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Bayesian Technique for Reducing Uncertainty in Fatigue Failure Model

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ABSTRACT

In this paper, Bayesian statistics is utilized to update uncertainty associated with the fatigue life relation. The distribution for fatigue strain at a constant load cycle is determined using the initial uncertainty from analytical prediction and likelihood functions associated with test data. The Bayesian technique is a good method to reduce uncertainty and at the same time provides a conservative estimate, given the distribution of analytical prediction errors and variability of test data. First, the distribution of analytical fatigue model error is estimated using Monte Carlo simulation with uniformly distributed parameters. Then the error distribution is progressively updated by using the test variability as a likelihood function, which is obtained from field test data. The sensitivity of estimated distribution with respect to the initial error distribution and the selected likelihood function is studied. The proposed method is applied to estimate the fatigue life of turbine blade. It is found that the proposed Bayesian technique reduces the scatteredness of fatigue life by almost 50%, while maintaining the conservative life estimate at a given fatigue strain. In addition, a good conservative estimate of fatigue life prediction has been proposed using a knockdown factor that is obtained from the distribution of lowest test data.

INTRODUCTION

In general, there are two different life prediction models in fatigue analysis: stress-life and strain-life models. The former is often used for high-cycle fatigue analysis in which the stress-strain relation is in the linear region. The latter is frequently used for low- and medium-cycle fatigue in which plastic deformation contributes to the fatigue life. Although the basic concept in the proposed Bayesian approach is the same, the strain-life model will be investigated in this paper.

In strain-life fatigue analysis, the total fatigue strain (ε_t) is decomposed by elastic strain (ε_e) and plastic strain (ε_p). For this analysis, the strain-life curve is defined [3] as:

$$\varepsilon_t = \varepsilon_e + \varepsilon_p = \frac{\bar{s}_f}{E}\left(2N\right)^b + \bar{e}_f\left(2N\right)^c \tag{1}$$

where $\bar{e}_f$ is the coefficient of fatigue ductility, $\bar{s}_f$ the coefficient of fatigue strength, b the exponent of fatigue strength, c the exponent of fatigue ductility, and E the Young's modulus.

The coefficients in the strain-life curve are obtained using curve-fitting of test data. However, due to variability in test and material, the results are often scattered. In the material handbook, for example, the standard values of the coefficients are available. However, a particular batch of material may have different properties. In addition, a particular machine may have different fatigue properties due to manufacturing process used and possibility of residual stresses. Thus, an important question is how to find more accurate life estimate for a specific machine when several test data are available. Traditionally, safety factors have been adopted as a measure to counter variability. But, presently, there is growing interest in replacing safety factor-based deterministic design with reliability-based design (e.g., Wirsching [7], SAE Aerospace Information Report 5080 [5]). In addition, the goal of test is often to find conservative estimate of the predicted life. Obviously taking the lowest test data can be a choice, but its variability will be high and in many cases, the lowest test data will not provide enough conservativeness [1]. Thus, another important question is how to find the best way of predicting conservative fatigue life of a machine.

The objective of this paper is to investigate the possibility of using the Bayesian statistics in order to reduce scatteredness of the fatigue life distribution when additional test data are available. In addition, a good way of conservatively estimating fatigue life is proposed using a knockdown factor a term introduced for correcting analytical predictions based on test results[8]) that is obtained from the distribution of lowest test data. The concept of

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knockdown factor has been adopted in aerospace structures.

INPUTS AND ASSUMPTIONS

It is important to understand the assumptions that are used in this paper. Some of them are for convenience, and others represent the lack of knowledge. If additional information is available, the latter can be improved.

It is first assumed that the coefficients in Eq. (1) are randomly distributed and their statistical distribution parameters are known. These distributions represent the prior knowledge or analytical prediction. This information can be obtained by studying strain-life test data and by estimating upper- and lower-bounds of test data from the mean curve. When this information is not available, it is possible to assume that these parameters ($\bar{s}_f$, $\bar{e}_f$, b, and c) are distributed uniformly with given bounds from their nominal values. This will serve as a prior knowledge of the fatigue failure distribution. Even if input parameters are uniformly distributed, the fatigue failure strain will not be uniformly distributed.

