analysis. Both the deterministic and random design variables are considered. The stochastic response surface using a polynomial chaos expansion is used to describe uncertainty propagation. It is shown that the polynomial chaos expansion with appropriate bases provides an accurate and efficient tool in evaluating the performance variance. The results are compared with the traditional linear approximation and Monte Carlo simulation. In addition, the sensitivity of the output variance, which is critical in the mathematical programming method, is calculated by consistently differentiating the polynomial chaos expansion with respect to the design variables. Lastly, the variance-based global sensitivity indices are calculated in order to estimate the effect of the input random variables on the output variance. Numerical examples are shown to verify accuracy of the sensitivity information and the convergence of the robust design problem.

I.   Introduction

Engineering system analysis often identifies the effect of input parameters on the output performance function. In many engineering applications, the values of input parameters are not deterministic but probabilistic, including tolerances, material properties, operating conditions, etc. Such uncertainties propagate through the system analysis and, as a result, the output performance also shows probabilistic distribution. In quality engineering, it has been realized that the deviation from the target value of performance due to the uncontrollable input variances/noises results in quality loss. Thus, robust design, which targets on making performance of product insensitive (robust) to the noise factors, has been pulling increasing attention in recent research activities.

Robust design, initially known as Taguchi parameter design [1, 2], is to design a product in such a way that the performance reliability is insensitive to the variation of variables that are beyond the control of design engineers. Wang & Kodiyalam [3] formulated robust design as an optimization problem by minimizing the variation of system response. Since the reduction of input variance is directly related to the manufacturing cost, the formulation of Chen and Du [4] compromises cost reduction with performance variance. A robust design can also be achieved by using traditional optimization techniques to minimize the performance sensitivities. Chen & Choi [5] formulated the robust design by minimizing a total cost function and sum of squares of magnitudes of first-order design sensitivities, which requires the evaluation of second-order sensitivity analysis. This is a different approach compared to the variance-based approach. It focuses on the local behavior of the system performance and can achieve local robustness. The final design by minimizing local sensitivity cannot guarantee the robustness of system globally if the input variances are considerable.

Traditionally, the performance variance is evaluated either using the Monte-Carlo simulation (MCS) or linear approximation. The computational cost of MCS and the lack of accuracy of the linear approximation have been issues in the robust design. In this paper, an efficient and accurate method of evaluating the performance variance is proposed using the polynomial chaos expansion [6, 7]. The proposed method has comparable accuracy with MCS, while requiring much less computational cost. By selecting appropriate bases of the surrogate model, the performance variance is calculated analytically. In addition, the derivatives of the performance variance with respect to design variables and input random parameters are calculated consistently with the variance calculation method, which is critical information for design optimization algorithm.

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[1] PH.D. Candidate, Member AIAA
[2] Assistant Professor, Member AIAA

In general, the robust design problem should not be formulated to reduce the variance alone. Even if robustness is a requirement from quality point of view, a good design should also satisfy the requirement of the performance. In most of cases, quality and performance requirement are two competing design objectives. Thus, the robust design problem becomes a multi-objective optimization problem. In multi-objective optimization, there will be multiple optimum designs in a sense that one objective function cannot be reduced further without increasing other objective functions. The optimal set is referred to as the Pareto optimal set and yields a set of possible answers from which the engineer may choose the desired values of the design variables.

The paper is structured as follows: Section 2 presents how to calculate the performance variance and its sensitivity using a surrogate model. Especially, the stochastic response surface using polynomial chaos expansion is used. Section 3 introduces global sensitivity indices, which describe the contribution of random inputs to the performance variance. In section 4 robust design for dynamic response of a cantilevered composite beam is used as a numerical example, followed by conclusions in Section 5.

II. Performance Variance and Its Sensitivity using SRS

A. SRS for Variance Calculation

In this paper, the stochastic response surface (SRS) [7-10] is used as a surrogate model since it is easy to address uncertainty properties in system response. The stochastic response surfaces can be view as an extension of classical deterministic response surfaces for model outputs constructed using uncertain inputs and performance data collected at heuristically selected collocation points. The polynomial chaos expansion uses Hermite polynomial bases for the space of square-integrable probability density function (PDF) and provides a closed form solution of model outputs from a significantly lower number of model simulations than those required by conventional methods such as modified Monte Carlo Methods and Latin Hypercube Sampling.

Let n be the number of random variables and p be the order of polynomial. The model output can then be expressed in terms of the vector of standard random variables (SRV) $\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_n\}^{\mathrm{T}}$ as:

$$g_p = a_0^p + \sum_{i=1}^{n} a_i^p \Psi_1(\xi_i) + \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij}^p \Psi_2(\xi_i, \xi_j) + \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} a_{ijk}^p \Psi_3(\xi_i, \xi_j, \xi_k) + \cdots \qquad (1)$$

where g_p is the approximated model output, $a_i^p, a_{ij}^p, \ldots$ are deterministic coefficients to be estimated, and $\Psi_p(\xi_i, \ldots, \xi_p)$ are multidimensional Hermite polynomials of degree p given by:

$$\Psi_p(\xi_i, \ldots, \xi_p) = (-1)^p e^{1/2 \xi^T \xi} \frac{\partial^p}{\partial \xi_i \ldots \partial \xi_p} e^{-1/2 \xi^T \xi} \qquad (2)$$

where $\boldsymbol{\xi} = \{\xi_k\}_{k=1}^{p}$ is the vector of p independent and identically distributed normal random variables that represent the model input uncertainties. The coefficients can be calculated using a regression method with samples of input/output pairs. The unique feature of the polynomial chaos expansion is that it uses the SRV and Hermite polynomial bases. Due to the property that the Hermite bases are orthogonal with respect to an inner product defined using Gaussian measures, the polynomial chaos expansion is convergent in the mean-square sense. In general, the approximation accuracy increases with the order of the polynomials and should be selected by reflecting the accuracy needs and computational constraints. In addition, the approximation in Eq. (1) is robust in a sense that the coefficients of the low-order approximation does not change significantly in the high-order approximation.

