Conservative Fatigue Life Estimation using Bayesian Update

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In this paper, Bayesian update is utilized to reduce uncertainty associated with the fatigue life relation. The distribution for fatigue strain at a constant life cycle is determined using the initial uncertainty from analytical prediction and likelihood functions from test data. The Bayesian technique is a good method to reduce uncertainty and at the same time, provides a conservative estimate, given the distribution of analytical prediction errors and variability of test data. First, the distribution of fatigue model error is estimated using Monte Carlo simulation with uniformly distributed parameters. Then the error distribution is progressively updated by using the test variability as a likelihood function, which is obtained from field test data. The sensitivity of estimated distribution with respect to the initial error distribution and the selected likelihood function is studied. The proposed method is applied to estimate the fatigue life of turbine blade. It is found that the proposed Bayesian technique reduces the scatteredness in life by almost 50%, while maintaining the conservative life estimate at a given fatigue strain. In addition, a good conservative estimate of fatigue life prediction has been proposed using a knockdown factor that is obtained from the distribution of lowest test data. Moreover, Bayesian update has been utilized to update the parameters of strain – life curve for a case of constant strain amplitude

Nomenclature

ε_t	=	total fatigue strain amplitude
ε_e	=	elastic strain amplitude
ε_p	=	plastic strain amplitude
e_f	=	co-efficient of fatigue ductility
s_f	=	co-efficient of fatigue strength
c	=	exponent of fatigue ductility
b	=	exponent of fatigue strength

I. Introduction

IN general, there are two different life prediction models in fatigue analysis: stress-life model and strain-life model. The former is often used for high-cycle fatigue analysis in which the stress-strain relation is in the linear region. The latter is frequently used for low- and medium-cycle fatigue in which plastic deformation contributes to the fatigue life. Although the basic concept in the proposed Bayesian approach is the same, the strain-life model will be investigated in this paper.

In strain-life fatigue analysis, the total fatigue strain (ε_t) is decomposed by elastic strain (ε_e) and plastic strain (ε_p). For this analysis, the strain-life curve is defined [1] as:

$$\varepsilon_t = \varepsilon_e + \varepsilon_p = \frac{s_f}{E}\left(2N\right)^b + e_f\left(2N\right)^c \tag{1}$$

where e_f is the coefficient of fatigue ductility, s_f the coefficient of fatigue strength, b the exponent of fatigue strength, c the exponent of fatigue ductility, and E the Young's modulus.

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The coefficients in the strain-life curve are obtained using curve-fitting of test data. However, due to variability in test and material, the results are often scattered. In the material handbook, for example, the standard values of the coefficients are available. However, a particular batch of material may have different properties. In addition, a particular machine may have different fatigue properties due to manufacturing process used and possibility of residual stresses. Thus, an important question is how to find more accurate life estimate for a specific machine when several test data are available. Several methods of incorporating test data to update the strain-life curve have been investigated. Williams et al[2] and Ronold et al[3] used linear regression to fit the strain-life curve using the given test data. To account for the variability of test cases, Ronolod et al[3] also provide the 95% confidence interval on these curves. Park et al[4] used an energy-based approach and a wide range of test data to predict the fatigue life with constant strain amplitude.

Traditionally, safety factors have been adopted as a measure to counter variability. However, applying a uniform safety factor can yield unreliable design because different parts of structure may have different distribution. Presently, there is growing interest in replacing safety factor-based deterministic design with reliability-based design (e.g., Wirsching[5], SAE Aerospace Information Report 5080[6]). Guo et al[7] propose a method to perform reliability analysis for windmills using test data by calculating the maximum likelihood function. Acar et al[8] concludes that reliability analysis that considers error and variability is far more effective than the deterministic, safety factor approach.

The goal of test is often to find conservative estimate of predicted life. Obviously it is easy to choose the lowest test data as a conservative estimate, but its variability will be high and in many cases, the lowest test data will not provide enough conservative estimate.[9] Thus, another important question is how to find the best way of predicting conservative fatigue life of a machine.

An, et al[9] proposes Bayesian update as a possible technique to reduce scatter in failure stress distribution and also to produce conservative estimate of the failure stress. Cross et al[10] uses Bayesian update technique and maintenance data to update the probabilistic rotorcraft structural life models. Guerin et al[11] use Bayesian update to update fatigue life for rubber boot seals in automobiles. Gogu et al[12] uses Bayesian update to update distributions of two elastic constants simultaneously.

In this paper, the Bayesian update technique is adopted to reduce scatter of the fatigue life distribution at constant strain amplitude when additional test data are available. Moreover, a good conservative estimate of fatigue life prediction is proposed using a knockdown factor that is obtained from the distribution of lowest test data. The concept of knockdown factor has been adopted in aerospace structures.