The confidence interval is a measure of our confidence in the analytically predicted value. It would be the lower and upper bounds of the strain-life test data. However, since only the distribution of each parameter is considered known, the bounds for life cycle value are determined from the distribution of all parameters.

Considering cases, where only one of the parameters varies at a time, while the rest of them are at their mean values, the sensitivity of life cycle value to each parameter has been determined, and found to be positive for all parameters. Hence, the upper and lower bounds for life cycle would be when all parameters are simultaneously at their algebraic maximum and minimum values, respectively. The bounds of the confidence interval are usually expressed as a percentage of analytical value. If the bounds are asymmetrically distributed about the analytical value, the maximum variation is considered.

The distribution of error is the distribution of life cycle due to the distributions of the parameters. This distribution of error is generated using the Monte Carlo Simulation (MCS). In this paper, the MCS is performed by generating 100,000 values for each parameter, governed by its variability. These randomly generated numbers are used to calculate 100,000 values of strain life, from which the distribution of error can be estimated.

Test variability is the scatteredness of the test data. Distribution of test variability is the histogram of strain-life test data at the known constant failure fatigue strain (ε_t) value. However, for the want of test data, the distribution of test variability is assumed to be normal with same parameters as that of the distribution of error.

Even if there is no limitation on the number of test data, three test data are assumed available through the test of a specific component. These three test data are randomly chosen within the variability limits with the analytical value as the mean.

BAYESIAN UPDATE FOR FATIGUE FAILURE STRAIN

NORMALIZATION:

Although the raw data and initial distribution can be used for Bayesian update, it is often more convenient to normalize all data and distributions. All the factors affecting the Bayesian update, such as the confidence interval, error distribution, test variability, and test results are normalized with respect to the initial value of fatigue life, $2N$. Since the confidence interval is expressed as a percentage of analytical value, it is not affected by normalization. The parameters of test variability are expressed as a fraction of mean value. In the normalized distribution, the standard deviation becomes identical to the coefficient of variance (COV).

LIKELIHOOD FUNCTION:

The likelihood function of a test result is the probability of obtaining that test result, given the value of actual fatigue life and the test variability. It is actually the ordinate value of probability distribution function (PDF) of test variability with the actual fatigue life as its mean, when the abscissa is equal to the test result.

The likelihood would be a single value, if the actual fatigue life is known. But, since only the bounds for the actual fatigue life are known, the likelihood function varies within that confidence interval. The likelihood of the given test result can be found by considering each point within the error bounds as the actual fatigue life. The likelihood function for a given test result would be the variation of these likelihood values with the actual fatigue life values. The likelihood function for each of the three test results can be determined in a similar fashion.

BAYESIAN UPDATE:

The Bayesian update is based on the theory of conditional probability, which states

$$p\left(true/test \right) = \frac{p\left(test/true \right) \times p(true)}{p(test)} \tag{2}$$

The expression for Bayesian update is very similar, i.e.

$$f^{upd}(2N) = \frac{f_{test}(2N) f^{ini}(2N)}{\int_{-\infty}^{\infty} f_{test}(2N) f^{ini}(2N) de_t} \tag{3}$$

Here, $f_{test}(2N)$ is the likelihood function for the given test result. It could also be seen as probability of obtaining the test result given the true value of fatigue life. $f^{ini}(2N)$ is the initial distribution of $2N$. For updating with the first test result, this distribution is taken as the error distribution. This updated distribution is used as the initial distribution for updating with second test result and so on.

The denominator is simply the integration of numerator. This is interpreted as the normalization of PDF such that the area under the distribution becomes one. Different techniques like 'Trapezoidal rule', 'Simpson's Rule', etc, can be used for this purpose. Trapezoidal rule has been used in this paper for numerical integration.

After the initial distribution is updated using the first test data, $f^{ini}(2N)$ is replaced by $f^{upd}(2N)$, and the above procedure is repeated for the next test results.

The mean of the distribution obtained after updating with final test result, is called the Updated Bayesian Life.