One important issue in robust design is to evaluate the performance variance. Traditionally, a linear approximation using Taylor series expansion is often employed for that purpose [11]. However, the error of approximation increases according to nonlinearity of the performance. In addition, the coupled effect of input variance cannot be counted in the linear model.

The advantage of the polynomial chaos expansion in Eq. (1) becomes clear in evaluating the variance. In general, the polynomial chaos expansion in a surrogate model provides an analytical solution for the variance. If the polynomial bases are generally defined as $\Psi_i(\boldsymbol{\xi})$ with $\boldsymbol{\xi}$ being the vector of standard random variable, the SRS in Eq. (1) can be re-written as

$$g(\boldsymbol{\xi}) = \sum_{i=1}^{N} a_i \cdot \Psi_i(\boldsymbol{\xi}) \tag{3}$$

where g is the approximated system performance and N is the number of coefficients in SRS. Since the above expression is linear with respect to the unknown coefficients, the performance variance can be written as

$$\mathrm{Var}(g) = \sum_{i=1}^{N} a_i^2 \cdot \mathrm{Var}[\Psi_i(\boldsymbol{\xi})] \tag{4}$$

Thus, the analytical expression of the performance variation can be obtained if the variations of the polynomial bases are available. When input variables are SRV, the analytical variations of Hermite bases can be found in Ghanem and Spanos [6].

B. Variance Sensitivity

The robust design problem in this paper is formulated as an optimization problem that minimizes the performance variation in Eq. (4). In gradient-based optimization algorithms, calculation of sensitivity information is a critical issue for saving the computational cost and making the algorithm to converge. The finite difference method requires a complete recalculation of the performance variation [12]. The goal is to calculate the gradient information without carrying out a complete recalculation of the performance variance. From the fact that the SRV in the polynomial chaos remains constant while the design changes, the regression coefficients only depend on design variables. In the proposed polynomial chaos expansion, thus, the gradient of the performance variance with respect to j-th design parameter, d_j, can be written as

$$\frac{\partial \,\mathrm{Var}(g)}{\partial d_j} = \sum_{i=1}^{N} 2a_i \frac{\partial a_i}{\partial d_j} \mathrm{Var}[\Psi_i(\boldsymbol{\xi})] \tag{5}$$

It is clear that the derivatives of regression coefficients are enough to calculate the derivative of performance variation. In the linear regression method, the coefficients of SRS are obtained from

$$\mathbf{a} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{g} \tag{6}$$

where $\mathbf{g} = [g_1, g_2, \ldots, g_M]^{\mathrm{T}}$ is the vector of performance functions at sampling points, and $\mathbf{X}$ is the matrix of bases at sampling points, defined as

$$\mathbf{X} = [\Psi_{\mathbf{i}}(\boldsymbol{\xi})] = \begin{bmatrix} \Psi_1(\xi_1) & \Psi_2(\xi_1) & \ldots & \Psi_N(\xi_1) \\ \Psi_1(\xi_2) & \Psi_2(\xi_2) & \ldots & \Psi_N(\xi_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_1(\xi_M) & \Psi_2(\xi_M) & \ldots & \Psi_N(\xi_M) \end{bmatrix}_{M \times N} \tag{7}$$

In the above equation, M is number of sampling points, and N is the number of bases. Then, the derivatives of the coefficients can be obtained from

$$\frac{\partial \mathbf{a}}{\partial d_j} = ((\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \frac{\partial \mathbf{g}}{\partial d_j} \tag{8}$$

The last term, $\partial \mathbf{g} / \partial d_j$, is the derivative of performance function at sampling points, which can be calculated using design sensitivity analysis (see Choi and Kim [13, 14]). By substituting Eq. (8) into Eq. (5), the derivative of performance variation can be obtained. This procedure of calculating sensitivity of the performance variation is much more efficient than the traditional finite difference method because most information, such as $\mathbf{a}$ and $\mathbf{X}$, is

already available from the performance variation calculation. The only term required for sensitivity analysis is $\partial \mathbf{g} / \partial d_j$.

When finite element analysis is used as a computational tool for calculating the performance function, sensitivity analysis provides an efficient tool for calculating the performance derivative. In the context of structural analysis, for example, the discrete system is often represented using a matrix equation of the form $[\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{F}\}$. The performance function $\mathbf{g}$ in Eq. (6) can be expressed as a function of the nodal solution $\{\mathbf{D}\}$. Thus, the sensitivity of the performance can be easily calculated if that of the nodal solution is available. When design variables are defined, the matrix equation can be differentiated with respect to them to obtain

$$[\mathbf{K}]\left\{\frac{\partial \mathbf{D}}{\partial d_j}\right\} = \left\{\frac{\partial \mathbf{F}}{\partial d_j}\right\} - \left[\frac{\partial \mathbf{K}}{\partial d_j}\right]\{\mathbf{D}\} \tag{9}$$

Equation (9) can be solved inexpensively because the matrix $[\mathbf{K}]$ is already factorized. The computational cost of sensitivity analysis is usually less than 20% of the original analysis cost so local sensitivity can in fact be obtained efficiently.