Although the abovementioned Bayesian technique is useful, but it is not the most general case because it updates the fatigue life of a fixed strain amplitude. Since materials can be subjected to fatigue of different strain amplitudes, it is logical to update the entire strain – life curve, rather than at a particular value of strain. This can be achieved by updating the material parameters of the strain – life curve. Although the traditional Bayesian technique updates the parameter of which the test data are available, the initial updating formula is modified such that the plastic strain parameters can be updated using the fatigue life test data.

II. Inputs and Assumptions

It is important to understand the assumptions that are used in this paper. Some of them are for convenience, and others represent the lack of knowledge. If additional information is available, the latter can be improved.

It is first assumed that the coefficients in Eq. (1) are randomly distributed and some information regarding their statistical distribution is known. This information represents the prior knowledge or analytical prediction. This information can be obtained by studying strain-life test data and by estimating upper- and lower-bounds of test data from the mean curve. When this information is not available, it is possible to assume that these parameters (sf, ef, b, and c) are distributed uniformly with given bounds from their nominal values. This will serve as a prior knowledge of the fatigue failure information. Even if input parameters are uniformly distributed, the fatigue failure strain will not be uniformly distributed.

The confidence interval is a measure of our confidence in the analytically predicted value. It would be the lower and upper bounds of the strain-life test data. However, since only the distribution of each parameter is considered known, the bounds for life cycle value are determined from the distribution of all parameters.

Considering cases, where only one of the parameter varies at a time, while the rest of them are at their mean values, the sensitivity of life cycle value to each parameter has been determined, and found to be positive for all parameters. Hence, the upper and lower bounds for life cycle would be when all parameters are simultaneously at their algebraic maximum and minimum values respectively. The bounds of the confidence interval are usually expressed as a percentage of analytical value. If the bounds are asymmetrically distributed about the analytical value, the maximum variation is considered.

The distribution of error is the distribution of life cycle due to the distributions of the parameters. This distribution of error is generated using the Monte Carlo Simulation (MCS). In this paper, the MCS is performed by generating 10^5 values for each parameter, governed by its variability. These randomly generated numbers are used to calculate 10^5 values of strain life, from which the distribution of error can be estimated.

Test variability is the measure the scatteredness of the test data. Distribution of test variability is the histogram of strain-life test data at the known constant failure fatigue strain (ε_t) value. However, for the want of test data, the distribution of test variability is assumed to be normal with same parameters as that of the distribution of error.

Even if there is no limitation on the number of test data, three test data are assumed available through the test of a specific component. These three test data are randomly chosen within the variability limits with the analytical value as the mean.

III. Bayesian Update for Fatigue Failure Strain

Once the confidence interval, error distribution, test variability are obtained, the Bayesian update can be performed to estimate the distribution of the failure fatigue strain. In addition, the obtained distribution can be used to estimate conservative failure fatigue strain.

A. Normalization:

Although the raw data and initial distribution can be used for Bayesian update, it is often more convenient to normalize all data and distributions. All the factors affecting the Bayesian update, such as the confidence interval, error distribution, test variability, and test results are normalized with respect to the initial value of fatigue failure strain ε_t. Since the confidence interval is expressed as a percentage of analytical value, it is not affected by normalization. Both the mean and standard deviation of test variability are divided by the mean value. In the normalized distribution, the standard deviation becomes identical to the coefficient of variance (COV).

B. Likelihood Function:

The likelihood function of a test result is the probability of obtaining that test result, given the value of actual fatigue strain and the test variability. It is actually the ordinate value of probability distribution function (PDF) of test variability with the actual fatigue strain as its mean, when the abscissa is equal to the test result.

The likelihood would be a single value, if the actual fatigue strain is known. But, since only the bounds for the actual fatigue strain are known, the likelihood function varies within that confidence interval. The likelihood of the given test result can be found by considering each point within the error bounds as the actual fatigue strain. The likelihood function for a given test result would be the variation of these likelihood values with the actual fatigue strain values.

The likelihood function for each of the three test results can be determined in a similar fashion.

C. Bayesian Update:

The Bayesian update is based on the theory of conditional probability, which states

$$p\left(true/test\right) = \frac{p\left(test/true\right) \times p(true)}{p(test)} \tag{2}$$

The expression for Bayesian update is something very similar, i.e.

$$f^{upd}\left(\varepsilon_t\right) = \frac{f_{1,test}\left(\varepsilon_t\right) f^{ini}\left(\varepsilon_t\right)}{\int_{-\infty}^{\infty} f_{1,test}\left(\varepsilon_t\right) f^{ini}\left(\varepsilon_t\right) d\varepsilon_t} \tag{3}$$

Here, $f_{1,test}(\varepsilon_t)$ is the likelihood function for the given test result. It could also be seen as probability of obtaining the test result given the true value of ε_t. $f^{ini}(\varepsilon_t)$ is the initial distribution of ε_t. For updating with first test result, this distribution is taken as the error distribution. This updated distribution is used as the initial distribution for updating with second test result and so on.