KNOCKDOWN FACTOR:

It is common in practice to take the minimum value of the test results as the actual fatigue life. Such consideration implicitly applies a knockdown factor on average fatigue life value. In this paper, an explicit knockdown factor is calculated from test statistics and multiplied to the mean value of the updated Bayesian life to calculate a conservative estimate of the fatigue life. It is known that the lowest of the test results follows an extreme value distribution [6]. The mean of this extreme value distribution is used for the knockdown factor.

Knock down factor calculation depends on material variability only. If 'f' is the cumulative distribution function (CDF) of material variability after normalization, the extreme value distribution is given by,

$$F_1^3 = 1-(1-f)^3 \tag{4}$$

The mean of this extreme value distribution (F_1^3) is the knockdown factor. [2]

The conservative estimate of fatigue life is the product of the Updated Bayesian Life and knockdown factor.

NUMERICAL EXAMPLE 1 – CONSTANT CYCLE TEST

As a test case, the Bayesian technique is applied to steel 4340 material, whose strain-life fatigue parameters are shown in Table 1 [4]. The strain-life curve for steel 4340 material is shown as 'Mean' curve in Figure 1. The uncertainty in the parameters leads to uncertainty in strain amplitude (ε_t). First the sensitivity of strain amplitude with respect to each of the parameters has been determined, and found to be positive for all parameters. The effect of the uncertainty in strain amplitude has been plotted through the 'Maximum' and 'Minimum' curves in Figure 1. These curves have been plotted considering each of the parameter at its algebraic maximum and minimum respectively, governed by its variability.

In the first example, it is assumed that the material is failed at 2N = 100,000 reversals and three test data are available at that reversal value. Since the crossover reversal for this material is in the order or 10,000, the fatigue strain is in the elastic region. For the material parameters in Table 1, the value of analytical fatigue strain is $\varepsilon_t = 0.0032$.

Table 1: Strain-life fatigue parameters for steel 4340

Parameter	Value
Elastic stiffness (E)	208,900 MPa
Fatigue ductility coefficient (e_f)	0.83
Fatigue ductility exponent (c)	-0.65
Fatigue strength coefficient (s_f)	1,713 MPa
Fatigue strength exponent (b)	-0.095

