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Inverse measure-based tail modeling approaches for structural reliability estimation

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Sampling-based reliability estimation with expensive computer models may be computationally prohibitive. One way to alleviate the computational expense in high reliability designs is to extrapolate reliability estimates from observed levels to unobserved levels. Classical tail modeling approaches provide a class of models to enable this extrapolation using asymptotic theory by approximating the tail region of the cumulative distribution function (CDF). This paper proposes an alternate tail extrapolation based on inverse measure, which can complement classical tail modeling. The proposed approach applies a nonlinear transformation to the CDF of the inverse measure and approximates the transformed CDF by a quadratic polynomial. Accuracy and the computational efficiency are competing factors in selecting sample size. Yet, as our numerical studies reveal, the accuracy lost to the reduction of computational power is very small in the proposed method. The method is demonstrated on two engineering examples and on true statistical distributions.

1. Introduction

Aerospace and space applications typically demand high reliability. In a probabilistic perspective, high reliability translates to small probability content in the tails of the statistical distributions. Safety analysis such as reliability analysis, especially when dealing with high reliability (or low failure probability) designs is mostly dependent on how the tails of the random variables are modeled. In few cases, the safety levels can vary by an order of magnitude with slight modifications in the tails of the basic variables.

Limitations in computational power prevent us in employing direct simulation methods to model the tails. Hence, estimating high reliability involves the challenging task of accurately modeling the tails of the statistical distribution with limited data.

Powerful theories and results developed based on extreme value theory are useful to model tails of the statistical distributions efficiently. The distinguished feature of extreme value analysis is the objective to quantify the stochastic behavior of a process at unusually large or small levels (Coles 2001).

In structural engineering, reliability is measured by quantities like probability of failure or reliability index. Recently, alternate safety measures like the inverse measures have cornered enough interest because of their multifaceted advantages (Ramu et al, 2006). Among the several advantages they exhibit, inverse measures like probabilistic sufficiency factor (PSF) are capable of providing information about additional cost required achieving a safe design, stable and accelerated convergence in optimization, better response surface approximations compared to surfaces fit to other reliability measures.

Reliability measures can be estimated using several techniques like First–Order Reliability Method (FORM- Enevoldsen and Sorensen, 1994), Monte Carlo simulation (Qu et al., 2003), stochastic response surface (Kim et al., 2004), and worst–case analysis (Sundaresan et al., 1993). Monte Carlo methods are computationally expensive. Moment based methods like FORM are limited to address single failure modes. Stochastic response surface represents central model and it is reported that using central models to estimate large percentiles such as those required in reliability constraint calculations can lead to significant inaccuracies (Maes and Huyse, 1995).

This paper presents an approach for reliability estimation using inverse measures and general tail–

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models available from extreme value theory in statistics In tail-models, the conditional cumulative distribution function (CDF) above a certain threshold is approximated using the generalized Pareto distribution (GPD) (Castillo, 1988). Here, we explore the usefulness of inverse measure for tail modeling and use it in conjunction with classical approaches. The approach applies a nonlinear transformation to the CDF of the inverse measure and approximates the relationship by a quadratic polynomial. The proposed method does not approximate the functional expression of the model output; rather approximates the tail of the cumulative distribution. Thus, it has an advantage of the system reliability analysis and design in which no single form of functional expression is available.

The paper is structured as follows. Section 2 discusses reliability estimation using inverse measures. Tail modeling concepts and how it can be applied to estimate the inverse measure is discussed in Section 3. Section 4 discusses the extrapolation technique to estimate inverse measures corresponding to lower failure probability followed by a demonstration on a cantilever beam and tests on true statistical distribution. Conclusions are presented in Section 5.

2. Reliability Estimation Using Probabilistic Sufficiency Factor

The inverse measure used here is the probabilistic sufficiency factor (PSF) introduced by Qu and Haftka (2001, 2003). PSF is a safety factor with respect to the target probability of failure and hence combines the concepts of safety factor and the probability of failure.

Let the capacity of the system be g_c (e.g., allowable strength) and the response be g_r. For the given vector $\mathbf{x}$ of input variables, the traditional safety factor is defined as the ratio of the capacity to the response, as

$$S(\mathbf{x}) = \frac{g_c(\mathbf{x})}{g_r(\mathbf{x})} \tag{1}$$

The system is considered to be failed when $S \leq 1$ and safe when $S > 1$.

In probabilistic approaches, it is customary to use a performance function or a limit state function instead of the safety factor to define failure (or success) of a system. For example, the limit state function can be expressed as

$$G(\mathbf{x}) = S(\mathbf{x}) - 1 \,. \tag{2}$$

The failure of the system is defined as $G(\mathbf{x}) \leq 0$, while the system is considered to be safe when $G(\mathbf{x}) > 0$. A performance function is often defined as the difference between capacity and response. However, the role of safety factor is clear in the definition in Eq. (2).