The denominator is simply the integration of numerator. This is nothing but a normalization of PDF such that the area under the distribution becomes one. Different techniques like trapezoidal rule or Simpson's rule can be used for this purpose. Trapezoidal rule has been used here for numerical integration.

After the initial distribution is updated using the first test data, $f^{ini}(\varepsilon_t)$ is replaced by $f^{upd}(\varepsilon_t)$, and the above procedure is repeated for the next test results.

The mean of the distribution obtained after updating with the third and final test result, is called the updated Bayesian strain.

D. Knockdown Factor:

It is common in practice to take the minimum value of the test results as the actual fatigue strain. However, the variability in the lowest test data will be significant. For example, if three test data are taken multiple times, the lowest test data will vary every time. In addition, the conservativeness of the lowest test data is often insufficient. Indeed, using the lowest test data is equivalent to multiplying a knockdown factor to the mean fatigue strain value. Only problem is that this knockdown factor is implicit and its variability is too high.

In the Bayesian update, the distribution of fatigue failure strain is updated using available test data with variability in the test. Once the final distribution, updated Bayesian strain, is obtained, its mean value is calculated from the distribution information. Then, a knockdown factor is multiplied to obtain the conservative estimate of the fatigue failure strain. The question is how to calculate the knockdown factor, which will be explained below.

It is known that the lowest test result follows an extreme value distribution[13]. The idea is that the knockdown factor is calculated using the mean value of the extreme value distribution. Calculation of the knockdown factor depends on material variability. Let f be the CDF of material variability after normalization, the extreme value distribution is given by,

$$F_1^{(3)} = 1-(1-f)^3 \tag{4}$$

The knockdown factor is the mean value of this extreme value distribution $F_1^{(3)}$ [14]. The conservative estimate of fatigue failure strain will be the product of the mean value of the updated Bayesian strain and knockdown factor.

IV. Numerical Example 1 – Constant CYCLE Test

As a first numerical example, the Bayesian technique is applied to steel 4340 material, whose strain-life fatigue parameters are shown as a solid curve in Table 1 [15]. The strain-life curve for steel 4340 material is shown in Fig 1. In the first example, it is assumed that the material is failed at 2N = 100,000 reversals and three test data are available at that reversal value. Since the crossover reversal for this material is in the order of 10,000, the fatigue strain is in the elastic region. For the material parameters in Table 1, the value of analytical fatigue strain is ε_t = 0.0032.

Table 1: Strain-life fatigue parameters for steel 4340

Parameter	**Value**
Elastic stiffness (E)	208,900 MPa
Fatigue ductility coefficient (e_f)	0.83
Fatigue ductility exponent (c)	-0.65
Fatigue strength coefficient (s_f)	1,713 MPa
Fatigue strength exponent (b)	-0.095

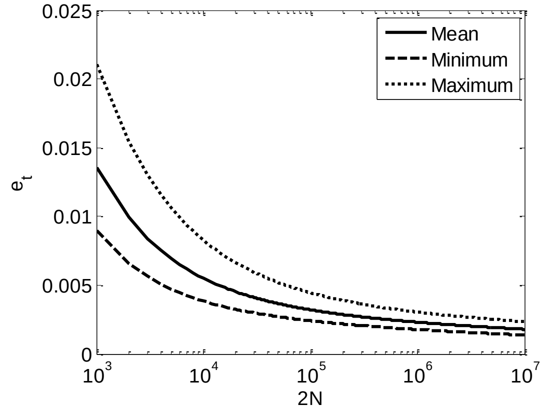

Figure 1. Strain life curve for Steel 4340.

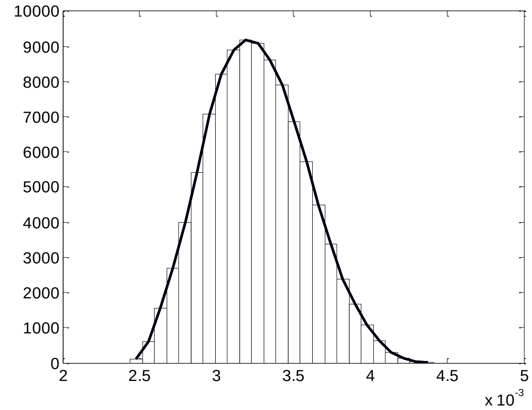

Figure 2. Histogram of fatigue strain at 2N = 100,000 reversals.

In order to find the initial error bounds of the fatigue model, all parameters affecting the fatigue strain are assumed to vary uniformly ±10% of their nominal value. When all the parameters are varying simultaneously, the value of ε_t varies between 0.0024 and 0.0045. Then, the error bounds are calculated considering the maximum deviation from the mean, i.e., 0.0045 – 0.0032 = 0.0013, which is about 39% of the mean. Hence, the error bounds are ±39% of the mean.