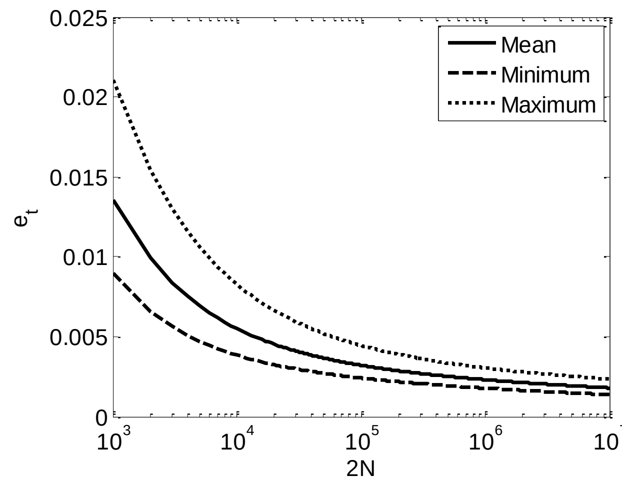

Figure 1: Strain-life curve for steel 4340

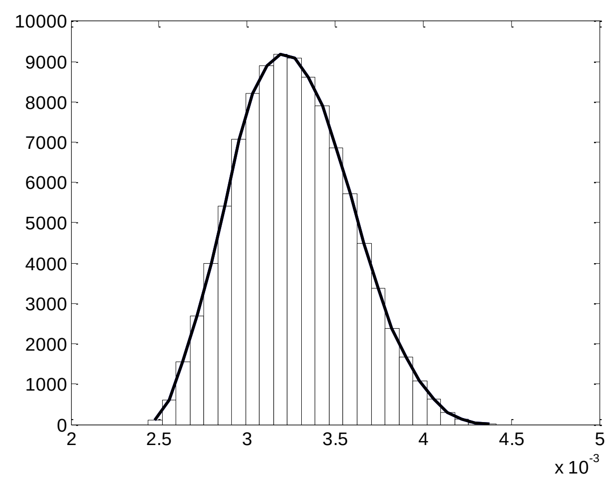

Figure 2: Histogram of fatigue strain at 2N = 100,000

In order to find the initial error bounds of the fatigue model, all parameters affecting the fatigue strain are assumed to vary uniformly ±10% of their nominal value. When all the parameters are varying simultaneously, the value of ε_t varies between 0.0024 and 0.0045. Then, the error bounds are calculated considering the maximum deviation from the mean, i.e., 0.0045 – 0.0032 = 0.0013, which is about 39% of the mean. Hence, the error bounds are ±39% of the mean.

In addition to the error bounds, detailed distribution of ε_t can be plotted using MCS. First, 100,000 samples of material parameters are randomly generated according to uniform distribution. Then, the histogram of ε_t is plotted by applying these samples to Eq. (1). Figure 2 shows the histogram of fatigue strain at $2N$ = 100,000 reversals. The solid curve connects the midpoint of each bin in the histogram. The PDF of fatigue strain is obtained by scaling down the curve, such that the area under the curve is unity. It turns out that the fatigue strain distribution has following parameters:

$$\begin{aligned} \text{Mean} &= 0.0033 \\ \text{SD} &= 0.00033 \end{aligned} \tag{5}$$

In Figure 3, the normal distribution with the same mean and standard deviation is plotted. It is clear that the histogram looks close to a normal distribution with same parameters. Although variability associated with fatigue test should be obtained from more rigorous method, it is assumed that the test variability is normally distributed with mean 0.0033 and COV 10%.

With error bounds and test variability, now it is possible to perform Bayesian update. Let us consider that three fatigue tests are performed. After normalizing by analytical fatigue strain, the three test data strains are 0.85, 1.05, and 1.15. These three normalized data correspond to strain values of 0.02805, 0.003465, and 0.003795, respectively.

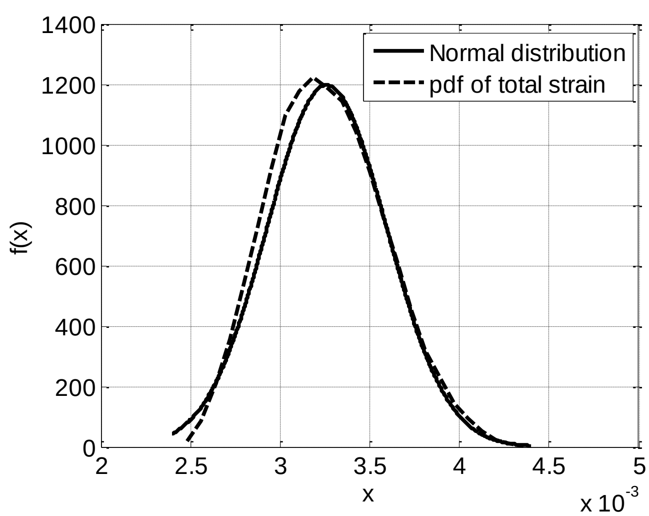

Figure 3: Fitting the distribution of fatigue strain using normal distribution

Figure 4 shows distribution of estimated fatigue strain at each stage of Bayesian update. The dotted lines show the likelihood function for each test result, while the solid lines show the updated distributions. The final updated distribution of the fatigue strain has the following parameters:

$$\begin{aligned} \text{Mean} &= 0.00327 \\ \text{SD} &= 0.00017 \end{aligned} \tag{6}$$

Note that the mean does not change significantly, while the standard deviation of the updated Bayesian distribution is about 50% of that of the original distribution. Thus, the three test data effectively reduce the uncertainty in the fatigue failure strain. Table 2 tabulates the variation of the parameters of the distribution of fatigue failure strain with each stage of Bayesian update.