When the vector $\mathbf{x}$ of input variables is random, $g_c(\mathbf{x})$ and $g_r(\mathbf{x})$ are random in nature, resulting in the safety factor being a random function. In such instances, the safety of the system can be enforced by using the following reliability constraint:

$$P_f := \Pr\left(G(\mathbf{x}) \leq 0\right) = \Pr\left(S(\mathbf{x}) \leq 1\right) \leq P_{f\,\text{target}} \,, \tag{3}$$

where P_f is the failure probability of the system and $P_{f\,\text{target}}$ is the target failure probability, which is the design requirement.

Reliability analysis calculates P_f with given random input $\mathbf{x}$, and reliability–based design optimization (RBDO) imposes Eq. (3) as a constraint. Since the magnitude of the probabilities in Eq. (3) tends to be small, the notion of reliability index is often employed. From the observation that the cumulative distribution is monotonic, the inverse transformation of the probability constraint in Eq. (3) is taken in the standard normal random space, to obtain

$$\beta := \left[-\Phi^{-1}(P_f)\right] \geq \left[-\Phi^{-1}(P_{f\,\text{target}})\right] := \beta_{\text{target}} \,, \tag{4}$$

where $\Phi(\bullet)$ is the cumulative distribution function (CDF) of the standard normal random variable, β the reliability index, and β_{target} the target reliability index. The reliability index is the value of standard normal random variable that has the same probability with P_f. The RBDO using Eq. (4) is called the Reliability Index Approach (RIA) (Enevoldsen, 1994; Tu *et al*., 1999)

The last inequality in Eq. (3) can be converted into equality, if the upper bound of the safety factor is relaxed (in this case it is one). Let the relaxed upper bound be s^*. Then, the last part of the reliability constraint in Eq. (3) can be rewritten, as

$$\Pr(S(\mathbf{x}) \le s^*) = P_{f\,\text{target}}\,. \tag{5}$$

The relaxed upper bound s^* is called the Probabilistic Sufficient Factor (PSF). Using PSF, the goal is to find the value of PSF that makes the CDF of the safety factor equals to the target failure probability. Finding s^* requires inverse mapping of CDF, from which the terminology of inverse measure comes.

The concept of PSF is illustrated in Figure 1. The shaded region represents the target failure probability. Since the region to the left of $S=1$ denotes failure, s^* should be larger than one in order to satisfy the basic design condition that the failure probability should be less than target failure probability. This can be achieved by either increasing the capacity g_c or decreasing the response g_r, which is similar to the conventional notion of safety factor, but now it is extended to probabilistic problems using PSF.

PSF gives a notion of how far the current design is from the safe design, in the performance space. This is analogical to reliability index being a measure of distance in the input variable space. The major

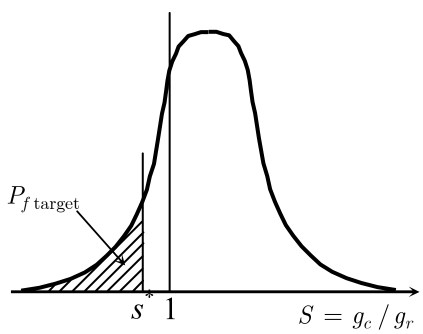

Figure 1: Probabilistic distribution of safety factor S. PSF is the value of the safety factor whose CDF corresponds to the target probability of failure.

difference is the measurement of distance in different spaces, the performance function (or output) space and the input space. A unique advantage of PSF is that design engineers, who are familiar to the deterministic design using the safety factor, can apply the similar notion to the probabilistic design.

The PSF s^* is the factor that has to be multiplied by the response or divided by the capacity so that the safety factor be raised to one. For example, a PSF of 0.8 means that g_r has to be multiplied by 0.8 or g_c be divided by 0.8 so that the safety factor increases to one. In other words, it means that g_r has to be decreased by 20% or g_c has to be increased by 25% in order to achieve the target failure probability.

The PSF can be computed using either Monte Carlo Simulation (MCS) or moment–based methods. If MCS with N samples is used to calculate PSF, the location n is first determines as the smallest integer larger than $N \times P_{f\,\text{target}}$. Then, the PSF is the n-th smallest safety factor, which is mathematically expressed as:

$$s^* = n^{th} \min_{i=1}^{N}(S(x_i))\,. \tag{6}$$

The calculation of PSF requires sorting the safety factors from the MCS samples and choosing the n-th smallest one.

3. Tail modeling and Inverse measures

Low failure probability problems (extreme value) require one to have sufficient data in the tails of the distribution which represent the extremes. But this is seldom possible and instead researchers use extreme value theory based tail modeling to predict the probability of extreme events. The theory comprises a principle for model extrapolation based on the implementation of mathematical limits as finite level approximations. This section discusses the tail modeling technique and how to apply it to find inverse measures.