In addition to the error bounds, detailed distribution of ε_t can be plotted using MCS. First, 100,000 samples of material parameters are randomly generated according to uniform distribution. Then, the histogram of ε_t is plotted by applying these samples to Eq. (1). Figure 2 shows the histogram of fatigue strain at 2N = 100,000 reversals. The solid line connects the midpoint of each bin in the histogram. The PDF of fatigue strain is obtained by scaling down the solid lined curve, such that the area under the curve is unity. It turns out that the fatigue strain has mean, m = 0.0033 and standard deviation, sd = 0.00033.

In Fig. 3, the normal distribution with the same mean and standard deviation is plotted. It is clear that the histogram looks close to a normal distribution with same parameters. Although variability associated with fatigue test should be obtained from more rigorous method, it is assumed that the test variability is normally distributed with mean 0.0033 and COV 10%.

With error bounds and test variability, now it is possible to perform Bayesian update. Let us consider that three fatigue tests are performed. After normalizing by analytical fatigue strain, the three test data strains are 0.85, 1.05, and 1.15. These three normalized data correspond to 0.02805, 0.003465, and 0.003795.

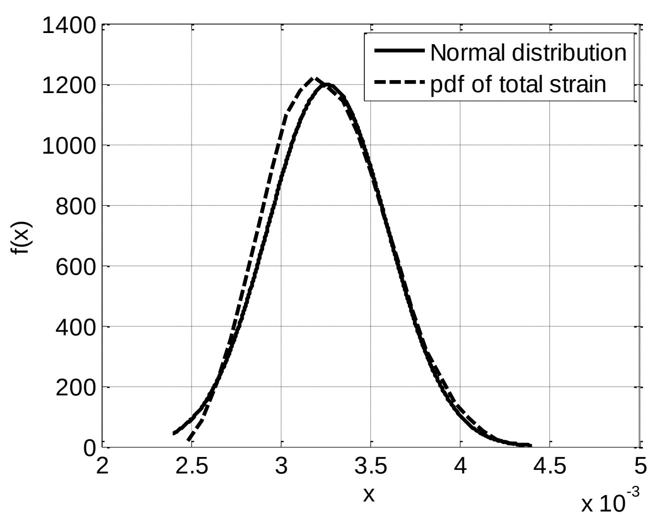

Figure 3. Fitting the distribution of fatigue strain using normal distribution.

Figure 4 shows distribution of estimated fatigue strain at each stage of Bayesian update. The dashed lines show the likelihood function for each test result, while the solid lines show the updated distributions. The final updated distribution of the fatigue strain has the following parameters:

Mean = 0.00327
SD = 0.00017

Note that the standard deviation of the updated Bayesian distribution is about 50% of that of the original distribution. Thus, the three test data effectively reduce the uncertainty in the fatigue failure strain.

Since the fatigue strain is distributed, it is better to provide a conservative estimate of the fatigue strain using a knockdown factor. When the test variability is normalized, the mean is shifted to 1.00, while retaining the COV. Hence, the knockdown factor for the normalized test variability, governed by $N(1, 0.1^2)$ is calculated to be 0.9127. Hence, the conservative estimate of fatigue failure strain at 100,000 reversals becomes ε_t = 0.00298.