Since the fatigue strain is distributed, it is better to provide a conservative estimate of the fatigue strain using a knockdown factor. When the test variability is normalized, the mean is shifted to 1.0, while retaining the COV. Hence, the knockdown factor for the normalized test variability, governed by $N(1.0, 0.1^2)$ is calculated to be 0.9127, using the mean value of extreme distribution in Eq. (4). Hence, the conservative estimate of fatigue failure strain at 100,000 reversals becomes $\varepsilon_t = 0.00297$.

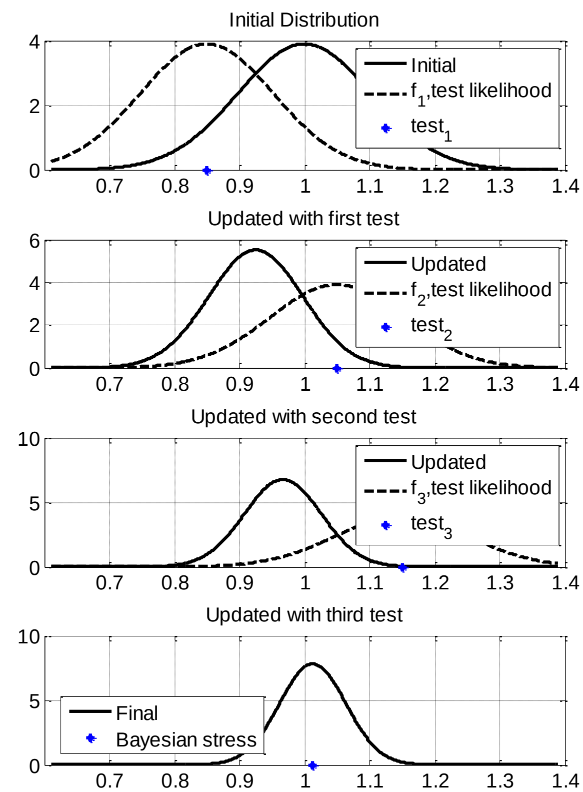

Figure 4: Bayesian update history of failure fatigue strain

Table 2: Variation of parameters with each stage of Bayesian update

	Mean	SD
Initial	1.0000	0.1014
After first test	0.925	0.0724
After second test	0.9667	0.0591
After third test	1.0125	0.0512

NUMERICAL EXAMPLE 2 – CONSTANT STRAIN TEST

For the constant strain test, it is assumed that the amplitude of strain applied to the machine is constant at 0.0055. For the material properties in Table 1, the value of analytical fatigue life is 10,000 reversals. Since the crossover reversal value for this material is 4,103 reversals, the assumed fatigue strain is in the elastic region.

Due to computational difficulties, the strain-life expression is solved for $y = \log_{10}(2N)$, rather than for the strain life, $2N$. Hence, the analytical value for '*y*' would be 4.00.

In order to find the initial error bounds of the fatigue model, all parameters affecting the fatigue strain are assumed to vary uniformly ±10% of their nominal value. The error bounds for '*y*' have been determined to be ±16% of the mean.

In addition to the error bounds, detailed distribution of '*y*' can be plotted using MCS. First, 100,000 samples of material parameters are randomly generated according to uniform distribution. Then, the histogram of '*y*' is plotted by applying these samples to Eq. (1). Figure 5 shows the PDF of '*y*' at fatigue strain, $\varepsilon_t = 0.0055$. It turns out that '*y*' has the following distribution parameters:

$$\begin{aligned} \text{Mean} &= 3.8937 \\ \text{SD} &= 0.1991 \end{aligned} \tag{7}$$

