Modeling the Effect of Structural Tests on Uncertainty in Estimated Failure Stress

Chanyoung Park[1], Taiki Matsumura[2], Raphael T. Haftka[3], Nam H. Kim[4]
Department of Mechanical and Aerospace Engineering, University of Florida
PO Box 116250, Gainesville, FL 32611-6250

and

Erdem Acar[5]
TOBB University of Economics and Technology, Söğütözü, Ankara 06560, Turkey

This paper investigates the effects of the number of coupon and element tests on uncertainty in element failure stress. In aircraft structural design, failure stress is first obtained from coupon tests, which is then used in predicting failure stress of structural element under combined loads. The mean and standard deviation of failure stress are expressed as a distribution due to errors in failure theory and variability due to finite number of coupon tests. This paper focuses on identifying the effect of the number of coupon and element tests on the distribution of failure stress of structural element. This paper assumes isotropic properties and the failure stress of structural element is assumed to be predicted by a failure theory (e.g. Von Mises), and initial distribution of this failure stress reflects uncertainties. Bayesian updating is used to reduce the uncertainties in the initial failure stress distribution by using element tests. The relation between the number of tests and the level of uncertainties is presented for a simple test case.

I. Introduction

THE traditional design practice of aircraft structures is based on safety factors and building-block test processes, which have evolved in tandem over many decades based on experience through trial and error in aircraft structures. For example, the Federal Aviation Administration (FAA) requires using A-basis (or B-basis) allowable failure stresses that are below 99% (or 90%) of the test failure stresses with 95% confidence. However, it is unclear how much the use of conservative failure stresses improves the safety of the system. Similarly, all aircraft structures are regularly inspected and repaired when large cracks are detected, but it is not quantified how much this process will improve safety. Although it is generally accepted that these processes improve product safety, only a few research results have tried to quantify their contribution to safety over the lifecycle of the product. Dhillon *et al*. [1] incorporated theses processes into evaluating reliability of industrial robots. Kale *et al*. [2] and Garbatov and Guedes Soares [3] used variable inspection schedules to maintain a constant level of reliability throughout the lifecycle. Kulkarni and Achenbach [4] modeled the effects of inspection schedule on the probability of failure using the probability of damage detection when uncertainty comes from the initial crack distribution.

Although there is a push to replace safety-factor-based design with probabilistic design, the latter cannot readily replace the former because current probabilistic design frameworks do not incorporate various uncertainty reduction measures (URMs) that happen after design. Instead, it uses only uncertainty information available at the design stage without considering future reductions through these processes, which is a major obstacle for the wide adoption of probabilistic design in most engineering industries. Therefore, it is important for the probabilistic design to include the effects of URM on structural safety; i.e., reliability.

---

[1] Graduate Research Assistant, AIAA Student Member
[2] Graduate Research Assistant, AIAA Student Member
[3] Distinguished Professor, AIAA Fellow
[4] Associate Professor, AIAA Member
[5] Assistant Professor, Mechanical Engineering, AIAA Member.

As a first step toward quantifying the effect of URMs, the objective of this paper is to calculate the contribution the effects of coupon tests and structural element tests on the uncertainty in element failure stresses. The fundamental difficulty in modeling the abovementioned building-block test processes is how to consider them in the design stage, especially when most of tests are not performed at the design stage; that is, how to model the uncertainty reduction effect of future tests. As a first step toward quantifying uncertainty reduction by building-block test processes, the proposed paper models the uncertainty reduction by structural element tests in the failure stress. The effect of future element tests is modeled using a distribution of distributions, and multi-layered Monte Carlo simulation with Bayesian inference is utilized to evaluate this effect. The goal is to show the relation between final uncertainty and the number of element tests. For the sake of simplicity, the true (unknown) distribution of failure stresses is assume to be Normal.

II. Uncertainty reduction by structural element tests

In aircraft structural design, most complex modern aircrafts are designed based on simulations, which may have substantial prediction errors and material variability; i.e., uncertainties. Once the conceptual design is completed, simulation models are refined and extensive tests should be undertaken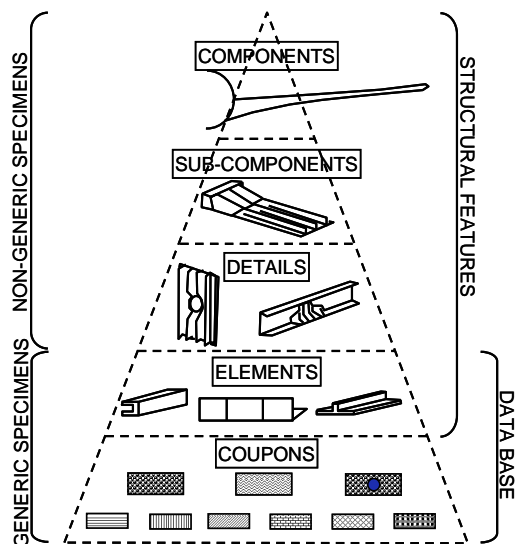 so that the structural safety can be achieved by taking steps to reduce uncertainties. For airliners, the building-block test is used as a process of error reductions through tests (see Figure 1). Before the design, dozens of coupon tests are carried out to measure stiffness and failure properties. A large number of coupon tests provide statistical information on the variability of the material properties. Then tests with structural elements are used to update failure prediction by adjusting the design failure stresses to the results of the tests. When going up the building-block pyramid, understanding deviations from analytical predictions becomes more difficult and the tests are more expensive; furthermore any modification can also be very expensive. The fundamental difficulty in the building-block tests is that since it is heuristically designed, it is difficult to quantify how much each level can reduce uncertainty, which is the main objective of this paper. Once the contribution of each level to the uncertainty is modeled, the design engineer can decide how much resources should be allocated to a specific URM, such as a particular set of tests, in order to achieve the target reliability at minimum cost.

Figure 1: Building-block approach for testing aircraft structural components. At each stage, analytical models can be adjusted to account for discrepancies between the model and test and, thus, reduce the errors in the analytical model used to predict the response at the next level.

In this paper, only coupon tests and element tests are considered to demonstrate the main concept of URMs. The goal is to reduce the uncertainty in element failure stress prediction. The effect of the number of coupon tests and element tests on final uncertainty in element failure stress is investigated. In the following subsections, detailed assumptions and procedures are presented.

A. Errors in estimating failure stress from coupon tests.

Coupon tests are conducted to obtain the statistical distribution of material strength properties, such as failure stress, and their corresponding design values (A-basis or B-basis). In this paper, it is assumed that the *true failure stress of coupons* $\tau_{c,true}$ follows a normal distribution as,

$$\tau_{c,true} \sim N\left(\mu_{c,true}, \sigma_{c,true}\right) \tag{1}$$

where $\mu_{c,true}$ and $\sigma_{c,true}$ are, respectively, the mean and standard deviation of the true failure stress of coupons.

Unfortunately, $\mu_{c,true}$ and $\sigma_{c,true}$ are unknown and can only be found with an infinite number of coupon tests. Instead, it is estimated with a finite number of specimens. In this paper, the estimated mean and standard deviation of failure stress with a finite number of specimens are called the *possible true mean failure stress*, $\mu_{c,Ptrue}$, and *possible true standard deviation of failure stress*, $\sigma_{c,Ptrue}$.

In this paper, it is assumed that the failure stress is normally distributed so that the true distribution is estimated by estimating two parameters; mean and standard deviation. Two methods can be used to estimate the parameters of the true distribution: Bayesian or sampling method. In this section, the latter is used because the two methods have little differences and the latter is simpler. Appendix A shows the detail of comparison between the two methods.