In engineering applications, rather than maxima, the interest is to address the excesses over threshold. In these situations, the generalized pareto distribution (GPD) arises as the limiting distribution. The concept of GPD is presented in Figure 2. Let y be a model output which is random and u be a large threshold of y. The observations of y that exceed u are called exceedance. The conditional distribution

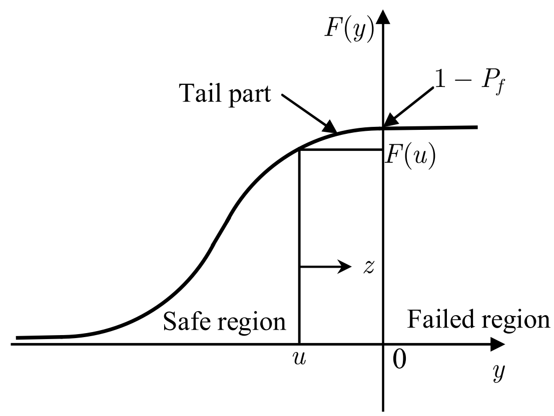

Figure 2: Tail modeling of F*(u)* using the threshold *u.* The region of y>0 is failure.

$F_u(z)$ of the exceedance given that the data y is greater than the threshold u, is modeled fairly well by the GPD. Here, $z = y - u$. Let approximation by GPD be denoted by $\hat{F}_{\xi,\sigma}(z)$. ξ and σ are the shape and scale parameters respectively. For a large enough u, the distribution function of (y-u), conditional on $y > u$, is approximately (Coles, 2001):

$$\hat{F}_{\xi,\sigma}(z) = \begin{cases} 1 - \left\langle 1 + \frac{\xi}{\sigma} z \right\rangle_+^{-\frac{1}{\xi}} & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{z}{\sigma} \right) & \text{if } \xi = 0 \end{cases} \quad (7)$$

In Eq (7), $\langle A \rangle_+ = \max(0, A)$ and $z > 0$. ξ plays a key role in assessing the weight of the tail. Eq (7) can be justified as a limiting distribution as u increases (Coles, 2001, pp:75-76). Tails can be classified based on ξ as:

$\xi > 0$, heavy tail (pareto type tail)
$\xi = 0$, medium tail (exponential type tail)
$\xi < 0$, light tail (Beta-type tails)

It is to be noted that conditional excess CDF $F_u(z)$ is related to the CDF of interest $F(y)$ through the following expression:

$$F_u(z) = \frac{F(y) - F(u)}{1 - F(u)} \tag{8}$$

From Eq (8), the CDF of y can be expressed as:

$$F(y) = (1 - F(u))F_u(z) + F(u) \tag{9}$$

When Eq (7) is substituted for $F_u(z)$ in the above expression, Eq (9) becomes:

$$F(y) = 1 - (1 - F(u))\left\langle 1 + \frac{\xi}{\sigma}(y - u)\right\rangle_+^{-\frac{1}{\xi}} \tag{10}$$

For simplicity of presentation, only the case of $\xi \neq 0$ is considered here. Once we obtain estimates of the parameters as $\hat{\xi}$ and $\hat{\sigma}$ using some parameter estimation method like maximum likelihood estimation method, least square regression that are discussed later in the chapter, it is possible to estimate the pth quantile by inverting Eq.(10) :

$$\widehat{y_p} = \widehat{F^{-1}(p)} = u + \frac{\widehat{\sigma}}{\widehat{\xi}}\left(\left(\frac{1 - F(p)}{1 - F(u)}\right)^{-\hat{\xi}} - 1\right) \tag{11}$$

If $P_{ftarget}$ refers to the target failure probability that we wish to design the structure, then the interest is to estimate the corresponding PSF in inverse reliability analysis. The PSF can be directly obtained from Eq (11) as:

$$\text{PSF} = u + \frac{\widehat{\sigma}}{\widehat{\xi}}\left[\left(\frac{P_{ftarget}}{(1 - F(u)}\right)^{-\hat{\xi}} - 1\right] \tag{12}$$

Extending the idea of tail modeling to structural applications one can use it effectively to estimate the failure probability. The failure probability of a structure is governed by the value the limit state function takes at the sample points. Considering the probability distribution of a limit state function, failure probability is essentially the probability content to the left (or right) of the limit state function value of zero. This can be estimated in the tail modeling context by substituting $y = 0$ in Eq. (10) as:

$$P_f = 1 - F(0) = (1 - F(u))\left\langle 1 - \frac{\xi}{\sigma}u\right\rangle_+^{-\frac{1}{\xi}} \tag{13}$$