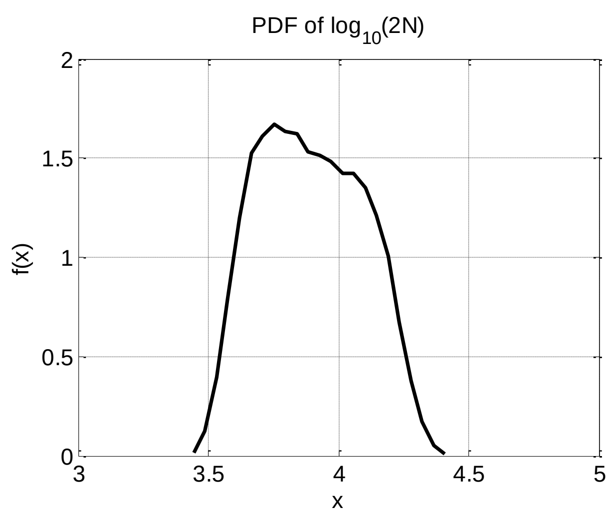

Figure 5: PDF of $\log_{10}(2N)$ at fixed strain $\varepsilon_t = 0.0055$

The PDF of $\log_{10}(2N)$ can be modeled as a product of PDF of a normal distribution and a polynomial. The normal distribution that has the same parameters as that of the PDF of $\log_{10}(2N)$ is shown as the curve with dotted lines in Figure 6. Figure 7 plots the product function determined from the two curves in Figure 6. An 8th degree polynomial is fitted to estimate the product function. The dashed lines show the polynomial in Figure 7.

For the want of test results, the variability of the test results have been assumed to a normal distribution with mean 3.8937 and COV 5.11%

The three additional test cases for '*y*' have been assumed to at 0.85, 1.05 and 1.15 times the analytical value.

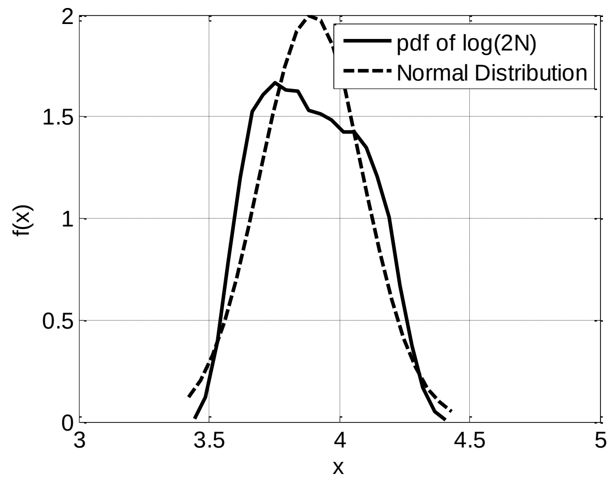

Figure 6: PDF of $\log_{10}(2N)$ and that of normal distribution with the same mean and standard deviation

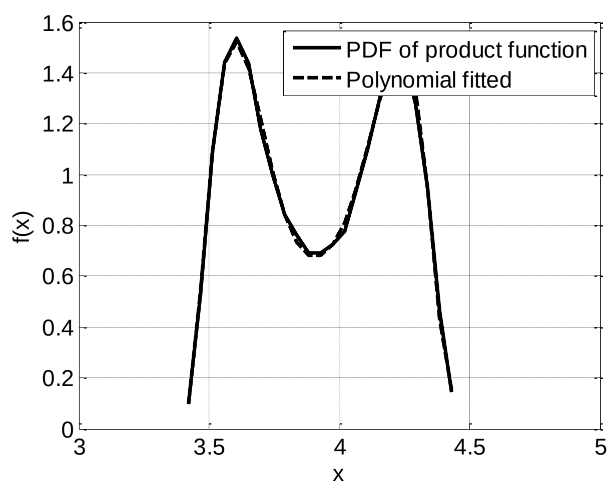

Figure 7: Product function that models the difference between PDF of $\log_{10}(2N)$ and that of normal distribution

Figure 8 shows distribution of estimated fatigue life at each stage of Bayesian update. The dashed lines show the likelihood function for each test result, while the solid lines show the updated distributions. The final updated distribution of the '*y*' has the following parameters:

$$\begin{aligned} \text{Mean} &= 4.0576 \\ \text{SD} &= 0.1140 \end{aligned} \tag{8}$$

Since the fatigue life is distributed, it is better to provide a conservative estimate of the fatigue life using knockdown factor. When the test variability is normalized, the mean is shifted to 1.00, while retaining the COV. Hence, the knockdown factor for the normalized test variability, governed by $N(1, 0.05112)$ is calculated to be 0.9567. Hence, the conservative estimate of fatigue failure life at strain amplitude of 0.0055 is $2N = 10^{3.8819} = 7619$ reversals.