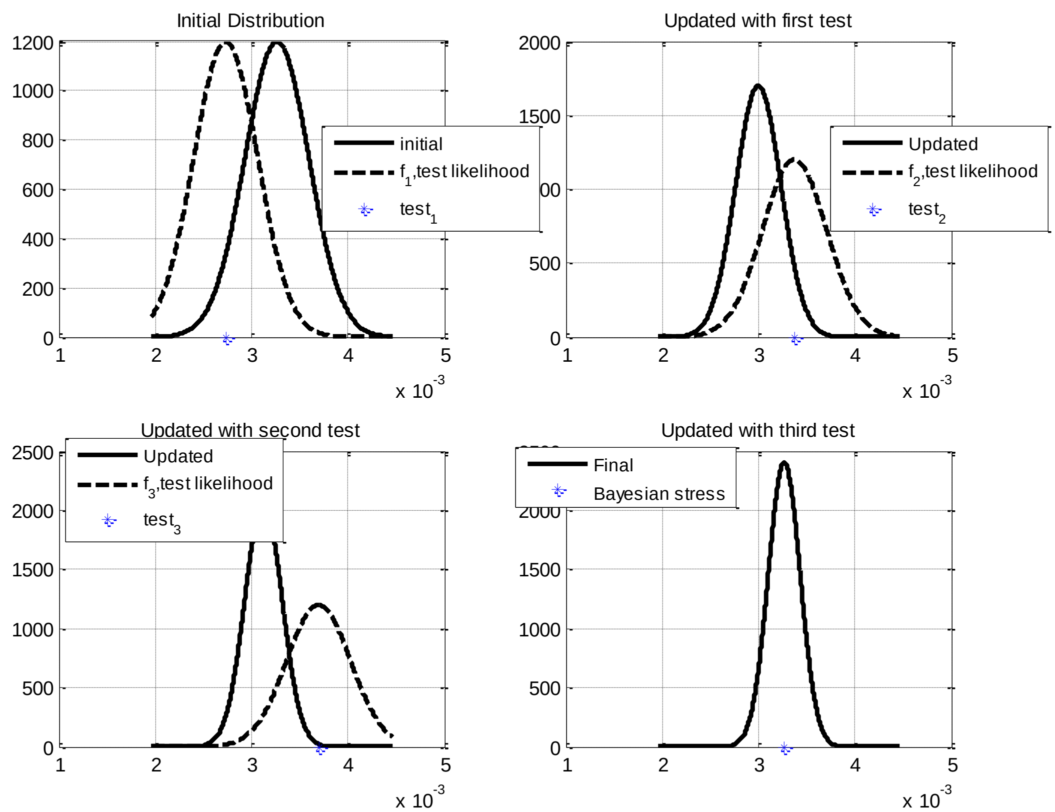

Figure 4. Bayesian update history of failure fatigue strain.

V. Numerical Example 2 – Constant Strain Test

For the constant strain test, it is assumed that the amplitude of strain applied to the machine is constant at 0.0055. For the material properties in Table 1, the value of analytical fatigue life is 10,000 reversals. Since the crossover cycle value for this material is 4,103 cycles, the assumed fatigue strain is in the elastic region.

Due to computational difficulties, the strain-life expression is solved for $y = \log_{10}(2N)$, rather than for the strain life, 2N. Hence, the analytical value for ‘y’ would be 4.00.

In order to find the initial error bounds of the fatigue model, all parameters affecting the fatigue strain are assumed to vary uniformly ±10% of their nominal value. Using Monte Carlo simulation, the error bounds for ‘y’ have been determined to be ±16% of the mean.

In addition to the error bounds, detailed distribution of 'y' can be plotted using MCS. First, 100,000 samples of material parameters are randomly generated according to uniform distribution of all parameters. Then, the histogram of 'y' is plotted by applying these samples to Eq. (1). Figure 5 shows the PDF of 'y' at fatigue strain, ε_t = 0.0055. It turns out that 'y' has mean, m = 3.8937 and standard deviation, sd = 0.1991.

For the want of test results, the variability of the test results have been assumed to be a normal distribution with mean 3.8937 and COV 5.11%.

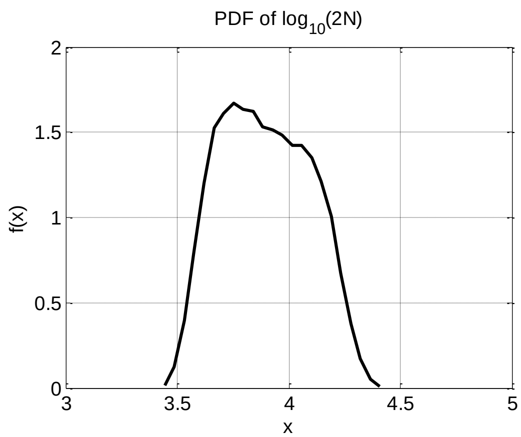

Figure 5: PDF of $\log_{10}$(2N)

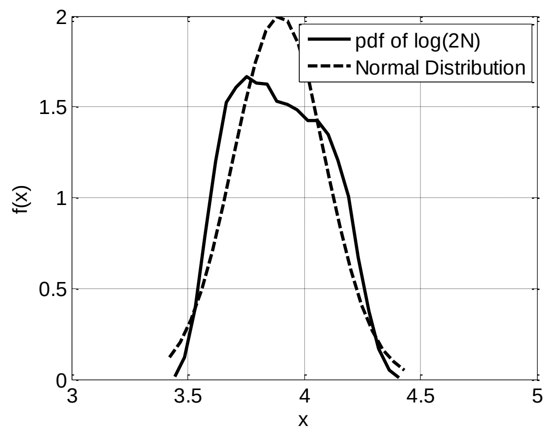

Figure 6: PDF of $\log_{10}$(2N) and normal distribution

The PDF of $\log_{10}$(2N) can be modeled as a product of PDF of a normal distribution and a polynomial. The normal distribution that has the same parameters as that of the PDF of $\log_{10}$(2N) is shown as the curve with dotted lines in Fig. 6. Figure 7 plots the product function determined from the two curves in Fig. 6. An 8th degree polynomial is fitted to estimate the product function. The dashed lines show the polynomial in Fig. 7.