C. Example – Cantilevered Beam

As an illustrative example, a cantilevered beam (Figure 1) is taken from literature [15, 16]. Two failure modes are considered in this example: (1) the maximum stress of the beam should be less than the strength of the material [Eq.(10)], and the tip deflection should be less than the allowable displacement [Eq.(11)]. These two constraints can be expressed by

$$g_1 = R - (\frac{600}{wt^2}Y + \frac{600}{w^2t}X) \geq 0 \tag{10}$$

$$g_2 = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{Y}{t^2}\right)^2 + \left(\frac{X}{w^2}\right)^2} \geq 0 \tag{11}$$

where R is the yield strength, E is the elastic modulus, X and Y are the independent horizontal and vertical loads shown in Figure 1. D_0 is the allowable tip displacement which is given as 2.25 in.
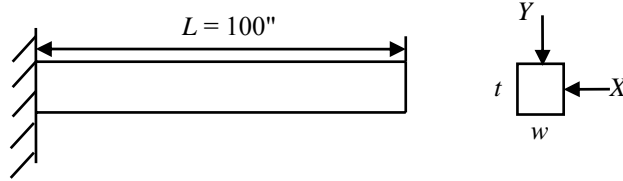

Figure 1: Cantilever beam subject to two direction loads

Two cross-sectional dimensions, w and t, are considered as controllable design variables. Five random variables are defined in Table 1.

Table 1: Random variables for cantilevered beam structure

Random variable	X	Y	R	E
Distribution type	Normal $(500,100^2)$ lb	Normal $(1000,100^2)$ lb	Normal $(40000,2000^2)$ psi	Normal $(29E6,(1.45E6)^2)$ psi

It is obvious that strength constraint defined in Eq. (10) is a linear function of the random inputs. For linear performance, the variance can be analytically obtained as

$$\mathrm{Var}(g_1) = \mathrm{Var}(R) + (\frac{600}{wt^2})^2 \, \mathrm{Var}(Y) + (\frac{600}{w^2t})^2 \, \mathrm{Var}(X) \tag{12}$$

Using this property, the accuracy of the proposed variance estimation in Eq. (4) can be verified. Table 2 shows the comparison between the variance from the SRS-based method and that from analytical approach. The variance is calculated at the deterministic optimal design ($w = 1.9574"$, $t = 3.9149"$).

Table 2: Variance estimation of linear performance (strength)

Analytical variance	Variance from SRS (3rd-order)
2.4E7	2.4E7

In the case of strength constraint, it is possible to find the analytical expression of the variance. However, in the case of nonlinear performance, such as deflection constraint in Eq. (10), there is no easy way of calculating analytical expression except for the first-order approximation. The linear approximation of the deflection constraint becomes

$$\mathrm{Var}(g_2)_{\mathrm{linear}} = \left(\frac{\partial g_2}{\partial X}\right)^2 \mathrm{Var}(X) + \left(\frac{\partial g_2}{\partial Y}\right)^2 \mathrm{Var}(Y) + \left(\frac{\partial g_2}{\partial E}\right)^2 \mathrm{Var}(E) \tag{13}$$

Due to the error involved in the linear approximation, MCS is the only method that can verify the accuracy of variance calculation. Since MCS is a sampling-based method, the estimated variance always has variability. Let σ^2 be the variance of a random variable and let s^2 be the unbiased estimator of σ^2. When n number of samples are used, the variance of the MCS-estimated performance variance can be predicted by [17]

$$\mathrm{Var}(s^2) = \frac{\sigma^4}{n}(\frac{\mu_4}{\sigma^4} - \frac{n-3}{n-1}) \tag{14}$$

where $\mu_4 = E(X - \mu)^4$ is the fourth central moment of random variable X. μ_4 / σ^4 is called kurtosis.

For the nonlinear performance in Eq. (11), the third-order SRS is used to approximate the deflection and the expression in Eq. (4) is used to evaluate the performance variance. Table 3 compares the variance obtained from these three methods. As expected, the linear approximation has about 1% error compared with MCS, while SRS-based variance is within the confidence range of MCS. The error in the linear approximation will increase proportional to the nonlinearity of the function.

Table 3: Variance estimation of nonlinear performance (deflection)

$\mathrm{Var}(g_2)_{\mathrm{MCS}}$ (500,000 samples)	$\mathrm{Var}(g_2)_{\mathrm{linear}}$	$\mathrm{Var}(g_2)_{\mathrm{SRS}}$
0.1947 (standard deviation = 3.9248E−4)	0.1966	0.1948

Based on the accuracy of the proposed method in calculating performance variance, the variance sensitivity in Eq. (5) is also tested using the cantilevered beam model. Table 4 and Table 5 show the variance sensitivities obtained from the proposed method compared with those from the central finite difference method (FDM). In FDM, the design variables are perturbed by 2% and the variance is recalculated using the SRS. When the performance is linear with respect to random variables, the analytical sensitivity can be obtained, for example, by differentiating Eq. (12) with respect to design variable. In Table 4, the sensitivity obtained from SRS agrees well with that from analytical sensitivity. The finite difference sensitivity shows a small error because the variance is still a nonlinear function with respect to the design variable.