The possible true failure stress of coupons, $\tau_{c,Ptrue}$, is again assumed to be normally distributed as,

$$\tau_{c,Ptrue} \sim N\left(\mu_{c,Ptrue}, \sigma_{c,Ptrue}\right) \tag{2}$$

The possible true failure stress $\tau_{c,Ptrue}$ is estimated using $\mu_{c,Ptrue}$ and $\sigma_{c,Ptrue}$ from n_c specimens. Obviously $\mu_{c,Ptrue}$ and $\sigma_{c,Ptrue}$ include sampling errors due to the finite number of specimens. Different analysts have different $\mu_{c,Ptrue}$ and $\sigma_{c,Ptrue}$ so that they are modeled as random variables, it is rational to consider them as a distribution and the possible true failure stress in Eq. (2) has a form of the distribution of distributions. Of course, a test with a small number of specimens has large uncertainty, and it will lead to a wide distribution. With more specimens, this wide distribution will shrink and gives more accurate estimation.

It will be discussed next how to obtain the distributions of $\mu_{c,Ptrue}$ and $\sigma_{c,Ptrue}$ for use in Eq. (2). However, first the final product is displayed with an example. Figure 2 shows a PDF of $\tau_{c,true}$, with a mean of 1.1 and a coefficient of variation of 0.07, as well as the PDFs of two $\tau_{c,Ptrue}$ with different numbers of specimens. For this example, test results of 30 specimens and 80 specimens were randomly sampled. The 30 specimen sample had a mean of 1.053 and a standard deviation of 0.096. From the calculations described later (in Eqs. (3) and (4)) the standard deviation of the mean was estimated to be 0.018 and the standard deviation of the standard deviation was estimated to be 0.013. The possible true distribution was obtained by double-loop Monte Carlo simulation. First we sampled means and standard deviations and then the actual failure stress from a normal distribution with the sampled mean and standard deviation. The resulting distribution, shown in Figure 2 had a mean of 1.053 and a standard deviation of 0.098. For the 80 sample case, the corresponding numbers were a mean of 1.113, standard deviation of 0.083, the standard deviation of the mean of 0.009 and the standard deviation of the standard deviation of 0.007. The resulting distribution, shown in Fig. 2 had a mean of 1.113 and a standard deviation of 0.085. As expected, the possible true distribution of coupon failure stress with 80 specimens gives good estimation of the true distribution of coupon failure stress. For the distribution with 30 specimens, the sample mean was an underestimate (not conservative), but the width of the distribution compensates.
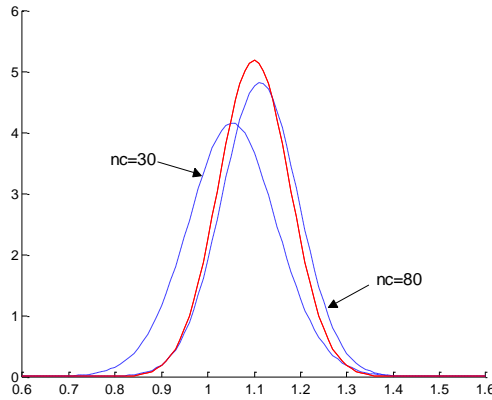

Figure. 2: True distribution, the distribution of $\tau_{c,Ptrue}$ with 30 specimens, and the distribution of $\tau_{c,Ptrue}$ with 80 specimens. (True mean =1.1, and true standard deviation=0.077)

Since it was assumed that $\tau_{c,true}$ is normally distributed, its sample mean also follows a normal distribution. We assume that an analyst estimates the distribution of the sample mean as $\mu_{c,Ptrue}$ with a test mean $\mu_{c,test}$ and a test standard deviation $\sigma_{c,test}$ from a coupon test with finite number of specimens as

$$\mu_{c,Ptrue} \sim N\left(\mu_{c,test}, \frac{\sigma_{c,test}}{\sqrt{n_c}}\right) \tag{3}$$

It is well known that the standard deviation $\sigma_{c,true}$ follows a chi-distribution of order $n_c - 1$ [10]. See Appendix B for detail of the chi-distribution. Since calculating true distribution of $\sigma_{c,true}$ needs the unknown $\sigma_{c,true}$, analysts estimate it as $\sigma_{c,Ptrue}$ with the test standard deviation $\sigma_{c,test}$. The standard deviation can be estimated by

$$\frac{\sqrt{n_c - 1}\, \sigma_{c,Ptrue}}{\sigma_{c,test}} \sim \chi(n_c - 1) \tag{4}$$

Estimated standard deviation of $\mu_{c,Ptrue}$ is calculated by

$$\text{Estimated standard deviation of } \mu_{c,Ptrue} = \frac{\sigma_{c,test}}{\sqrt{n_c}} \tag{5}$$

Estimated standard deviation of $\sigma_{c,Ptrue}$ is standard deviation of chi-distribution of order n_c.

$$\text{Estimated standard deviation of } \sigma_{c,Ptrue} = \sqrt{\sigma_{c,test}^2 - 2\frac{\sigma_{c,test}^2}{(n_c - 1)}\frac{\Gamma^2(n_c/2)}{\Gamma^2((n_c - 1)/2)}} \tag{6}$$

Those estimated standard deviations of mean and standard deviation are approximated standard deviations of the estimated mean $\mu_{c,Ptrue}$ and the estimated standard deviation $\sigma_{c,Ptrue}$. When the sample standard deviation $\sigma_{c,test}$ is replaced with the true standard deviation $\sigma_{c,true}$, Eqs. (5) and (6) give true standard deviation of the sample mean and the sample standard deviation.

Note that Eq. (2) describes a distribution used by the designer whose parameters are also random and this will be referred to as a distribution of distributions, with an example illustrated in Fig. 2. This distribution is needed to satisfy the FAA requirements that the design stress value is sufficiently conservative. For example, when no redundancy is present in the structure, the A-basis is used which determines the failure stress from the value of a material property exceeded by 99% of the population with a 95% lower tolerance bound [12].

B. Errors in estimating failure stress of structural element tests

The second level in the building-block test sequence in Fig. 1 is structural element tests, where structural elements with different joints are tested to validate the failure theory used in multi-dimensional stress states. Here it is assumed that Bayesian updating will be used to integrate the element test results with the information available from coupon tests. It is assumed that the *true failure stress of element* $\tau_{c,true}$ follows a normal distribution as,

$$\tau_{e,true} \sim N(\mu_{e,true}, \sigma_{e,true}) \tag{7}$$

where $\mu_{e,true}$ and $\sigma_{e,true}$ are, respectively, the mean and standard deviation of the true failure stress of elements.

The Bayesian updating estimates the mean and standard deviation of the distribution. In the following we discuss the prior distribution used for the mean and standard deviation.

(1) Estimating the prior distribution of element mean failure stress

In the failure theory, there exists a relation between one-dimensional failure stress (e.g., $\tau_{c,true}$ from coupon tests) and failure stress due to multi-dimensional stress state (e.g., $\tau_{e,true}$ from element tests). In general, this relation can be represented using the prediction factor k_{true} as

$$\tau_{e,true} = k_{true}\tau_{c,true} \tag{8}$$

where k_{true} is a ratio between the unidirectional failure stress and the failure stress under combined loading. Since most failure theories are not perfect, the true prediction factor is unknown, and the calculated prediction factor, k_{calc}, is given. For a given failure theory and a given loading direction, k_{calc} can be uniquely determined.

From the relation in Eq. (8), it is assumed that $\mu_{e,true}$ can be calculated from $\mu_{c,true}$ as

$$\mu_{e,true} = k_{true}\mu_{c,true}. \tag{9}$$

If the failure theory is perfect (k_{true}) and the number of coupon tests is infinite ($\mu_{c,true}$), then the true mean failure stress of element ($\mu_{e,true}$) can be obtained, but this is impractical. Two errors are involved in predicting $\mu_{e,true}$; the

first due to the finite number of specimens and the other due to imperfect failure theory. The possible true mean failure stress of structural element $\mu_{e,Ptrue}$ can be obtained by Eq. (10) in terms of two possible true variables.

$$\mu_{e,Ptrue} = k_{Ptrue}\mu_{c,Ptrue} \tag{10}$$

where k_{Ptrue}, due to the imperfect failure theory, is defined in terms of the error in prediction e_k as

$$k_{Ptrue} = (1-e_k)k_{calc} \tag{11}$$

Although one true value of e_k exists, the analyst does not know the true value The error e_k is assumed to be uniformly distributed with bounds $\pm b_k$ reflecting the estimated accuracy of the failure theory. The minus sign in Eq. (11) is selected in order to make a positive error conservative. Readers are referred to An *et al.* [11] for additional discussion of the error bounds.