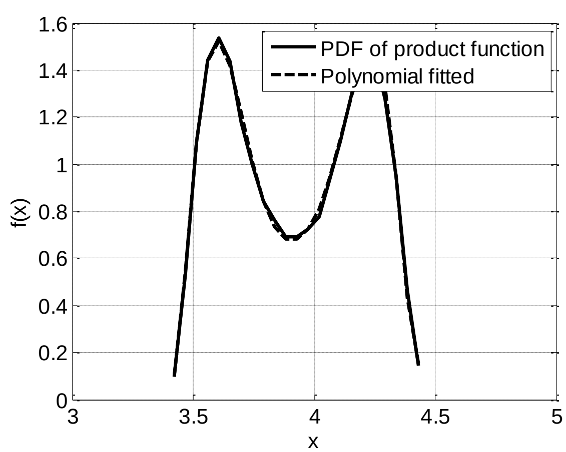

Figure 7. Product function that models the factor between PDF of $\log_{10}$(2N) and that of normal distribution

The three test cases for 'y' have been assumed to be 0.85, 1.05 and 1.15 times the analytical value. Figure 8 shows distribution of estimated fatigue life at each stage of Bayesian update. The dashed lines show the likelihood function for each test result, while the solid lines show the updated distributions. The final updated distribution of the normalized 'y' has the following parameters:

Mean = 1.0144
SD = 0.0285

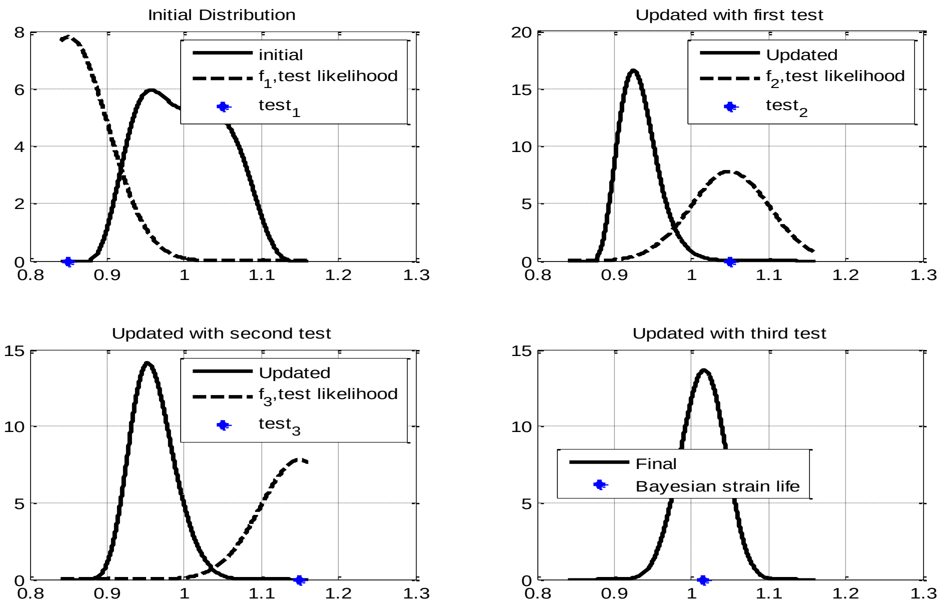

Figure 8. Bayesian update history of failure fatigue life.

Since the fatigue life is distributed, it is better to provide a conservative estimate of the fatigue life using knockdown factor. When the test variability is normalized, the mean is shifted to 1.00, while retaining the COV. Hence, the knockdown factor for the normalized test variability, governed by $N(1,\ 0.0511^2)$ is calculated to be 0.9567. Hence, the conservative estimate of fatigue failure life at strain amplitude of 0.0055 is 2N = 6,713 reversals.

VI. Effect of Initial Distribution

The Bayesian update for the fatigue life has been performed in Example 2 using the initial distribution obtained from the initial distribution of the four parameters. In this section, the effect of this initial distribution is investigated by comparing with the Bayesian updating starting from the uniformly distributed initial fatigue life. The Bayesian update history of fatigue life with a uniformly distributed prior is shown in Fig. 9. Table 2 compares the results from these two Bayesian updates. It is noted that when the actual PDF of life is used as prior, the mean of final distribution decreases by 0.23%, and the standard deviation of the final distribution reduces by 3.39%. This suggests that having a better knowledge of the prior distribution results in a better coefficient of variance for the distribution after update. Although the improvement seems small, this can be interpreted to differences of 760 reversals in the range of 95% confidence intervals.

Table 2: Comparison of parameters of distribution after Bayesian update when the prior distribution is uniform or the actual PDF

Parameter	**Prior Distribution**	
	Uniform Distribution	**Actual Distribution**
Mean of final distribution	1.0167	1.0144
SD of final distribution	0.0295	0.0285