Table 4: Sensitivity of variance for linear performance (strength)					
$\partial \mathrm{Var}/\partial w$ (SRS)	$\partial \mathrm{Var}/\partial w$ (FDM)	$\partial \mathrm{Var}/\partial w$ (Analytical)	$\partial \mathrm{Var}/\partial t$ (SRS)	$\partial \mathrm{Var}/\partial t$ (FDM)	$\partial \mathrm{Var}/\partial t$ (Analytical)
−3.6784E7	−3.6801E7	−3.6785E7	−1.2261E7	−1.2265E7	−1.2261E7

Table 5: Sensitivity of variance for nonlinear performance (deflection)			
$\partial \mathrm{Var}/\partial w$ (SRS)	$\partial \mathrm{Var}/\partial w$ (FDM)	$\partial \mathrm{Var}/\partial t$ (SRS)	$\partial \mathrm{Var}/\partial t$ (FDM)
−0.6538	−0.6544	−0.0712	−0.0712

III.   Variance-based Global Sensitivity Analysis

The sensitivity of variance in the previous section is the derivative with respect to deterministic design variables. In some cases, such as tolerance analysis, the output variance is controlled by changing the variance of input variables. Since controlling the input variance often accompanies manufacturing cost or additional tests, it is important to find the most effective input variable to the performance variance. An index called global sensitivity [18-22] can be used for that purpose. The global sensitivity indices are the contributions of input random variables to the performance variance. Variance-based methods are rigorous and theoretically sound approaches [18-22] for global sensitivity calculation. This section describes the fundamentals of the variance-based approach and illustrates how the polynomial chaos expansions are particularly suited for this task.

The variance based methods: (i) decompose the model output variance as the sum of partial variances, and then, (ii) establish the relative contribution of each random variable (global sensitivity indexes) to the model output variance. In order to accomplish step (i), the model output is decompose as a linear combination of functions of increasing dimensionality as described by the following expression:

$$g(\mathbf{x}) = a_0 + \sum_{i=1}^{n} a_i f_i(x_i) + \sum_{i=1}^{n}\sum_{j>i}^{n} a_{ij} f_{ij}(x_i, x_j) + \ldots + a_{12\ldots n} f_{12\ldots n}(x_1, x_2, \ldots, x_n) \tag{15}$$

The above decomposition is subject to the restriction that the integral of the weighted product of any two different functions is zero. Formally,

$$\int \cdots \int p(\mathbf{x}) f_{i_1,\ldots,i_s}(x_{i_1},\ldots,x_{i_s}) f_{j_1,\ldots,j_s}(x_{j_1},\ldots,x_{j_s}) d\mathbf{x} = 0, \quad \text{for} \quad i_1,\ldots,i_s \neq j_1,\ldots,j_s \tag{16}$$

where $p(\mathbf{x})$ is the joint probability distribution function (PDF) of the vector $\mathbf{x}$ of input random variables. Depending on distribution type of input variables, there exists a family of polynomials that satisfy the above requirement. If, for example, the weighting function is the uniform distribution for the random variables or the Gaussian probability distribution, the functions of interest can be shown to be Legendre and Hermite orthogonal polynomials, respectively.

Once a performance function is decomposed in the form of Eq. (15), The variance can now be calculated using a well-known result in statistics. The result establishes that the variance of the linear combination of random variables (x_i) can be expressed as:

$$\mathrm{Var}\left(b_0 + \sum_{i=1}^{n} b_i x_i\right) = \sum_{i=1}^{n} b_i^2 \, \mathrm{Var}(x_i) + 2\sum_{i=1}^{n}\sum_{j>i}^{n} COV(x_i, x_j) \tag{17}$$

Hence, the performance variance can be shown to be:

$$\mathrm{Var}(g) = \sum_{i=1}^{n} a_i^2\,\mathrm{Var}(f_i) + \sum_{i=1}^{n}\sum_{j>i}^{n} a_{ij}^2\,\mathrm{Var}(f_{ij}) + \ldots + a_{12\ldots n}^2\,\mathrm{Var}(f_{12\ldots n}) \tag{18}$$

There are no covariance terms in Eq. (18) because of the orthogonal property shown in Eq. (16).

In general the global sensitivity can be decomposed into main factors and interactions between different input random variables. The global sensitivity index, S_i, that considers only main factor is called main sensitivity index, which associated with each of the random variables. From Eq. (18), the main sensitivity indices can be calculated by

$$S_i = \frac{a_i^2\,\mathrm{Var}(f_i(x_i))}{\mathrm{Var}(g)}, \quad i = 1, 2, \ldots n \tag{19}$$

A sensitivity index that considers the interaction between two or more factors is called interaction sensitivity index. From Eq. (18), the interaction sensitivity indices can be calculated by

$$S_{ij\ldots} = \frac{a_{ij\ldots}^2\,\mathrm{Var}(f_{ij\ldots}(x_i, x_j, \ldots))}{\mathrm{Var}(g)} \tag{20}$$

As denoted by Chan and Saltelli [18], the summation of all sensitivity indices, involving both main and interaction effect of *i*-th random variable, is called *total sensitivity index* (S_i^{total}). From Eq. (18), the total sensitivity indices can be calculated by

$$S_i^{\text{total}} = \frac{S_i + \sum_{j=1}^{n} S_{ij} + \sum_{j=1}^{n}\sum_{k=1}^{n} S_{ijk} + \ldots}{\mathrm{Var}(g)}, \quad i = 1, 2, \ldots n \tag{21}$$

Sobol [20] suggested to use total sensitivity indices to fix unessential variables. If total sensitivity index for certainty variable extremely small compare to 1, that means the contribution of the variable is negligible and the variable can be fixed.