It was assumed that $\mu_{c,Ptrue}$ is calculated using the normal distribution with $\mu_{c,test}$ and $\sigma_{c,test}$ in Eq. (3). That is, although the error e_k and the true mean coupon failure stress have each a single true value, the distributions of possible true variables reflect the uncertainty associated with our lack of knowledge of what these values are (so-called epistemic uncertainty).

The process of obtaining the distribution of possible mean failure stress of structural element $\mu_{e,Ptrue}$ based on Eq. (10) is shown in Fig. 3. It is assumed that the distribution of $f_{k,Ptrue}(k_{Ptrue})$ is obtained using the error bounds b_k. The possible true distribution $f_{\mu c,Ptrue}(\mu_{c,Ptrue})$ is the normal distribution defined by Eq. (3). Using the two possible distributions, the distribution of possible true element mean failure stress $f_{\mu e,Ptrue}(\mu_{e,Ptrue})$ is obtained.
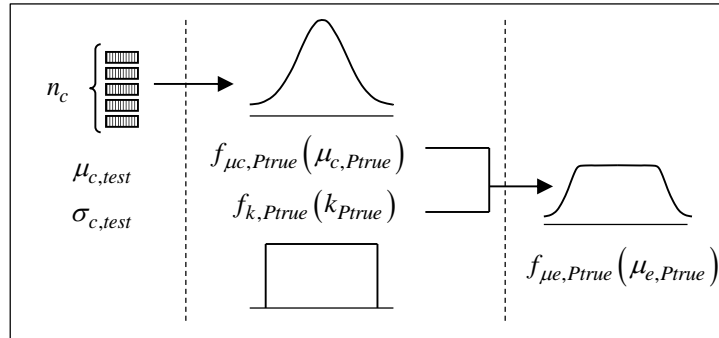

Figure. 2: Process of estimating initial element mean failure stress

The combined possible true distribution $f_{\mu e,Ptrue}(\mu_{e,Ptrue})$ is calculated with Eq. (12).

$$f_{\mu e,Ptrue}(\mu_{e,Ptrue}) = \int_{-\infty}^{\infty} f_{\mu e,Ptrue}(\mu_{e,Ptrue}\mid\mu_{c,Ptrue}) f_{\mu c,Ptrue}(\mu_{c,Ptrue})\,d\mu_{c,Ptrue} . \tag{12}$$

In the following, the two PDFs in the integrand will be explained.

First, in the case of $k_{calc}=1$ for simplicity, it was assumed that the possible true distribution $f_{k,Ptrue}(k_{Ptrue})$ is a uniform distribution with bounds b_k.

$$f_{k,Ptrue}(k_{Ptrue}) = \begin{cases} \dfrac{1}{2b_k} & \text{if } |k_{Ptrue}-1| \le b_k \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

By using Eq. (10), the possible true PDF $f_{\mu e,Ptrue}(\mu_{e,Ptrue})$ can be obtained from all possible combinations of random variables generated from $f_{k,Ptrue}(k_{Ptrue})$ and random variables generated from $f_{\mu c,Ptrue}(\mu_{c,Ptrue})$. It is reasonable to consider a case that a $\mu_{c,Ptrue}$ is given from the possible true distribution of coupon mean. In the case of the one given $\mu_{c,Ptrue}$, the possible true element failure stress which is calculated from the $\mu_{c,Ptrue}$ can be regarded as a conditional PDF $f_{\mu e,Ptrue}(\mu_{e,Ptrue}\mid\mu_{c,Ptrue})$ which is a uniform distribution with an error bound of b_k and mean of the uniform distribution is the same with the given $\mu_{e,Ptrue}$ because this PDF depends on the value of $\mu_{c,Ptrue}$. It represents the prediction error of a given failure theory.

5

American Institute of Aeronautics and Astronautics

$$f_{\mu e,Ptrue}\left(\mu_{e,Ptrue} \mid \mu_{c,Ptrue}\right) = \begin{cases} \dfrac{1}{2b_k \mu_{c,Ptrue}} & \text{if } \left|\dfrac{\mu_{e,Ptrue}}{\mu_{c,Ptrue}} - 1\right| \le b_k \\[2mm] 0 & \text{otherwise} \end{cases} \tag{14}$$

Simply, Eq. (14) is a possible true PDF of $\mu_{e,Ptrue}$ for a given $\mu_{c,Ptrue}$ so that the possible true PDF of element mean failure stress $f_{\mu e,Ptrue}\left(\mu_{e,Ptrue}\right)$ is calculated through considering all possible values $\mu_{c,Ptrue}$ with Eq. (14).

Second, by using Eq. (3) PDF of the possible true distribution of $\mu_{c,calc}$ is calculated from coupon test results as

$$f_{\mu c,Ptrue}\left(\mu_{c,Ptrue}\right) = N\left(\mu_{c,Ptrue} \mid \mu_{c,test}, \frac{\sigma_{c,test}}{\sqrt{n_c}}\right) \tag{15}$$

where the notation $N(x \mid a,b)$ denotes the value of the normal PDF with mean a and standard deviation b at x. Figure 3 illustrates $\mu_{e,true}$ a unique value, the estimated PDF of $\mu_{c,Ptrue}$, and the conditional PDF of $\mu_{e,Ptrue}$ for given $\mu_{c,Ptrue}$.
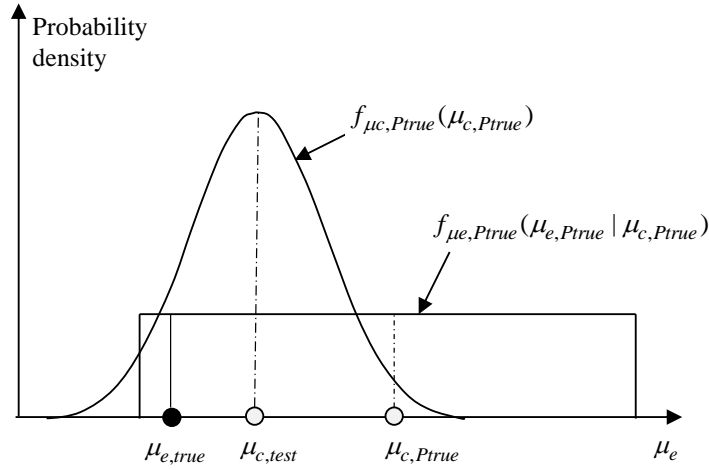

Figure. 3: The possible true distributions of mean failures stress of specimens and the conditional true distribution of the possible true distribution of element mean failure stress.