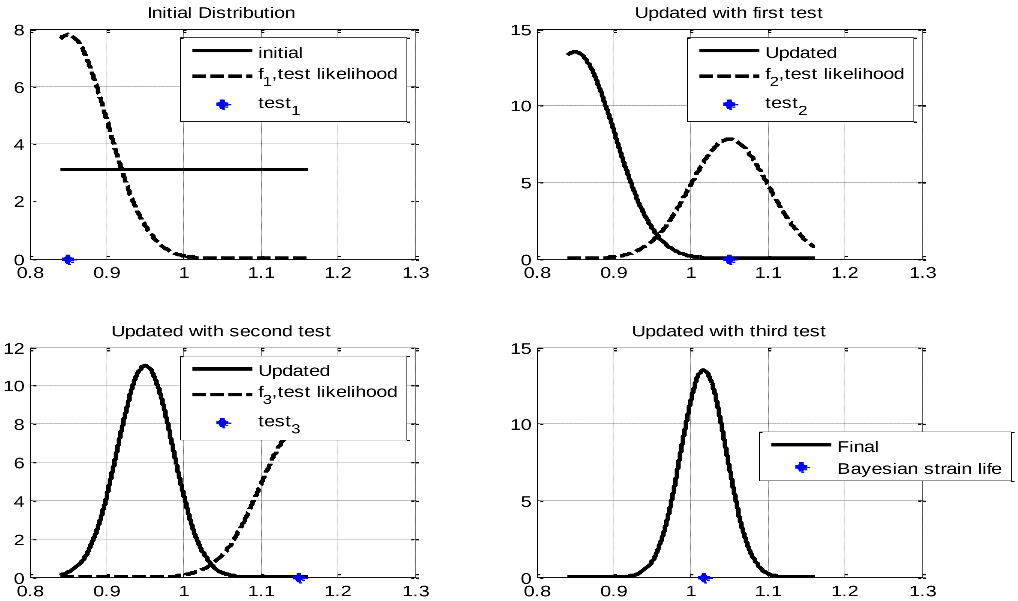

Figure 9. Bayesian update history of failure fatigue life with a uniform distribution for prior information.

VII. UPDATE OF FATIGUE STRAIN PARAMETERS

In the previous sections, the distribution of fatigue life or strain is updated using test data. However, in predicting material's fatigue behavior, it is necessary to estimate the fatigue parameters. The update of the strain – life curve parameters is performed at constant strain amplitude of 0.0055. This strain amplitude corresponds to 10^4 reversals, which classifies this case as Low Cycle Fatigue (LCF). It has been seen that the plastic strain parameters (e_f, c) greatly affect the LCF part of the strain – life curve than the elastic strain parameters (s_f, b). Hence, it is logical to update just the distribution of plastic strain parameters.

The test cases for the number of reversals at a constant strain amplitude of 0.0055 has been randomly chosen as [4169, 13804, 30200] reversals from the initial distribution of fatigue life. Bayesian update has been utilized to update the plastic strains (e_f, c) simultaneously, or in other words, the Joint PDF of (e_f, c). The initial Joint PDF of (e_f, c) is assumed to be uniformly distributed within ± 10% of their analytical values in Table 1.

In order to perform Bayesian update, it is necessary to calculate the Likelihood function for each of the 3 test cases. Two cases of variability are considered in calculating the likelihood function. Case 1 corresponds to variability in test; different tests may have different fatigue life at a given strain amplitude. Case 2 corresponds to variability induced by other two elastic parameters (s_f, b).

For Case 1, the variability in test has been assumed to be lognormal with 15% COV. To construct the likelihood function, a 100 × 100 grid of (e_f, c) values within their bounds is considered. For each point in the grid, the mean of the life distribution has been determined with the elastic strain parameters at their analytical values.

For Case 2, the variability in life is induced by the random elastic strain parameters (s_f, b), which are varying uniformly between ± 10% of their analytical value. For each grid point of (e_f, c), the distribution of life is constructed accounting for the variability in (s_f, b) by performing a 10^5 MCS. It has been found that the life distribution is always lognormal with a constant SD of 0.07 for all combinations of (e_f, c) considered.

The likelihood function values for both cases are the probability density of the life distribution at the location of the test case. Once the likelihood function is constructed, the Bayesian update is performed. The posterior distribution after the first test is used as the prior for the update with the second test and so on.

Since distributions of two parameters are updated here, the conditional probability distribution of one parameter, when the other is fixed at its mean value is plotted. Figures 10 and 11 show the distribution of parameters, e_f and c, respectively, when the other parameter is its mean value for Case 1. Figures 12, 13 show similar plots for the two parameters for Case 2.

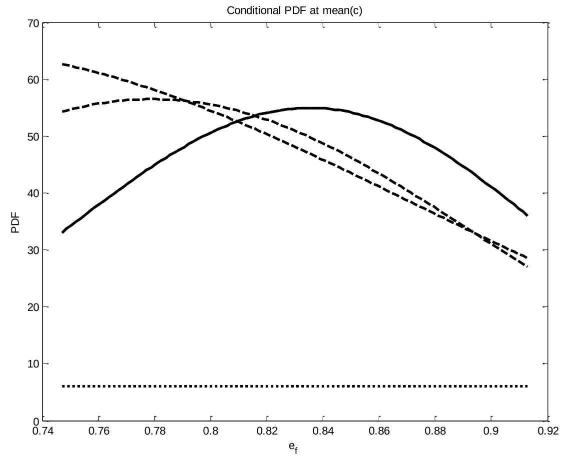

Figure 10. Conditional PDF of ef - Case 1.