Performance of this approach is based on the choice of the threshold value u In theory, the threshold should be selected where the actual upper tail starts. Selection of threshold is a tradeoff between bias and variance. If the threshold selected is low, then some data points belong to the central part of the distribution and do not provide a good approximation to the tails. On the other hand, if the threshold selected is too high, the number of data available for the tail approximation is much less and this might lead to excessive scatter in the final estimate. The proper selection of threshold is very important because it has important repercussions on the estimated value of the shape factor (Caers and Maes, 1998, McNeil and Saladin, 1997) and hence on the final estimates such as the quantile, extreme values etc. There are many exploratory techniques like the mean excess plot which help in selecting the threshold. But in a simulation study, it is impractical to perform interactive data analysis required by the exploratory techniques to choose the threshold. Boos (1984) suggests that the ratio of Nex (number of tail data) over N (total number data) should be 0.02 ($50<N<500$) and the ratio should be 0.1 for $500<N<1000$. Hasofer (1996) suggests using $Nex = 1.5\sqrt{N}$. Caers and Maes (1998) propose to use a finite sample mean square error (MSE) as a criterion for estimating the threshold. They use the threshold value that minimizes the MSE. In a similar fashion Beirlant et al (1996) find an optimal threshold by minimizing an approximate expression for asymptotic mean square error. The other methods include plotting the quantile, shape or scale factor or any quantity of interest with respect to different thresholds and look for a stability in the curve (Bassi et al, Coles pp:84-86).

There are several parameter estimation methods like the maximum likelihood (MLE) method and regression approach to estimate the parameters ξ and σ. MLE is a popular statistical method that is used to make inferences about the parameters of the underlying probability distribution of a given data set. The likelihood of a set of data is the probability of obtaining that particular set of data, given the chosen probability distribution model. ML estimation starts with writing the likelihood function which contains the unknown distribution parameters. The values of these parameters that maximize the likelihood function are called the maximum likelihood estimators.

The method of least squares assumes that the best-fit curve of a given type is the curve that has the minimal sum of the deviations squared (*least square error*) from a given set of data. The parameters are obtained by solving the following minimization problem

$$\underset{\xi,\sigma}{Min} \sum_{i=1}^{N} \left(\hat{F}_{\xi,\sigma}(z) - EmpCDF \right)^2 \tag{14}$$

The GPD CDF can be obtained by using Eq. (7). The empirical CDF are plotting positions which are computed as:

$$P_i = \frac{i}{N+1} \quad ,\text{i=1}...N \tag{15}$$

where N is the total number of samples and P is the plotting position. Least square regression requires no or minimal distributional assumptions. Unlike MLE, there is no basis for testing hypotheses or constructing confidence intervals.

4. Extrapolation schemes and simultaneous application of tail models to estimate inverse measure for highly safe designs

In this section, an extrapolation scheme is proposed to estimate the PSF for low target failure probability using MCS which is sufficient only to estimate the PSF for substantially higher failure probability (lower target reliability index). This is based on approximating the relationship between the PSF and the reliability index by a quadratic polynomial. It is to be noted that when dealing with normal distribution, this relationship is linear. The PSF for each reliability index in a range of small reliability indices is obtainable using smaller sample size MCS. A quadratic polynomial is fit to the PSF in terms of the natural logarithm of the reliability index in this range. Once the polynomial is obtained, the PSF corresponding to any higher reliability index can be estimated using it. Hence, once the PSF for each reliability index in a range of low reliability indices is obtained, the problem reduces to a data fitting problem.

The PSF extrapolation method and the tail modeling approach are conceptually similar. The major difference in perceiving the two methods is that the tail modeling techniques model the CDF of PSF whereas the extrapolation scheme approximates the trend of PSF in terms of reliability index. Tail modeling approaches enable us to address the problem of finding the probability of failure at unobserved level corresponding to a particular level of safety. Whereas, the extrapolation scheme allows us to estimate the PSF that corresponds to an unobserved level of failure probability. Since several advantages are reported by working with inverse measures, it is logical to justify an attempt to perform tail modeling in the performance space along with inverse measures to estimate quantities at unobserved levels.

The extrapolation scheme and the tail modeling methods are demonstrated on a cantilever beam example. Next, a simultaneous application of the methods is proposed. Finally the method is tested on true statistical distributions.

Cantilever beam example

Consider the cantilevered beam design problem, shown in Figure 3 (Wu *et al.*, 2001). The objective is to minimize the weight or equivalently the cross sectional area, $A = w \cdot t$ subject to two reliability constraints, which require the reliability indices for strength and deflection constraints to be larger than three. The expressions of two performance functions are given as

Strength:
$$y_s = R - S = R - \left(\frac{600}{w^2 t} F_X + \frac{600}{wt^2} F_Y \right) \tag{16}$$

Tip Displacement:
$$y_d = D_O - D = D_O - \frac{4L^3}{Ewt} \sqrt{\left(\frac{F_Y}{t^2} \right)^2 + \left(\frac{F_X}{w^2} \right)^2} \tag{17}$$

where R is the yield strength, F_X and F_Y are the horizontal and vertical loads and w and t are the design parameters. L is the length and E is the elastic modulus. R, F_X, F_Y, and E are random in nature and are defined in Table 1.