With Eqs. (14) and (15), the convolute form in Eq. (12) can be directly integrated with the finite range of PDF in Eq. (14) as

$$f_{\mu e,Ptrue}\left(\mu_{e,Ptrue}\right) = \int_{\frac{\mu_{e,Ptrue}}{(1-b_e)}}^{\frac{\mu_{e,Ptrue}}{(1+b_e)}} \frac{1}{2b_e \mu_{c,Ptrue}} N\left(\mu_{c,Ptrue} \mid \mu_{c,test}, \frac{\sigma_{c,test}}{\sqrt{n_c}}\right) d\mu_{c,Ptrue} . \tag{16}$$

Appendix C shows the detail of convolution integral. This PDF is a prior distribution of mean failure stress of structural element and will be updated using the Bayesian method.
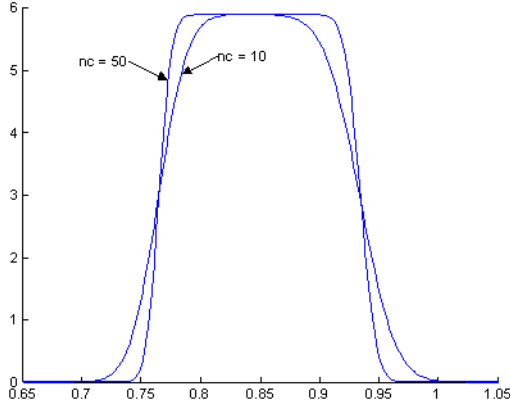
Figure 4. Illustration of distribution of element mean failure stress and its dependence on the number of coupons with 10% error bounds $\mu_{\sigma_{cf}}^{true}$=0.85, $\sigma_{\sigma_{cf}}^{true}$=0.068, and k_{2d}^{calc}=1. As the number of coupon tests increases, the distribution converges to uniform distribution because of reduced uncertainty about the true mean coupon failure stress.

(2) Predicting the prior distribution of the standard deviation of element failure stress

In predicting the mean failure stress, we considered the error in failure theory and the effect of finite number of specimens. However, in the case of standard deviation, the same standard deviation is used from the coupon test, neglecting the effect of additional scatter from the failure theory. When a random variable is normally distributed, the sampling distribution of standard deviation follows a chi-distribution [10] as shown in Eq. (4). Standard deviation of element failure stress $\sigma_{e,true}$ can be directly obtained from the $\sigma_{c,true}$ as

$$\sigma_{e,true} = (1-e_{\sigma,true})\sigma_{c,true} \tag{17}$$

where $e_{\sigma,true}$ is the true error between $\sigma_{e,true}$ and $\sigma_{c,true}$. Due to the lack of knowledge of the relation between $\sigma_{e,true}$ and $\sigma_{c,true}$ (epistemic uncertainty), and the finite number of specimens, the possible true standard deviation of element failure stress $\sigma_{e,Ptrue}$ can be obtained by Eq. (18) in terms of two possible true variables.

$$\sigma_{e,Ptrue} = (1-e_{\sigma,Ptrue})\sigma_{c,Ptrue} \tag{18}$$

In Eq. (18), $e_{\sigma,Ptrue}$ is used because of the lack of knowledge of the relation between the true element standard deviation of failure stress $\sigma_{e,true}$ and the true coupon standard deviation of failure stress $\sigma_{c,true}$ and the possible true standard deviation of failure stress $\sigma_{c,Ptrue}$ is used due to the finite number of specimens. The $e_{\sigma,Ptrue}$ is assumed to be uniformly distributed with zero mean. Again, due to the finite number of specimens, $\sigma_{c,Ptrue}$ has a form of a distribution. The distribution is estimated as in Eq. (4) which is the chi-distribution with order of n_c-1.

The process of obtaining the possible standard deviation of failure stress of structural element $\sigma_{e,true}$ based on Eq. (18) is shown in Fig. 5. It is assumed that the possible true distribution $f_{e,Ptrue}(e_{\sigma,Ptrue})$ is a uniform distribution with mean of zero and bound b_σ. Here, it was assumed that the analyst can estimate the bounds b_σ of the possible true PDF $f_{e,Ptrue}(e_{\sigma,Ptrue})$ on the magnitude of the error.

The possible true distribution of standard deviation of failure stress of structural element $f_{\sigma e,Ptrue}(\sigma_{e,Ptrue})$ is the Chi-distribution which defined by Eq. (4). Using the two possible distributions, the distribution of possible true element failure stress of standard deviation $f_{\sigma e,Ptrue}(\sigma_{e,Ptrue})$ is obtained.
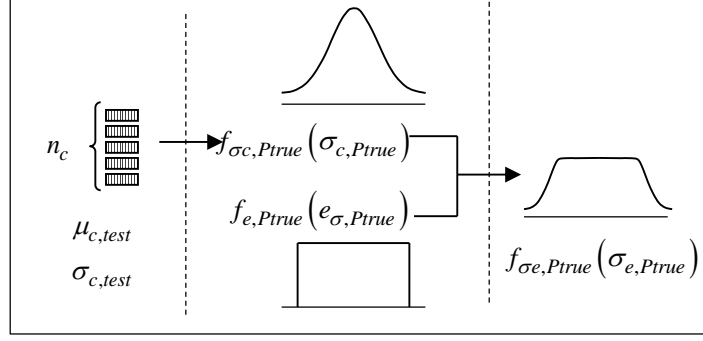
American Institute of Aeronautics and Astronautics

Figure. 5: Process of estimating standard deviation of failure stress

In the same manner as Eq. (12), the combined possible true distribution $f_{\sigma e, Ptrue}\left(\sigma_{e, Pture}\right)$ is obtained as

$$f_{\sigma e, Ptrue}\left(\sigma_{e, Pture}\right) = \int_{-\infty}^{\infty} f_{\sigma e, Ptrue}\left(\sigma_{e, Ptrue} \mid \sigma_{c, Ptrue}\right) f_{\sigma c, Ptrue}\left(\sigma_{c, Ptrue}\right) \mathrm{d}\sigma_{c, Ptrue} \tag{19}$$

In the following, the two PDFs in the integrand will be explained as it was done for estimating the mean value.

For given $\sigma_{c, Ptrue}$, $\sigma_{e, Ptrue}$ has a uniform distribution which is centered around $\sigma_{c, Ptrue}$. The possible true distribution of $\sigma_{e, Ptrue}$ for given $\sigma_{c, Ptrue}$ is defined as a conditional PDF:

$$f_{\sigma e, Ptrue}\left(\sigma_{e, Ptrue} \mid \sigma_{c, Pcalc}\right) = \begin{cases} \dfrac{1}{2b_{\sigma}\sigma_{c, Ptrue}} & \text{if } \left|\dfrac{\sigma_{e, Ptrue}}{\sigma_{c, Ptrue}} - 1\right| \le b_{\sigma} \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

However $\sigma_{c, Ptrue}$ is varying because of the finite number of specimens. Chi-distribution is used to define PDF of $\sigma_{c, Ptrue}$ for the given number of coupon tests n_c and the test standard deviation of coupon test $\sigma_{c, test}$. The PDF of chi-distribution is defined as

$$f_{\chi}\left(\chi \mid n_c - 1\right) = \frac{2^{1-(n_c-1)/2}\,\chi^{n_c-2}e^{-\chi^2/2}}{\Gamma\left((n_c - 1)/2\right)} \tag{21}$$

where the notation $f_{\chi}\left(\chi \mid n_c - 1\right)$ denotes the value of chi PDF with χ and the number of coupon tests n_c. From Eq. (21), the PDF of $\sigma_{c, Ptrue}$ for the given number of specimens n_c is obtained as

$$f_{\sigma c, Ptrue}\left(\sigma_{c, Ptrue}\right) = f_{\chi}\left(\chi \mid n_c - 1\right)\frac{\sqrt{n_c - 1}}{\sigma_{c, test}} \text{ and } \chi = \frac{\sqrt{n_c - 1}}{\sigma_{c, test}}\sigma_{c, Ptrue} \tag{22}$$

In Eq. (19), $f_{\sigma e, Ptrue}\left(\sigma_{e, Ptrue} \mid \sigma_{c, Ptrue}\right)$ has finite integrand range so that Eq. (19) can be rewritten as

$$f_{\sigma e, Ptrue}\left(\sigma_{e, Ptrue}\right) = \int_{\frac{\sigma_{e, Ptrue}}{(1-b_{\sigma})}}^{\frac{\sigma_{e, Ptrue}}{(1+b_{\sigma})}} \frac{1}{2b_{\sigma}\sigma_{c, Ptrue}} f_{\sigma c, Ptrue}\left(\sigma_{c, Ptrue}\right) \mathrm{d}\sigma_{c, Ptrue} . \tag{23}$$

This PDF is an initial distribution of standard deviation of element failure stress.