IV. Robust Design – Two Layer Beam

A. Dynamic Response of Composite Beam

The robust design problem formulation is demonstrated using a cantilevered, composite beam, shown in Figure 2. When an electric field is applied to the piezoelectric part, it will generate bending moment and deform the beam. On the other hand, when the base is oscillating with a specific frequency, the deformation of the beam will induce electric field, which can be used as an energy reclamation device. System dynamic response of the composite beam is highly coupled and the closed-form solution is difficult to obtain [23-25]. In this paper a lumped element modeling technique (LEM, [26]) is used to obtain the approximate solution for the system. Under the quasi-static assumption, the LEM can estimate the first fundamental natural frequency with accuracy. First, the effective mechanical compliance (C_e) and the effective mass (M_e) can be calculated by lumping the total strain energy and kinetic energy, respectively. The detailed procedure is summarized in Appendix. The first natural frequency is then calculated using the following expression [26].

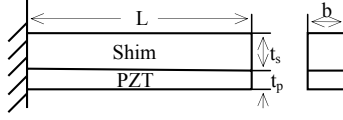$$f_n = \frac{1}{2\pi}\sqrt{\frac{1}{C_e M_e}} \tag{22}$$

Figure 2: Piezoelectric cantilevered composite beam

When the composite beam is used as an energy reclamation device, the maximum efficiency can be obtained when the excitation frequency and the natural frequency are resonant. Thus, the design goal is to find the design variables such that the natural frequency is as close as possible to the excitation frequency. However, due to the uncertainty of the material properties, the performance function (natural frequency) in Eq. (22) is not a deterministic quantity. Thus, the additional design goal is to minimize the variance of the natural frequency due to the input random variables.

B. Robust Design for Two Layer Beam

When a robust design problem is formulated in a way such that only the variance of the output is minimized, the optimization problem may find an inappropriate design without considering the mean value of the performance. Thus, it would be appropriate to consider both the variance and the mean value simultaneously. In this paper, the robust design problem is formulated as two-objective optimization: one for the variance and the other for the mean value. When two objectives are competing with each other, there will be no single optimum design. Instead, a Pareto optimal front can be constructed, which represents the best combination between the competing objective functions. Due to the uncertainty in inputs, all constraints are modeled as reliability constraints.

In the composite beam problem, the goal is to design a structure with natural frequency close to the prescribed value. Considering the uncertainties involved in input variables, however, the natural frequency at any design will have certain variation, which should also be minimized. In addition, the reliabilities for the stress and deflection constraints should be considered. In the reliability-based robust design, the reliability constraints are imposed by pushing the mean value to the certain levels of standard deviation in the conservation direction. Thus, the robust design problem is formulated as

$$\text{Minimize} \quad g_1 = \left| \mu_f - f_0 \right| \text{ and } g_2 = \sqrt{\text{Var}(f)}$$
$$\text{s.t.} \quad (\mu_\sigma - R) + k\sqrt{\text{Var}(\sigma - R)} \leq 0 \tag{23}$$
$$(\mu_w - D_0) + k\sqrt{\text{Var}(w - D_0)} \leq 0$$

where μ_f is the mean of the first natural frequency; f_0 is the excitation frequency; σ is the maximum stress; R is the material strength, which is assumed as 11,743Pa; w is the tip deflection and D_0 is the allowable maximum tip deflection, which is 7.138 nanometer; and k is the user-defined constant for specific target reliability level. It is assumed that uncertainties only exist in the material properties such as elastic modulus and material density. Table 6 lists the random parameters of these quantities. All random variables are assumed to be normally distributed and the standard deviation for the elastic modulus is 10% of the mean value and that of the density is 5%.

Table 6: Random parameters for the composite beam structure

Random variable	Mean	Standard deviation
Young's modus of shim (E_s)	169 GPa	16.9 GPa
Density of shim (ρ_s)	2330 kg/m^3	116 kg/m^3
Young's modus of PZT (E_p)	60 GPa	6 GPa
Density of PZT (ρ_p)	7500 kg/m^3	375 kg/m^3

In the composite beam problem, three design variables are defined: beam length L, shim thickness t_s, and PZT layer thickness t_p. The robust design problem involves three deterministic design variables and four random parameters. For given design variables, the SRS for the performance functions, such as natural frequency, stress, and tip deflection, are constructed according to Eq. (1). Then, the performance variances are calculated from Eq. (4) and

variance sensitivity from Eq. (5). The values and sensitivities of the two objectives functions g_1 and g_2 with respect to the three design variables are summarized in Table 7 at the initial design ($t_s = 6\mu m$, $t_p = 0.2\mu m$, $L = 1000\mu m$). Table 7 shows that for given design, the mean of the frequency will change at least 15 times more than the frequency variance. Thus, it is easier to change the mean values than to change the frequency variance. This observation leads to the idea of controlling the input variances directly rather than controlling the design variables in the following section.