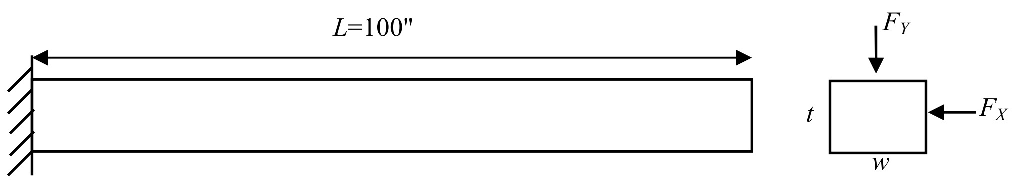

Figure 3: Cantilever beam subjected to horizontal and vertical random loads

Table 1: Random variables for the cantilevered beam problem

Random Variable	F_X	F_Y	R	E
Distribution	Normal (500,100)lb	Normal (1000,100)lb	Normal (40000,2000) psi	Normal (29E6,1.45E6) psi

Here we consider a system failure case. That is, both the failure modes are considered simultaneously. The approximated tail of the CDF for the cantilever beam system reliability example is presented in Figure 3. The number of samples used is 1E5 and the quantile selected is 0.9. The threshold is selected in terms of CDF of the safety factor. This 0.9th quantile value is 0.85. The fit based on GPD approximation is superimposed on the empirical data. The ordinate can be viewed as the failure probability levels. Hence, if an inverse normal transformation is performed on the ordinate and the axes swapped, Figure 3 takes the form of the plot in Figure 4. The idea of the extrapolation technique is to approximate the relationship depicted in Figure 4 by a quadratic polynomial. 1e5 samples are used here for demonstration purpose. This is seldom possible in real time with computationally intensive models. In order to consider a reasonable real situation, we consider the same example with 500 samples.

The objective is to the estimate PSF corresponding to low failure probabilities by simultaneously applying the extrapolation method and classical tail modeling techniques. Though the methods are conceptually same, they burgeon from different theories or assumptions and exhibit their own limitations. Since neither of the methods can be applied to all the problems, we propose to use both the methods simultaneously to model the tail data. With respect to parameter estimation in the tail modeling approach, the ML method might work well sometimes and the regression approach might work better (Hasking and Wallis, 1987). Both ML and regression techniques are used for parameter estimation in classical tail modeling approach. In the extrapolation technique, one can use the exceedance data alone or use the entire data. Because of the symmetry of reliability index, it suffices to use only half of the data (else, we'll have to