C. Bayesian Updating Approach

Element tests are used to update the prior distribution. Bayesian updating is used to reduce the uncertainty from failure theory and finite number of coupon tests. The updated joint distribution of the mean and standard deviation of the element failure stress is given as

$$f^{upd}\left(\mu_{e, Ptrue}, \sigma_{e, Ptrue}\right) = \frac{f_{1, test}\left(\mu_{e, Ptrue}, \sigma_{e, Ptrue}\right) f^{init}\left(\mu_{e, Ptrue}, \sigma_{e, Ptrue}\right)}{\int_{-\infty}^{\infty} f_{1, test}\left(\mu_{e, Ptrue}, \sigma_{e, Ptrue}\right) f^{init}\left(\mu_{e, Ptrue}, \sigma_{e, Ptrue}\right) d\mu_{e, Ptrue} d\sigma_{e, Ptrue}} \tag{24}$$

where $f^{init}(\mu_{e, Ptrue}, \sigma_{e, Ptrue})$ is the prior PDF of the element parameters. Because the prior of the mean and the prior of the standard deviation were obtained independently, the prior of the joint distribution is the product of the individual distributions

$$f^{init}\left(\mu_{e, Ptrue}, \sigma_{e, Ptrue}\right) = f_{\mu e, Ptrue}\left(\mu_{e, Ptrue}\right) \cdot f_{\sigma e, Ptrue}\left(\sigma_{e, Ptrue}\right) \tag{25}$$

American Institute of Aeronautics and Astronautics

in Eq. (24) $f_{1,test}\left(\mu_{e,Ptrue},\sigma_{e,Ptrue}\right)$ is the likelihood of obtaining the first element test results for given $\mu_{e,Ptrue},\sigma_{e,Ptrue}$. It reflects the assumption that the element failure stress is normally distributed, so that the probability of obtaining the first test result $\tau_{e,test}^{1}$, is

$$f_{1,test}\left(\mu_{e,Ptrue},\sigma_{e,Ptrue}\right)=N\left(\tau_{e,test}^{1};\mu_{e,Ptrue},\sigma_{e,Ptrue}\right) \tag{26}$$

Note that it is not a probability distribution but conditional probability distribution. Subsequent tests are handled by the same equation with the updated distribution as the initial one.

III.   Illustrative Examples

In these examples, the effects of the number of coupon tests and the number of element tests on the remaining uncertainty will be illustrated. True distributions of coupon test and element test are assumed to be normal. Structural test results are generated by sampling the true distributions. 201×201 grid is used to calculate a PDF of the possible true mean and standard deviation of coupon tests with regard to the number of coupon tests. In this study, a joint PDF of the possible true coupon mean and standard deviation are calculated so that analysis of the joint PDF is needed.

Table 1. True distributions of structural tests

Test	Distribution	Parameters
Coupon test	Normal	$\mu=1.1$, $\sigma=0.077$ (cov=0.07)
Element test	Normal	$\mu=1.1$, $\sigma=0.099$ (cov=0.09)

Table 2. illustrates the effect of the number of coupon tests on the remaining uncertainty for a single set of samples. s.

Table 2. Analysis of the joint PDF of coupon mean and standard deviation; obtained from a single set of samples

Number of coupon tests	10	30	50	70	90
Estimated mean of $\mu_{c,Ptrue}$	1.0696	1.1053	1.1042	1.0852	1.1014
Estimated mean of $\sigma_{c,Ptrue}$	0.0974	0.0783	0.0803	0.0736	0.0763
Sample mean	1.0696	1.1053	1.1042	1.0852	1.1014
Sample standard deviation	0.1001	0.0790	0.0807	0.0738	0.0765
True mean $\mu_{c,true}$			1.1		
True standard deviation $\sigma_{c,true}$			0.077		
Estimated std. of $\mu_{c,Ptrue}$	0.0317	0.0144	0.0114	0.0088	0.0081
Estimated std. of $\sigma_{c,Ptrue}$	0.0232	0.0103	0.0081	0.0063	0.0057
Sample std. of $\mu_{c,Ptrue}$ [a]	0.0244	0.0141	0.0109	0.0092	0.0081
Sample std. of $\sigma_{c,Ptrue}$ [b]	0.0179	0.0101	0.0078	0.0065	0.0058

[a] Sample std. of $\mu_{c,Ptrue}$ is calculated using Eq. (5) replacing $\sigma_{c,test}$ with $\sigma_{c,true}$ for comparison.

[b] Sample std. of $\sigma_{c,Ptrue}$ is calculated using Eq. (6) replacing $\sigma_{c,test}$ with $\sigma_{c,true}$ for comparison, see appendix B for detail.

In Table 2, for the particular samples used, 30 samples provide more accurate estimate of the mean (estimate is equal to sample mean) than 70 samples, and more accurate estimate of the standard deviation than 50 samples. However, the estimates of the uncertainty are is substantially smaller for 70 than for 30. In addition, the estimated mean of $\mu_{c,Ptrue}$ converges to the true value of 1.1 as the number of coupon tests (n_{c}) increases. The estimated standard deviation of the $\mu_{c,Ptrue}$ and the estimated standard deviation of $\sigma_{c,Ptrue}$ converge to the true value so that it can be said that the quality of estimated uncertainty is good. When $n_{c}=10$, uncertainty in the mean of $\mu_{c,Ptrue}$ is almost three times larger than uncertainty in the mean of $\mu_{c,Ptrue}$ of $n_{c}=90$. Also uncertainty in the standard deviation

of $\mu_{c,Ptrue}$ ($n_c = 10$) is three times larger than uncertainty in the standard deviation of $\mu_{c,Ptrue}$ ($n_c = 90$); it means that large number of tests can give good estimation. The estimated uncertainty of $\mu_{c,Ptru}$ reduces as the number of specimens increases. In the same manner as the mean, estimated uncertainty of $\sigma_{c,Ptrue}$ is more than four times of the standard deviation of $\mu_{c,Ptrue}$ ($n_c = 90$). Also the estimated standard deviation of the mean $\mu_{c,Ptrue}$ and mean $\sigma_{c,Ptrue}$ are shown and the estimated standard deviations can be a good approximated uncertainty of the estimated mean $\mu_{c,Ptrue}$ and mean $\sigma_{c,Ptrue}$.

In this study, parameters of element test are estimated from the joint PDF of coupon test parameters. Due to the imperfect failure theory in estimation, the estimation has error. Each estimated element mean and element standard deviation has error, distributions of e_μ and e_σ are given below in Table 3.

Table 3. Error distributions of element tests

Error	Distribution	Bounds
$e_{\mu c}$	Uniform	±10%
e_σ	Uniform	±50%

201×201 grid is used to calculate a PDF of the possible true mean and standard deviation of element tests n_e with regard to the number of coupon tests n_c. In this study, a joint PDF of the possible true element mean and standard deviation are calculated so that analysis of the joint PDF is needed. Mean of coupon mean, mean of coupon standard deviation, standard deviation of coupon mean, and standard deviation of coupon standard deviation are shown in Table 4 for one particular set of element test results per coupon sample size. For the particular set used with 10 coupoon tests, the third element test with failure stress of 1.0927 happens to be very accurate, while the first and fifth tests are very inaccurate. . Contents are sorted by the number of coupon tests and the number of element tests. True parameters are used to generate random variables and used as a reference value for assessment purpose in Table 4.

Since the element parameters are estimated from coupon test results, the estimated parameters have less uncertainty than the sampling uncertainties of element test. In Table 4, mean of $\mu_{e,Ptrue}$ converges to the true value of 1.1 as the number of coupon tests (n_e) increases. As the number of element tests increases, the mean of $\sigma_{e,Ptrue}$ converges to the true value of 0.099 as well. It is observed that the estimated parameter with large number of coupon tests is less accurate than the estimated parameter with small number of coupon tests for the same number of element tests due to the randomness.