Table 7: Sensitivities of objective functions at the initial design ($t_s = 6\mu m$, $t_p = 0.2\mu m$, $L = 1000\mu m$)

g_1(Hz)	g_2(Hz)	$\partial g_1/\partial t_s$ (Hz/m)	$\partial g_1/\partial t_p$ (Hz/m)	$\partial g_1/\partial L$ (Hz/m)	$\partial g_2/\partial t_s$ (Hz/m)	$\partial g_2/\partial t_p$ (Hz/m)	$\partial g_2/\partial L$ (Hz/m)
834.88	144.54	−491.02	562.05	5.335	27.15	−36.55	−0.29

Since two objective functions are competing with each other, there will be no single optimum design. In such a case, the value of one objective function is fixed and then the minimum value of the other objective function can be found. By repeating this procedure for different values, a Pareto optimal front can be constructed. Figure 3 shows the Pareto optimal front of the two-objective optimization problem in Eq. (23). All points in the Pareto front are optimum design in a sense that one objective function cannot be reduced further without increasing the other objective function.
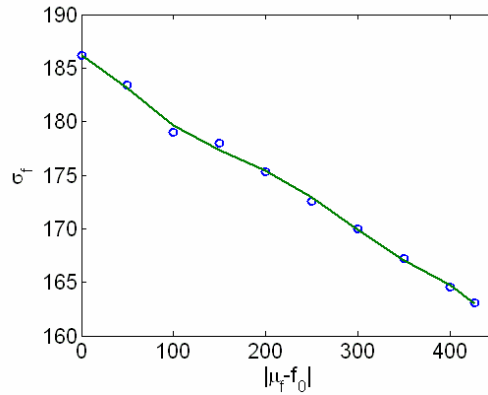

Figure 3. Pareto optimal front for the robust design of the composite beam

C. Global Sensitivity Analysis

In Figure 3, the change in the mean value (abscissa) is more significant than that in the standard deviation (ordinate), which is consistent with the observation in Table 7. This result indicates that when the design variables are deterministic, it is relatively easier to change the mean value rather than the performance variance. The performance variance can be changed more effectively by controlling input variance. However, controlling input variance accompanies manufacturing cost or large number of coupon tests. Thus, in practice, it is important to find the contribution of random variables to the performance variance, and then, to spend more resources in controlling the most significant random variable.

In Table 8, the contribution of input random variables to the performance variance is summarized in terms of total sensitivity indices in Eq. (21). It can be found that the contributions of ρ_s and E_s are more than 99% of the performance variance. Thus, it will be meaningful to reduce the variance of the shim rather than that of the PZT.

Table 8: Total sensitivity indices for the composite beam structure ($t_s = 6\mu m$, $t_p = 0.2\mu m$, $L = 1000\mu m$)

$S_{E_p}^{\text{total}}$	$S_{\rho_p}^{\text{total}}$	$S_{E_S}^{\text{total}}$	$S_{\rho_S}^{\text{total}}$
0.00%	0.96%	85.87%	13.17%

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V. Robust Design by Tolerance Control

In the previous section, the input variances were considered as uncontrollable variables and only deterministic design variables were considered. However, in tolerance design, the design variables are fixed, while the variances of random variables are changed to reduce the output variances further. However, in such a problem, the optimum design will reduce all input variances to zero. Thus, the robust design will turn out to be zero variance.

In practice, reducing input variance requires cost. Different costs are anticipated in reducing the variance of different inputs. The cost of controlling individual input variance can be represented by a cost-tolerance model [27]. Thus, a more appropriate robust design problem will be: for a given investment how much individual variance should be reduced in order to minimize the performance variance. Based on the total budget of controlling input variability, the robust design problem can be written as

$$
\begin{aligned}
\text{Minimize} \quad & \text{Var}\left[g(\sigma_1, \sigma_2, ..., \sigma_n) \right] \\
\text{s.t.} \quad & \sum_{i=1}^{n} C_i(\sigma_i) \leq C_{\text{total}}
\end{aligned}
\tag{24}
$$

where σ_i is the standard deviation of i-th random variable, C_i is the cost function of controlling i-th standard deviation, C_{total} is the total investment.

Similar to the robust design problem with deterministic design variables, the optimization problem in Eq. (24) requires the derivative of the performance variance. The only difference now is that the derivative is taken with respect to the input variance. By substituting j-th design variable d_j in Eq. (5) to j-th random parameter σ_j, the gradient of output variance in Eq. (24) with respect to j-th random parameter can be written as

$$
\frac{\partial \text{Var}(g)}{\partial \sigma_j} = \sum_{i=1}^{N} 2a_i \frac{\partial a_i}{\partial \sigma_j} \text{Var}[\Psi_i(\boldsymbol{\xi})]
\tag{25}
$$

Similarly, the derivatives of the coefficients can be obtained from

$$
\frac{\partial \mathbf{a}}{\partial \sigma_j} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \frac{\partial \mathbf{g}}{\partial \sigma_j}
\tag{26}
$$

Since all random variable are assumed to be independent, we have the following chain rule of differentiation:

$$
\frac{\partial \mathbf{g}}{\partial \sigma_j} = \frac{\partial \mathbf{g}}{\partial T_j^{-1}(\xi_j)} \cdot \frac{\partial T_j^{-1}(\xi_j)}{\partial \sigma_j}
\tag{27}
$$

where T_j is the transformation of j-th random variable from original random space to standard normal space:

$$
\xi_j = T_j(x_j)
\tag{28}
$$

Therefore, the sensitivity of performance variance with respect to random parameter can be obtained by combining Eqs. (5), (8) and (27) if the derivative $\partial \mathbf{g} / \partial T_j^{-1}(\xi_j) = \partial \mathbf{g} / \partial x_j$ is available.