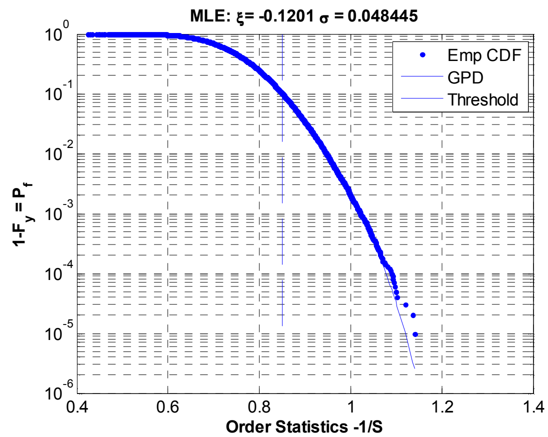

Figure 3: GPD fit to the tail of critical safety factor data

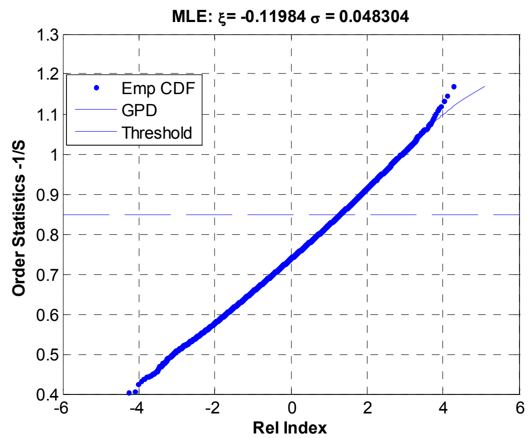

Figure 4: Extrapolation approach. Relationship between reliability index and PSF

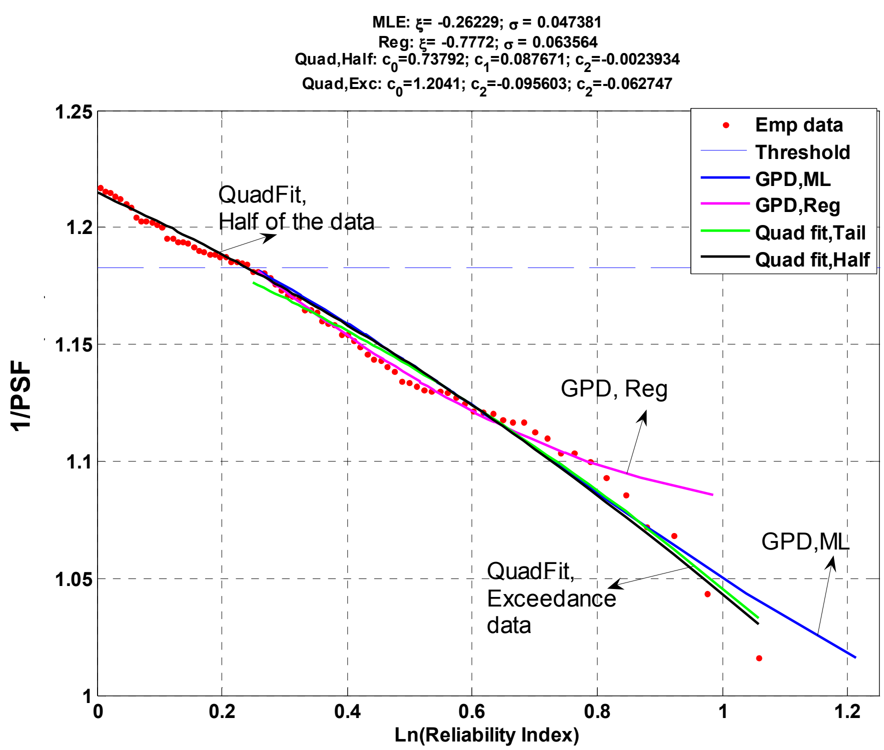

Figure 5: Simultaneous application of tail models. Cantilever beam system reliability case. 500 samples

deal with logarithms of negative data). A plot for the system reliability case of the cantilever beam that uses the 4 different fits is presented in Figure 5.

The fits presented in Figure 5 are for one repetition. In order to understand the uncertainty in the fit, a thousand replications were performed and the mean and standard deviation of the estimates were recorded for all the four fits. A pictorial representation of the spread of data can be obtained using a box plot. The box plots are presented for two reliability index values, 4 and 4.2 in Figure 6. The minimum of the sample is the bottom of the lower whisker. By default, an outlier is a value that is more than 1.5 times the interquartile range away from the top or bottom of the box. Based on Figures 5 and 6 and it can be concluded that the fit to half of the data performs better than other methods. The fit based on regression parameters seems to skew after a certain set of data points. This is mainly influenced by the extreme points that are volatile.

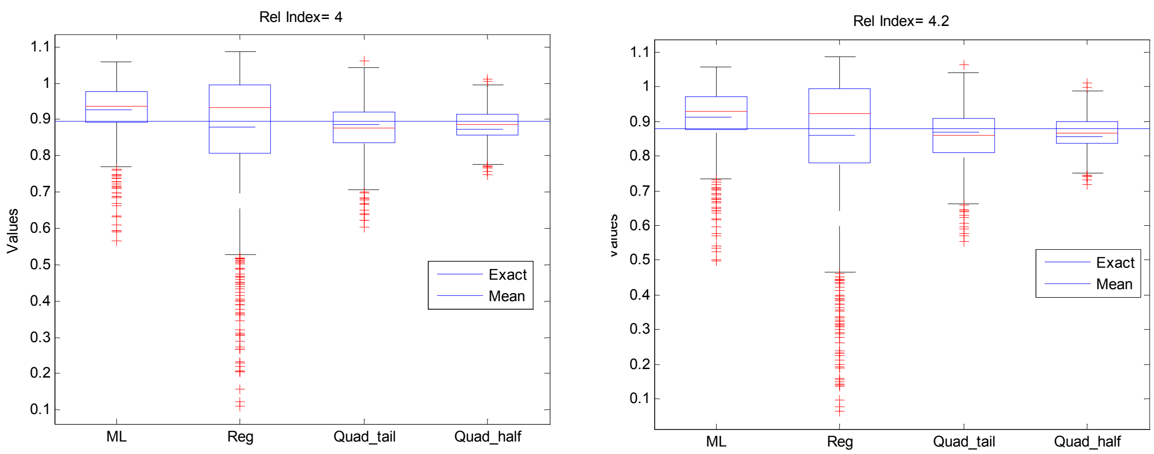

Figure 6: Box plot representation for the 4 data fit techniques. Cantilever beam system reliability case. 500 samples. 1000 repetitions.
The lower and upper lines of the box are the 25th and 75th percentiles of the sample. The distance between the top and bottom of the box is the interquartile range. The line in the middle of the box is the sample median. The whiskers are lines extending above and below the box. They show the extent of the rest of the sample (unless there are outliers).

The results presented and the consequent conclusions belong to the cantilever beam system reliability case alone. They cannot be generalized for any other example. In order to understand the performance of these methods individually with different distributions, the simultaneous application technique is tested on true statistical distributions. The different distributions that were tested are presented in Table 2. For a fixed mean and COV of the data, the objective is to estimate the inverse measure for fixed probability content. Once this is obtained, one can normalize the initial mean and use it to estimate inverse measures for different plotting positions that are used in the construction of empirical CDF. The steps followed in using the simultaneous application of tail models to true distribution is presented in Appendix 1. The measures of error in the estimated values in comparison to the exact values are the relative error and incremental relative errors. The expressions are:
Relative Error :

$$\frac{Exact_i - Method_i}{Exact_i} \quad i \rightarrow \text{Rel Index}(3.2:0.2:4.2) \tag{18}$$