As a matter of course, analyst does not know the true element mean $\mu_{e,true}$ and standard deviation $\sigma_{e,true}$ so that standard deviations of estimated parameters have to be estimated as a quantified uncertainty of the estimated mean and standard deviation of element failure stress and the estimated standard deviation of the estimated mean $\mu_{e,true}$ and the estimated mean $\sigma_{e,true}$ are given in Table 4.

Table 4. Analysis of the joint PDF of coupon mean and standard deviation ($n_c =10$, 50, and 90) for a single set of element test results.

Number of coupon tests	Number of element tests	1	2	3	4	5
10	Element failure stress	0.9646	1.0606	1.0927	1.1910	1.2609
	Estimated mean $\mu_{e,Ptrue}$	1.0250	1.0327	1.0458	1.0750	1.1029
	Estimated mean $\sigma_{e,Ptrue}$	0.0935	0.0892	0.0869	0.0990	0.1112
	Estimated std. of $\mu_{e,Ptrue}$	0.0555	0.0476	0.0428	0.0433	0.0425
	Estimated std. of $\sigma_{e,Ptrue}$	0.0342	0.0314	0.0290	0.0274	0.0260
50	Element failure stress	0.9668	1.1434	1.1948	1.0713	1.2334

Estimated mean $\mu_{e,Ptrue}$	1.0476	1.0725	1.1021	1.0953	1.1200
Estimated mean $\sigma_{e,Ptrue}$	0.0828	0.0900	0.0947	0.0914	0.0956
Estimated std. of $\mu_{e,Ptrue}$	0.0470	0.0483	0.0474	0.0424	0.0402
Estimated std. of $\sigma_{e,Ptrue}$	0.0242	0.0209	0.0189	0.0191	0.0176
Element failure stress	1.0410	0.9756	1.1130	1.2001	1.2643
Estimated mean $\mu_{e,Ptrue}$	1.0680	1.0380	1.0535	1.0846	1.1171
Estimated mean $\sigma_{e,Ptrue}$	0.0736	0.0713	0.0755	0.0876	0.0956
Estimated std. of $\mu_{e,Ptrue}$	0.0504	0.0367	0.0370	0.0409	0.0403
Estimated std. of $\sigma_{e,Ptrue}$	0.0226	0.0221	0.0206	0.0174	0.0146
True mean $\mu_{e,true}$			1.1		
True std. $\sigma_{e,true}$			0.099		

(The left column value "90" spans the Element failure stress through the last Estimated std. rows.)

Since the results in Table 4 refelct the idiosyncrasies of the single samples used, we repeated the calculations 500,000 times, and the results are shown in In Table 5. RMS error and MA (mean absolute) error of the mean $\mu_{e,Ptrue}$ and mean $\sigma_{e,Ptrue}$ are shown. Coupon test and element test are needed to estimate mean and standard deviation of element failure stress. The errors reduce as the number of coupon test and the number of element tests increase. However, the effect of the number of element tests is more noticeable. This reflects the fact that even after 10 coupon tests, the error due to inaccurate mean coupon failure stress is much smaller than the error in the failure theory that is addressed by element tests. The standard deviation of the mean of the coupon failure stress is seen in Table 3 to reduce from 0.024 to 0.008 by going from 10 coupons to 90. The element failure calculation is uniform with 10% bounds, which corresponds to a standard deviation of 0.058, and so it dominates the total error, and it can be reduced only by element tests. Consequently, for this case,

With large MCS, the RMS error is a very close value to the true standard deviation of estimated mean $\mu_{e,Ptrue}$ and standard deviation of $\sigma_{e,Ptrue}$. It is observed that the estimated standard deviations in Table 4. give approximate values of the RMS errors, comparison between the estimated standard deviations in Table 4 and RMS errors from 500,000 MCS is shown in Table 5. The estimated standard deviation of the mean $\mu_{e,Ptrue}$ gives conservative estimation of RMS error, the estimated standard deviation of the mean $\sigma_{e,Ptrue}$ gives slightly un-conservative estimation of RMS error from time to time.

Table 5. Comparison between estimated uncertainties for a single set of element test results and measured errors from 500,000 MCS (n_c =10, 50, and 90)

Number of coupon test	Number of element tests	0	1	2	3	4	5
10	Estimated std. of $\mu_{e,Ptrue}$	-	0.0555	0.0476	0.0428	0.0433	0.0425
	Estimated std. of $\sigma_{e,Ptrue}$	-	0.0342	0.0314	0.0290	0.0274	0.0260
	RMS error of mean $\mu_{e,Ptrue}$ [a]	0.0724	0.0480	0.0455	0.0420	0.0391	0.0366
	RMS error of mean $\sigma_{e,Ptrue}$ [b]	0.0419	0.0316	0.0294	0.0272	0.0252	0.0235
	MA error of mean $\mu_{e,Ptrue}$ [c]	-	0.0400	0.0373	0.0342	0.0317	0.0297
	MA error of mean $\sigma_{e,Ptrue}$ [d]	-	0.0277	0.0253	0.0228	0.0208	0.0193
50	Estimated std. of $\mu_{e,Ptrue}$	-	0.0470	0.0483	0.0474	0.0424	0.0402
	Estimated std. of $\sigma_{e,Ptrue}$	-	0.0242	0.0209	0.0189	0.0191	0.0176

	1	2	3	4	5
RMS error of mean $\mu_{e,Ptrue}$ [a]	0.0653	0.0417	0.0426	0.0409	0.0388	0.0369
RMS error of mean $\sigma_{e,Ptrue}$ [b]	0.0336	0.0246	0.0237	0.0224	0.0210	0.0198
MA error of mean $\mu_{e,Ptrue}$ [c]	-	0.0359	0.0358	0.0340	0.0321	0.0303
MA error of mean $\sigma_{e,Ptrue}$ [d]	-	0.0231	0.0212	0.0192	0.0174	0.0160
90 Estimated std. of $\mu_{e,Ptrue}$	-	0.0504	0.0367	0.0370	0.0409	0.0403
Estimated std. of $\sigma_{e,Ptrue}$	-	0.0226	0.0221	0.0206	0.0174	0.0146
RMS error of mean $\mu_{e,Ptrue}$ [a]	0.0646	0.0409	0.0423	0.0408	0.0388	0.0369
RMS error of mean $\sigma_{e,Ptrue}$ [b]	0.0326	0.0238	0.0230	0.0217	0.0204	0.0192
MA error of mean $\mu_{e,Ptrue}$ [c]	-	0.0354	0.0358	0.0340	0.0322	0.0304
MA error of mean $\sigma_{e,Ptrue}$ [d]	-	0.0227	0.0207	0.0186	0.0169	0.0154

[a] Root mean square error of the mean $\mu_{e,Ptrue}$ using Eq. (27), the RMS error with large number of samples can be used as a true standard deviation of $\mu_{e,Ptrue}$, it is a target value of the estimated std. of the mean $\mu_{e,Ptrue}$. Results are based on 500,000 MCS.

[b] Root mean square error of the mean $\sigma_{e,Ptrue}$ using Eq. (28), the RMS error with large number of samples can be used as a true standard deviation of $\sigma_{e,Ptrue}$, it is a target value of the estimated std. of the mean $\sigma_{e,Ptrue}$. Results are based on 500,000 MCS.

[c] Mean of absolute error sum of the mean $\mu_{e,Ptrue}$ using Eq. (29). This value can be used to determine how biased the estimated mean $\mu_{e,Ptrue}$ is. Results are based on 500,000 MCS.