As an illustration of the effectiveness and convergence properties of the proposed approach, the cantilevered beam model (Figure 1) in Section II is used. Based on the accurately estimated performance variance in Section II, the variance sensitivities with respect to input variances are calculated using proposed method in Eq. (25). Table 9 and Table 10 show the sensitivities obtained from the proposed method along with those from the finite difference method. Since the analytical sensitivity is available for the linear performance, Table 9 also lists the analytical sensitivity. It turns out that the proposed, SRS-based sensitivity calculation method provides accurate sensitivity information. Since the proposed method only requires the calculation of performance sensitivity at sampling points [Eq. (27)], the computational cost will be much less than that of the finite difference method.

Table 9: Sensitivity of variance for linear performance (strength)

$\partial \mathrm{Var}/\partial \sigma_X$ (SRS)	$\partial \mathrm{Var}/\partial \sigma_X$ (FDM)	$\partial \mathrm{Var}/\partial \sigma_X$ (Analytic)	$\partial \mathrm{Var}/\partial \sigma_Y$ (SRS)	$\partial \mathrm{Var}/\partial \sigma_Y$ (FDM)	$\partial \mathrm{Var}/\partial \sigma_Y$ (Analytic)	$\partial \mathrm{Var}/\partial \sigma_R$ (SRS)	$\partial \mathrm{Var}/\partial \sigma_R$ (FDM)	$\partial \mathrm{Var}/\partial \sigma_R$ (Analytic)
3.2e5	3.2e5	3.2e5	8.0e4	8.0e4	8.0e4	4000	4000	4000

Table 10: Sensitivity of variance for nonlinear performance (deflection)

$\partial \mathrm{Var}/\partial \sigma_X$ (SRS)	$\partial \mathrm{Var}/\partial \sigma_X$ (FDM)	$\partial \mathrm{Var}/\partial \sigma_Y$ (SRS)	$\partial \mathrm{Var}/\partial \sigma_Y$ (FDM)	$\partial \mathrm{Var}/\partial \sigma_E$ (SRS)	$\partial \mathrm{Var}/\partial \sigma_R$ (FDM)
3.41e-3	3.41e-3	5.77e-5	5.77e-5	2.66e-8	2.66e-8

With performance variance and its sensitivity, the robust design problem with input variance control in Eq. (24) can be solved efficiently. Consider the robust design problem that minimizes the variance of natural frequency with strength and deflection constraints, as

$$
\begin{aligned}
\text{Minimize} \quad & \sqrt{\mathrm{Var}(\omega)} \\
\text{s.t.} \quad & E(g_1) - k\sigma(g_1) \geq 0 \\
& E(g_2) - k\sigma(g_2) \geq 0 \\
& \sum_{i=1}^{n} C_i(\sigma_i) \leq C_{tot}
\end{aligned}
\tag{29}
$$

where g_1 and g_2 are strength and deflection constraints in Eq. (10) and (11), respectively. In this optimization problem, the deterministic design variables, w and t, are pre-determined ($w = 2.73$, $t = 3.50$) from the previous optimization. Now, the optimization is performed by changing the standard deviations of input random variables. In Eq. (29), ω is the first natural frequency of the beam defined as

$$
\omega = (\beta L)^2 \sqrt{\frac{EI}{\rho A L^4}} = \frac{\beta^2 t}{2}\sqrt{\frac{E}{3\rho}}
\tag{30}
$$

and $E(\cdot)$ and $\sigma(\cdot)$ represent the expect value and standard deviation of random output, respectively, and $C_i(\sigma_i)$ is the cost-tolerance function for the i-th random variable. For a specific boundary condition, the term, β, is constant. Thus, the objective function to control the variance of natural frequency is modified to

$$
\text{Minimize} \quad \sqrt{\mathrm{Var}\left(\frac{\omega}{\beta^2}\right)} = \sqrt{\mathrm{Var}\left(\frac{t}{2}\sqrt{\frac{E}{3\rho}}\right)}
\tag{31}
$$

Table 11 lists random variables and cost-tolerance functions [27] for the random variables.

Table 11: Random variables and cost-tolerance functions

Variables	X	Y	R	E	ρ
Distribution	$N(500,\sigma_1^2)$lb	$N(1000,\sigma_2^2)$lb	$N(40000,\sigma_3^2)$psi	$N(29E6,\sigma_4^2)$psi	$N(0.28,\sigma_5^2)$
Cost-tolerance	$1.5+200/\sigma_1$	$1.5+200/\sigma_2$	$1.5+1.6*10^7/\sigma_3^2$	$200\mathrm{Exp}(-\sigma_4*10^{-6})$	$18\mathrm{Exp}(-100\sigma_5)$
σ_i	$25\leq\sigma_1\leq200$	$50\leq\sigma_2\leq400$	$1000\leq\sigma_3\leq4000$	$10^6\leq\sigma_4\leq3*10^6$	$0.01\leq\sigma_5\leq0.05$

To demonstrate the robust design, total cost of controlling variance at initial design has been chosen as cost constraint. Thus, the design goal is to minimize the performance variance, while maintaining the same cost with initial variance control. Table 12 shows that the standard deviation of natural frequency reduced from 452.5 Hz to 325 Hz by redistributing the input variances. Since the natural frequency is independent of the applied loads and the two constraints are not active, the final design increased the variances of the first three random variables. The optimum design maintains the variance of the elastic modulus and halves the density, which is more cost effective than reducing the variance of the elastic modulus.