Incremental relative Error:

$$\frac{(Exact_3 - Exact_i) - (Method_3 - Method_i)}{(Exact_3 - Exact_i)} \quad i \rightarrow \text{Rel Index(3.2:0.2:4.2)} \tag{19}$$

Table 2: Statistical distributions used to test the simultaneous application of tail models

Distribution	Parameters	
	a	b
Normal	μ	σ
LogNormal	$\ln(\mu)-0.5(b^2)$	$\sqrt{\ln\left[1+\left(\frac{\sigma}{\mu}\right)^2\right]}$
Uniform	$\mu-\frac{\sqrt{12}}{2}\sigma$	$\mu+\frac{\sqrt{12}}{2}\sigma$
Extreme Type 1	$\mu-\frac{0.577}{b}$	$\frac{\pi}{\sigma\sqrt{6}}$
Gamma	$\left(\frac{\mu}{\sigma}\right)^2$	$\frac{\sigma}{\sqrt{a}}$
Single parameter distributions		
Exponential	$\mu-\sigma$	μ
Rayleigh	$\mu-\sqrt{\frac{\pi}{2}}(b)$	$\frac{\sigma}{\sqrt{2-\frac{\pi}{2}}}$

Table 3: Error metrics for different tail models. Lognormal distribution

ξ(ML): -0.07 ξ (Reg): -0.05 Capacity: 23.11 Threshold: 0.9

Extrapolated PSF values

Rel Index	**3**	**3.2**	**3.4**	**3.6**	**3.8**	**4**	**4.2**
Exact	1.0000	1.0605	1.1246	1.1926	1.2647	1.3412	1.4223
MLE	0.9797	1.0286	1.0777	1.1268	1.1755	1.2236	1.2709
Reg	1.0013	1.0561	1.1120	1.1687	1.2260	1.2836	1.3413
Quad-Half	1.0199	1.0804	1.1435	1.2091	1.2773	1.3481	1.4214
Quad-Tail	1.0324	1.0904	1.1483	1.2058	1.2629	1.3194	1.3754
Lin-Tail	0.9947	1.0415	1.0883	1.1350	1.1818	1.2286	1.2754
% Rel Error							
MLE	2.03	3.00	4.17	5.52	7.06	8.77	10.64
Reg	-0.13	0.41	1.12	2.00	3.06	4.30	5.69
Quad-Half	-1.98	-1.88	-1.68	-1.39	-1.00	-0.51	0.06
Quad-Tail	-3.24	-2.82	-2.10	-1.11	0.15	1.62	3.30
Lin-Tail	0.53	1.79	3.23	4.83	6.55	8.39	10.33
%Inc Rel Error							
MLE		19.03	21.28	23.62	26.03	28.51	31.03
Reg		9.31	11.14	13.08	15.12	17.26	19.47
Quad-Half		-0.12	0.77	1.72	2.73	3.79	4.91
Quad-Tail		4.06	7.00	9.97	12.93	15.88	18.79
Lin-Tail		22.62	24.89	27.11	29.29	31.43	33.52

The error metrics for the lognormal distribution using different tail models are presented in Table 3. It is observed that the quadratic fit to the half data and exceedance data performed better compared to other techniques. A similar exercise was performed for all the tabulated distribution. 2 different parameters and two different thresholds for each set of parameters are considered. The outcome based on best performance is presented in Table 4 and Table 5. A detailed table of the performance is presented in Appendix 2.

Table 4: Number of cases in which each technique performed as one of the best two fits. μ=10, σ=3

P_{ft}=0.00135, 500 samples					
Threshold			**Techniques**		
	Quad-Tail	**Lin-Tail**	**Quad-Half**	**Reg**	**MLE**
Incremental Error					
0.9	7	3	4	3	1
0.95	4	3	4	6	1
Relative Incremental Error					
0.9	6	2	3	2	1
0.95	2	3	5	2	1

Table 5: Number of cases in which each technique performed as one of the best two fits. μ=10, σ=8

P_{ft}=0.00135, 500 samples					
Threshold	**Techniques**				
	Quad-Tail	**Lin-Tail**	**Quad-Half**	**Reg**	**MLE**
Incremental Error					
0.9	5	2	3	4	2
0.95	1	2	4	7	2
Relative Incremental Error					
0.9	3	2	2	4	2
0.95	1	2	4	6	2

From the above tables it is clear that no particular distribution can be considered best for all distributions. Even for a particular distribution the performance of different technique varies based on the parameters. Therefore, it is more reasonable to use all the methods simultaneously. One can attempt to estimate an error metric in the tail model similar to PRESS errors (Predicted REsidual Error Sum of Squares) in response surface techniques and use this measure to estimate the accuracy of different methods and choose the best one.