[d] Mean of absolute error sum of the mean $\sigma_{e,Ptrue}$ using Eq. (30). This value can be used to determine how biased the estimated mean $\sigma_{e,Ptrue}$ is. Results are based on 500,000 MCS.

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\mu_{e,Ptrue}^{i}-\mu_{e,true}\right)^{2}} \tag{27}$$

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\sigma_{e,Ptrue}^{i}-\sigma_{e,true}\right)^{2}} \tag{28}$$

$$\frac{1}{N}\sum_{i=1}^{N}\left|\mu_{e,Ptrue}^{i}-\mu_{e,true}\right| \tag{29}$$

$$\frac{1}{N}\sum_{i=1}^{N}\left|\sigma_{e,Ptrue}^{i}-\sigma_{e,true}\right| \tag{30}$$

Figure 6. shows that ME error and RMS error associated with the number of element tests (1, 2, 3, 4, and 5) and the number of coupon tests (10, 50 and 90). It is observed that effect of the number of element is sufficient for reducing uncertainty in both the estimated mean $\mu_{e,Ptrue}$ and standard deviation $\sigma_{e,Ptrue}$. Increasing the number of the coupon tests is an effective way to reduce uncertainty in the estimated mean $\sigma_{e,Ptrue}$. However increasing the number of the coupon tests is less effective way to reduce uncertainty in the estimated mean $\mu_{e,Ptrue}$ is substantial with 1 or 2 element tests but the effect becomes a little with more than 2 element tests.
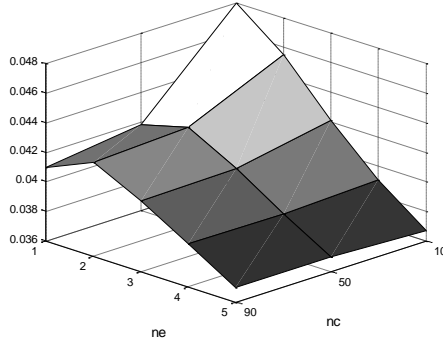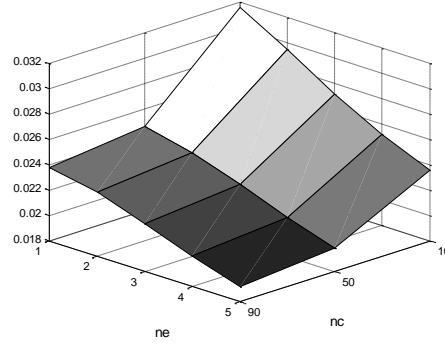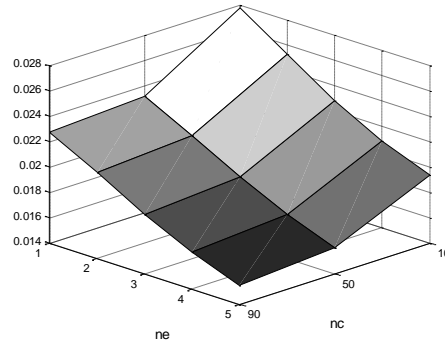
(c) RMS error of mean $\mu_{e,Ptrue}$ (d) RMS error of mean $\sigma_{e,Ptrue}$
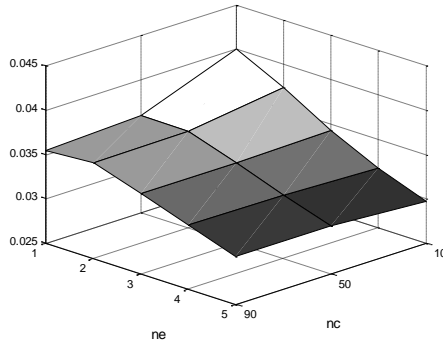
(a) MA error of mean $\mu_{e,Ptrue}$ (b) MA error of mean $\sigma_{e,Ptrue}$

Figure 6: Calculated errors with 500,000 MCS
(ne is the number of element tests, and nc is the number of coupon tests)

IV. Concluding remarks

In aircraft design, some selected parameters are based on our engineering judgment rather than published data. We have been trying to quantify the effects of structural tests which has been done. The effects of aircraft structural tests on aircraft structural safety particularly the effects of the number of coupon tests and the number of structural element tests are explored. In this study, following conclusions can be drawn.

As the number of coupon tests is increased, initial prediction of mean failure stress of structural element becomes accurate and it provides a good prior. Also the estimated uncertainties can be given and the estimated uncertainties become accurate as the number of coupon tests increases.

Error in the estimation of mean and standard deviation of element failure stress can be substantially reduced with element tests. Effect of the number of element tests is substantial for reducing error in prediction of mean failure stress of structural elements.

In this paper, accuracies of the estimated mean and standard deviation failure stress of structural tests are estimated through estimating the standard deviation of the estimated element parameters.

APPENDIX A: Estimating Failure Stress of Coupon Test

A. Estimating failure stress of coupon test using Bayesian updating

A maximum likelihood distribution of coupon failure stress is used to estimate a distribution of coupon failure stress.

$$N\left(\mu_{c,Maxl}, \sigma_{c,Maxl}\right) \tag{A1}$$

where $\mu_{c,Maxl}$ and $\sigma_{c,Maxl}$ are a maximum likelihood mean of coupon failure stress and a maximum likelihood standard deviation of coupon failure stress, respectively. The maximum likelihood parameters is calculated using the possible true distribution of mean and standard deviation of coupon failures stress which obtained by Bayesian updating as

13
American Institute of Aeronautics and Astronautics

$$f_{c,Ptrue}\left(\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)=\frac{f_{1,test}\left(\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)f^{init}\left(\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)}{\int_{-\infty}^{\infty}f_{1,test}\left(\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)f^{init}\left(\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)d\mu_{c,Ptrue}d\sigma_{c,Ptrue}} \tag{A2}$$

Initial distribution f^{init} is assumed a uniform distribution; the range is wide enough because no information was given before the tests. Likelihood function of ith test is defined as Eq. (A3).

$$f_{i,test}\left(\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)=N\left(\tau_{c,test}^{i}\mid\mu_{c,Ptrue},\sigma_{c,Ptrue}\right) \tag{A3}$$

Here $f_{1,test}(\mu_{c,Ptrue},\sigma_{c,Ptrue})$ is a function reflecting possible variability of the first test of $\tau_{c,test}^{1}$. Note that it is not a probability distribution but conditional probability distribution. Subsequent tests are handled by the same equation with the updated distribution as the initial one. After updating as the number of coupon tests, the final joint PDF $f_{c,Ptrue}(\mu_{c,Ptrue},\sigma_{c,Ptrue})$ is obtained and then maximum likelihood mean $\mu_{c,Maxl}$ and standard deviation $\sigma_{c,Maxl}$ are obtained from the joint PDF.

The maximum likelihood distribution of coupon failure stress is most likely estimation of failure stress but it is one possibility and the other distribution might be true so that consideration of possible true distributions is needed. The possible true distribution of the coupon failure stress $f_{\tau_{c,Ptrue}}(\tau_{c,Ptrue})$ is a distribution of all possible true failure stress and the possible true distribution is defined as

$$f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\mid\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)$$
$$f_{c,Ptrue}\left(\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)d\mu_{c,Ptrue}d\sigma_{c,Ptrue} \tag{A4}$$

It was already assumed that the true distribution of failure stress have a normal distribution, one expected failure stress distribution can be constructed using a normal distribution with a pair of $\mu_{c,Ptrue}$ and $\sigma_{c,Ptrue}$ come from $f_{c,Ptrue}(\mu_{c,Ptrue},\sigma_{c,Ptrue})$. The distribution of the one expected failure stress distribution for given $\mu_{c,Ptrue}$ and $\sigma_{c,Ptrue}$ is defined as a conditional PDF:

$$f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\mid\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)=N\left(\tau_{c,Ptrue}\mid\mu_{c,Ptrue},\sigma_{c,Ptrue}\right) \tag{A5}$$

The possible true distribution of the coupon failure stress $f_{\tau_{c,Ptrue}}(\tau_{c,Ptrue})$ in Eq. (A4) can be calculated using numerical integration. However, the integration range in Eq. (A4) is infinite so that wide enough integration range ($>5\sigma$ of each parameters) was used instead of the infinite range. Gaussian integration was used for the numerical integration.