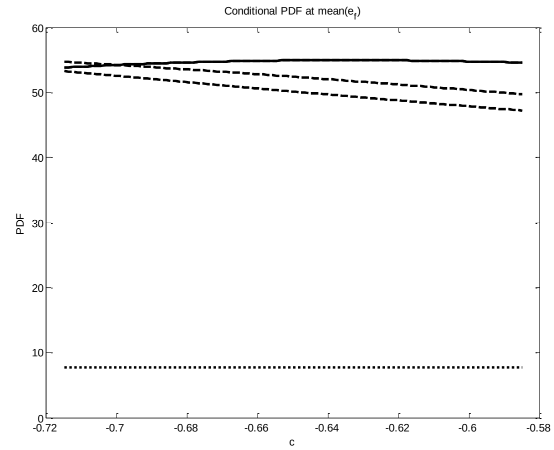

Figure 11. Conditional PDF of c – Case 1.

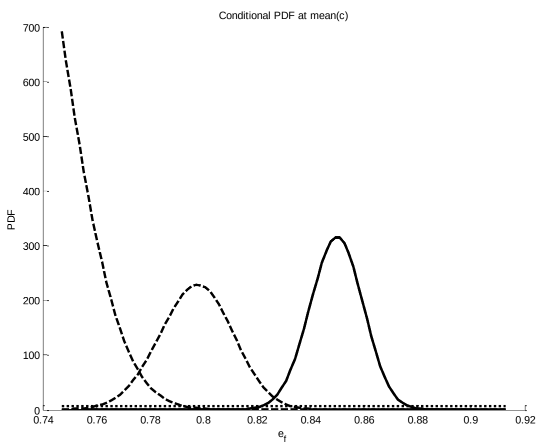

Figure 12. Conditional PDF of ef – Case 2.

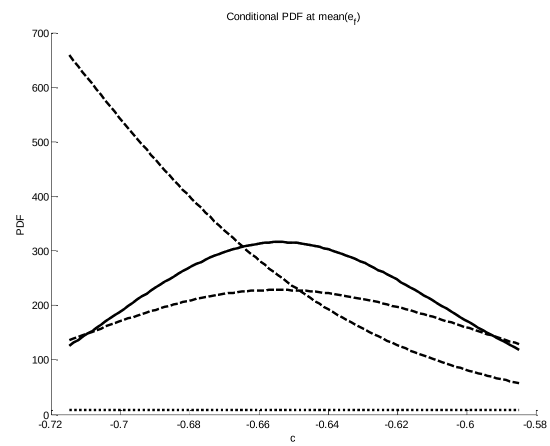

Figure 13. Conditional PDF of c – Case 2.

In all the four plots above, the dotted line shows the initial distribution. The dashed line shows the conditional PDF after each update, and the solid line shows the conditional PDF after update with the third test.

Table 3, 4 tabulates the mean and SD of (e_f, c) after each update for Case 1, 2 respectively.

Table 3: Mean and SD of (e_f, c) after each update for Case 1

After Update	Initial	With first test	With Second test	With Third test
Mean (e_f)	0.83	0.8196	0.8212	0.8313
SD(e_f)	0.0479	0.0464	0.0456	0.0450
Mean (c)	-0.65	-0.6512	-0.6509	-0.6497
SD (c)	0.0375	0.0375	0.0375	0.0374

Table 4: Mean and SD of (e_f, c) after each update for Case 2

After Update	Initial	With First test	With Second test	With Third test
Mean (e_f)	0.83	0.7583	0.7975	0.8499
SD (e_f)	0.0479	0.0096	0.0160	0.0127
Mean (c)	-0.65	-0.6728	-0.6498	-0.6497
SD (c)	0.0375	0.0326	0.0375	0.0375

It is noted that the distribution of e_f is affected more by the variability due to the elastic parameters than by variability due to test results. The distribution of parameter, c, remains practically constant throughout the update.

VIII. CONCLUSIONS AND FUTURE PLANS

- Bayesian update has been demonstrated as a good method to reduce uncertainty in number of reversals (2N) for constant strain amplitude case and also to reduce uncertainty in strain amplitude for constant life case, with the knowledge of test cases.
- It has been demonstrated that the Bayesian update reduces the standard deviation of the initial distribution by almost half.
- Bayesian update provides a more conservative estimate of the failure fatigue strain in Example 1 and of life in Example 2 than the average of the test cases considered in respective Examples.
- Having a better knowledge for the prior leads to less scatter in distribution obtained after Bayesian update
- Bayesian update has also been demonstrated as a method to update the parameters of strain – life curve for a case of constant strain amplitude test

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