5. Conclusions

This paper discussed about using classical tail modeling techniques to estimate reliability measures in the context of structural reliability. These methods are based on the idea of approximating the tail of the CDF by a GPD. Maximum likelihood and regression methods were used to estimate the parameters of GPD. A PSF based tail extrapolation technique is proposed that can complement the classical tail modeling. The methods are demonstrated on a cantilever beam example and true statistical distributions. It is observed that no single method can be universally used for all distributions. The performance of each technique varies depending on the distribution and parameters. A simultaneous application of tail models is proposed. Error metrics similar to PRESS can be used to estimate the accuracy of each method and choose the best fit.

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Appendix 1

Simultaneous application of tail models to true statistical distribution
Fixed μ; COV ; $P_{ftarget}$

1. Find the initial parameter values(a_0 and b_0) for each distribution. These parameters can be used to generate random numbers of response.
2. Find capacity C ($P_{ftarget}$ th quantile).In the case of single parameter distributions, add shift factor to the C.
3. Find normalized mean using $\hat{\mu} = \frac{\mu}{C}$. The normalized mean is the mean of the inverse mesure
4. Use $\hat{\mu}$ and COV to find new parameters (a and b) for all the distributions.
5. Generate y = 500 LHS on (0, 1) [allows to generate equally distributed samples in the y axis]
6. Find S using a,b and inverse CDF functions
7. Estimate plotting positions → $P = \frac{i}{N+1}$
8. Plotting S vs P provides the empirical CDF.
9. The tail of the CDF can be approximated by GPD. ML and regression approaches are used for parameter estimation
10. Apply the inverse standard normal cumulative distribution function to the plotting positions to get reliability indices and approximate the relationship between reliability indices and PSF using a quadratic fit to the data.

Appendix 2

Table A2.1: Best fits based on error metrics for various distributions
Case 1: μ=10, σ=3 ; Case 2: μ=10, σ=8

		Relative Error				**Inc Relative Error**			
	Cases	**1**		**2**		**1**		**2**	
Distribution	Threshold	Method	Behaviour	Method	Behaviour	Method	Behaviour	Method	Behaviour
Normal	0.9	LT	Over Est	LT	Over Est	LT	Over Est,	LT	Constant
		QT	Over Est	QT	Over Est	QT	Over Est	QT	OverEst
	0.95	Reg	Changes	Reg	changes	QH	Over Est	QH	OverEst
		LT	Over Est	LT	Over Est	LT	Over Est	LT	Constant
		QT	Over Est	QH	Over Est				
LogNormal	0.9	QH	Changes	MLE	UnderEst	QH	Changes	MLE	UnderEst
		QT	Changes	Reg	Over Est	QT	Constant	Reg	Over Est
	0.95	QH	Changes	MLE	UnderEst	QH	Changes	MLE	UnderEst
		QT	Over Est			QT	OverEst	QT	Over Est
				Reg	Over Est			Reg	Over Est
Gamma	0.9	QH	Over Est	Reg	Changes	QH	Over Est	Reg	UnderEst
		QT	Over Est	QH	OverEst	QT	Changes	QH	OverEst
		LT	Changes	QT	Changes				
	0.95	QH	Over Est	Reg	OverEst	QH	OverEst	Reg	OverEst
		Reg	Changes	QH	OverEst			QH	OverEst
		LT	Changes			LT	Constant		
Uniform	0.9	Reg	OverEst	MLE	UnderEst	Reg	Constant	MLE	UnderEst
		MLE	Constant	Reg	OverEst	ML	Constant	Reg	UnderEst
		QT	Constant						
	0.95	Reg	OverEst	MLE	UnderEst	Reg	Constant	MLE	UnderEst
		MLE	Constant	Reg	OverEst	ML	Constant	Reg	UnderEst
		QT	Constant	QT	UnderEst				
Exponential	0.9	Reg	Over Est	Reg	OverEst	Reg	OverEst	Reg	OverEst
		QH	Over Est	QH	OverEst	QH	OverEst	QH	OverEst
		QT	Changes	QT	Changes	QT	Constant		
	0.95	Reg	Over Est	Reg	OverEst	QH	OverEst	Reg	OverEst
		QH	Over Est	QH	OverEst	QT	OverEst	QH	OverEst
		QT	Over Est						
EV	0.9	QH	Changes	QH	UnderEst				
		QT	Over Est	QT	OverEst	QT	Over Est	QT	OverEst
	0.95	Reg	Change	Reg	Change	Reg	Constant		underEst
		QH	Change	QH	UnderEst	QH	Constant		underest
Rayleigh	0.9	QT	OverEst	QT	overEst	QT	Over Est	Reg	OverEst
		LT	OverEst	LT	OverEst	LT	Changes	QH	changes
	0.95	Reg	Changes	Reg	Changes			Reg	UnderEst
		LT	OverEst	LT	OverEst	LT	Over Est	LT	OverEst

LT – Linear fit to tail data, QT- Quadratic fit to tail data, QH – Quadratic fit to half data, MLE- Maximum likelihood estimate, Reg- Regression
OverEst – Over estimated, UnderEst – Under Estimated, Changes – Sign changes