Table 3 tabulates the variation of the parameters of the distribution of fatigue failure strain with each stage of Bayesian update.

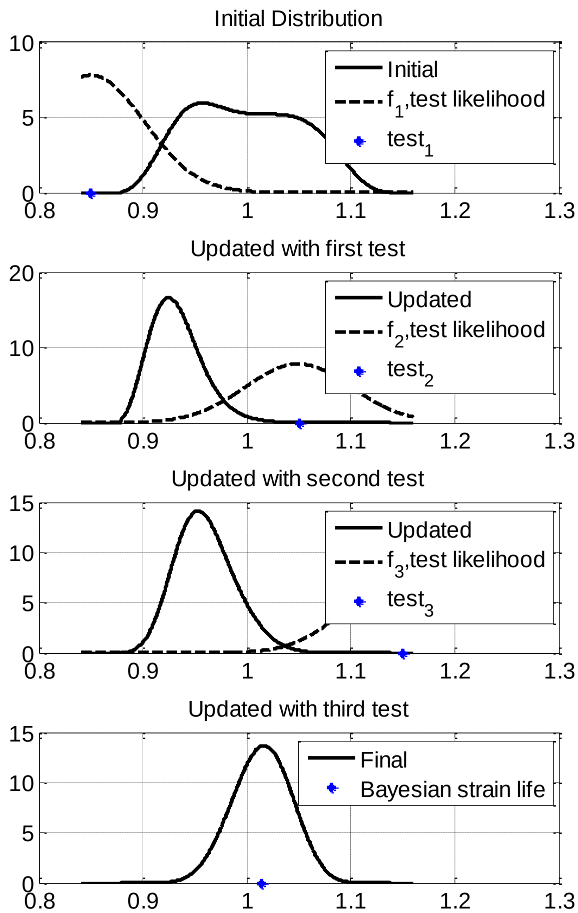

Figure 8: Bayesian update history of failure fatigue life

Table 3: Variation of parameters with each stage of Bayesian update

	Mean	SD
Initial	1.0000	0.0511
After first test	0.9326	0.0252
After second test	0.9601	0.0288
After third test	1.0144	0.0285

EFFECT OF INITIAL DISTRIBUTION

The Bayesian update for the fatigue failure life has been performed in Example 2 above, considering the PDF of life as initial distribution. The importance of prior or initial distribution could be emphasized by performing a Bayesian update with prior as uniform distribution between error bounds and comparing results. The Bayesian update history of failure fatigue life with a uniform distributed prior is shown in Figure 9. Table 4 compares the results of these two Bayesian updates.

It is noted that when the actual PDF of life is used as prior, the mean of final distribution decreases by 0.23% and the standard deviation of the final distribution reduces by 3.39%. This suggests that having a better knowledge of the prior distribution results in a better coefficient of variance for the distribution after update.