Table 12: Random variables and cost-tolerance functions

Design Variables	Initial design	Optimal design
σ_1	100	200
σ_2	100	400
σ_3	2000	4000
σ_4	$1.45*10^6$	$1.45*10^6$
σ_5	0.02	0.010762
Objective	452.5	325.0
C_{total}	17.8274	17.8274

VI. Conclusions

In this paper, SRS-based variance calculation is proposed to facilitate robust design application. Accurate variance sensitivity analysis is presented for the gradient-based optimizer. A simple cantilevered beam with two failure modes, one is linear and another is nonlinear, is used to illustrate the accuracy and robustness of variance calculation.

Robust design for the natural frequency of a cantilevered, composite beam showed that controlling deterministic design variables makes less change of the performance variance than that of the performance mean, we found it is more important to control the input variance itself rather than the design variable in our specific problem. Global sensitivity is then introduced to address which random variables should be paid more attention to reduce total performance variance.

Finally, a cost model based robust design is proposed to control the input variance, an alternative way of tolerance design. Design sensitivity analysis of performance variance with respect to input variance has been proposed in mathematical programming. Cantilever beam model is used to illustrate the effectiveness of tolerance design.

VII. Appendix

A. Bending Moment:

As indicated in Figure 2, the cantilever composite beam subjects to a bending moment (M_0) at the ends of the piezoceramic. This is caused by induced strain from applied voltage [28]. Figure A-1 replaces the mass of the composite beam as an equivalent uniform load (q) due to its weight. R and M_r are the reaction force and bending moment at the clamp.
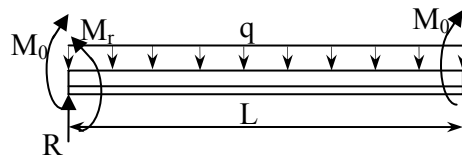

Figure A-1: Free body diagram of two-layer beam

Thus, the bending moment in the composite beam can be expressed as

$$M(x|0 \le x \le L) = M_r + Rx - M_0 - q\frac{x^2}{2} \tag{32}$$

where $R = qL$ and $M_r = qL^2/2$.

B. Geometric Properties of Composite Beam

Before we calculate the effective compliance and lumped mass, geometric properties such as location of neutral axis and flexural rigidity of the composite beam are required in static analysis of the beam.

If we define c_2 as the location of the neutral axis from the bottom of piezoceramic and $(EI)_c$ as equivalent flexural rigidity in composite beam, they can be calculated by the following two expressions:

$$c_2 = \frac{E_s t_s \left(t_p + \frac{t_s}{2}\right) + E_p \frac{t_p^2}{2}}{E_s t_s + E_p t_p}, \tag{33}$$

$$(EI)_c = E_s I_{sc} + E_p I_{pc} \tag{34}$$

where I_{sc} and I_{pc} are the moment of inertia of the shim and PZT layer with respect to its own neutral axis, respectively.

C. Effective Compliance for Composite Beam:

To find the effective compliance for the composite beam in Eq. (22), we need to use total potential energy in the beam as shown Eq. (35):

$$PE = \frac{(EI)_c}{2} \int_0^L \left(\frac{d^2 w(x)}{dx^2}\right)^2 dx \tag{35}$$

where

$$w(x) = -\frac{q}{24(EI)_c} x^4 + \frac{qL}{6(EI)_c} x^3 - \frac{qL^2}{4(EI)_c} x^2 \tag{36}$$

Equation (36) is obtained by conventional Euler-Bernoulli beam theory. Thus, by lumping the overall potential strain energy at the tip, an effective short circuit mechanical compliance for the composite beam will be calculated as

$$C_e = \frac{(w_{tip_F})^2}{2PE} \tag{37}$$

D. Effective Mass for Composite Beam:

In order to calculate the effective lumped mass in Eq. (22), total kinetic energy in the composite beam [Eq. (38)] will be used.

$$KE = \frac{\rho_{Lc}}{2} \int_0^L \dot{w}(x)^2 \, dx \tag{38}$$

where ρ_{Lc} is the equivalent mass density of the composite beam and $\dot{w}(x)$ is the velocity in the beam.

For a simple harmonic motion, the velocity of the beam are related to the displacement by

$$\dot{w}(x) = j\omega w(x) \qquad (39)$$

$\dot{w}(x)$ is then expressed as

$$\dot{w}(x) = \frac{w(x)}{w_{tip_F}} \dot{w}_{tip_F} \qquad (40)$$

Effective mass for the composite beam from its deflection shape is obtained by lumping the kinetic energy of the beam at its tip:

$$M_e = \frac{2KE}{(\dot{w}_{tip_F})^2} = \frac{\rho_{Lc}}{w_{tip_F}^2} \int_0^L w(x)^2 \, dx \qquad (41)$$

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