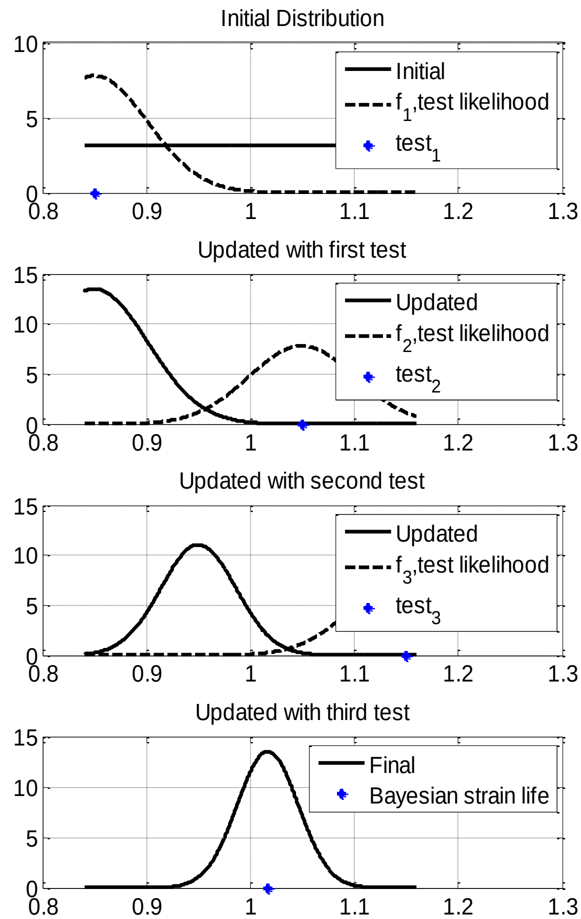

Figure 9: Bayesian update history of failure fatigue life with a uniform distribution for prior information

Table 4: Comparison of parameters of distribution after Bayesian update when the prior distribution is uniform or the actual PDF

Parameter	**Prior Distribution**	
	Uniform Distribution	**Actual Distribution**
Mean of final distribution	1.0167	1.0144
SD of final distribution	0.0295	0.0285

EFFECT OF TEST CASES

The normalized test cases considered in the example 2 above, are 0.85, 1.05 and 1.15. The Bayesian update performed with an altogether different set of normalized test results such as 1.01, 1.02 and 1.02 yields an interesting result. Figure 10 compares the distribution and mean of fatigue failure life after Bayesian update for the two cases of tests discussed above. Case 1 refers to the test result set of [0.85, 1.05, 1.15]. Case 2 refers to the test result set of [1.01, 1.02, 1.02]. It is noted that both the distributions are one and the same.

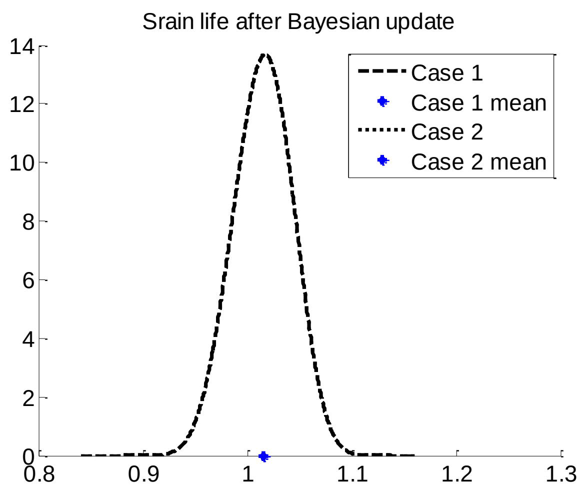

Figure 10: Strain life after Bayesian update for two different set of test cases considered for the update

It is noted that the average of the two sets of test results is same, which is 1.0167.Now, Case 1 has two test results in the extremes of the error bounds, and hence, the likelihood function of those results are truncated. Case 2 has all test results centered on 1.00 and hence their complete likelihood function is utilized in Bayesian update. Yet, the strain life distribution after Bayesian update is one and the same for both cases. Hence, it can be concluded that the distribution of fatigue failure life after Bayesian update will depend on the average of the normalized test cases considered.

The dependence of Bayesian update on the average of the three test cases have been further verified by considering a set of identical test values, [1.0167, 1.0167, 1.0167] which resulted in an identical distribution as shown in Figure 10.

It has also been verified that the changing the order in which the test cases are considered for the update does not affect the distribution of strain life after Bayesian update.

CONCLUSIONS

- Bayesian update has been demonstrated as a good method to reduce uncertainty in number of reversals (2N) for constant strain amplitude case and also to reduce uncertainty in strain amplitude for constant life case, with the knowledge of test cases.
- It is seen that the standard deviation of the distribution obtained after the Bayesian update, is almost half of that of the initial error distribution.
- Since only the average of the three test cases is the main parameter, there is no effect of having repetitive values for the test cases.
- Also, changing the order, in which the test cases are considered for the Bayesian update, doesn't affect the actual fatigue life.
- Having a better knowledge for the prior leads to less scatteredness in distribution obtained after Bayesian update.

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