B. Estimating failure stress of coupon test using sample data

An approximation was used to simply estimate a distribution of coupon failure stress.

$$N\left(\mu_{c,test},\sigma_{c,test}\right) \tag{A6}$$

where $\mu_{c,test}$ is a test mean and $\sigma_{c,test}$ is a test standard deviation. It is the best choice for given sample however this estimation is one of possibility so that consideration of possible true distributions is needed. The possible true distribution of the coupon failure stress $f_{\tau_{c,Ptrue}}(\tau_{c,Ptrue})$ is a distribution of all possible true failure stress and the possible true distribution and defined as

$$f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\mid\mu_{c,Ptrue},\sigma_{c,Ptrue}\right)$$
$$f_{\mu_{c,Ptrue}}\left(\mu_{c,Ptrue}\right)f_{\sigma_{c,Ptrue}}\left(\sigma_{c,Ptrue}\right)d\mu_{c,Ptrue}d\sigma_{c,Ptrue} \tag{A7}$$

It was already assumed that true distribution of failure stress is a normal distribution so that Eq. (A5) is also used for the conditional PDF in Eq. (A7). Due the true distribution is a normal distribution, a sample mean of coupon failures stress is according to a normal distribution and a sample standard deviation is according to chi-distribution. The possible true distribution of sample mean is defined by

$$f_{\mu_{c,Ptrue}}\left(\mu_{c,Ptrue}\right)=N\left(\mu_{c,Ptrue}\mid\mu_{c,test},\frac{\sigma_{c,test}}{\sqrt{n_{c}}}\right) \tag{A8}$$

The possible true distribution of standard deviation of coupon failures stress is defined using chi-distribution.

$$f_{\sigma_{c,Ptrue}}\left(\sigma_{c,Ptrue}\right) = \chi\left(x \mid n_c - 1\right)\frac{\sqrt{n_c - 1}}{\sigma_{c,test}} \text{ and } x = \frac{\sqrt{n_c - 1}}{\sigma_{c,test}}\sigma_{c,Ptrue} \tag{A9}$$

where $\chi(n_c - 1)$ is chi distribution with $n_c - 1$ degrees of freedom.

C. Numerical example

Comparisons of two approaches will be given in this section. As assumed, 90 coupon test results are generated from a normal distribution with mean of 1.1 and standard deviation of 0.077. The estimated failures stress distribution using two approaches will be shown in one graph with true distribution. Eq. (A1) is used for Bayesian approach, and Eq. (A6) is used for sampling approach to estimate failure stress of coupon test, the estimated distributions are shown when the number of coupon tests (n_c) is 10, 30, 50, 70 and 90. Samples are sequentially chosen, for $n_c = 10$ first 10 samples are used. In Fig. A1, blue broken line is for the Bayesian approach, blue solid line is for the sampling approach, and red solid line is for the true distribution of failure stress.
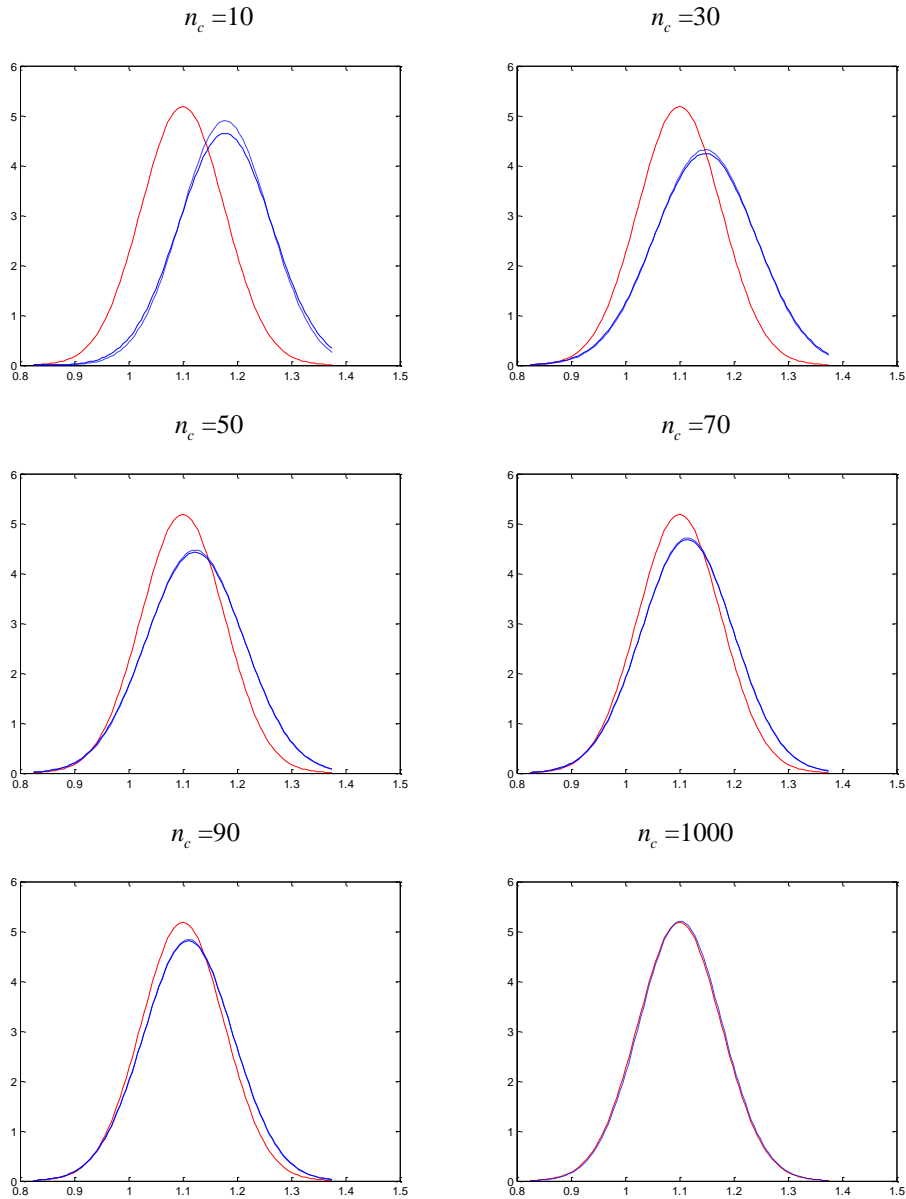
$$n_c = 10 \qquad\qquad n_c = 30$$

$$n_c = 50 \qquad\qquad n_c = 70$$

$$n_c = 90 \qquad\qquad n_c = 1000$$

Table A1. Estimated mean and standard deviation using Bayesian approach

	10	30	50	70	90
Maxl $\mu_{c,Ptrue}$ [a]	1.1780	1.1472	1.1226	1.1146	1.1102
Maxl $\sigma_{c,Ptrue}$ [b]	0.0814	0.0924	0.0892	0.0846	0.0824
Mean $\mu_{c,Ptrue}$ [c]	1.1780	1.1472	1.1225	1.1146	1.1103
Mean $\sigma_{c,Ptrue}$ [d]	0.1006	0.0983	0.0925	0.0868	0.0841
Std. $\mu_{c,Ptrue}$ [e]	0.0331	0.0181	0.0132	0.0104	0.0089
Std. $\sigma_{c,Ptrue}$ [f]	0.0288	0.0137	0.0097	0.0076	0.0064

[a] Maximum likelihood of possible true mean

[b] Maximum likelihood of possible true standard deviation

[c] Mean of possible true mean

[d] Mean of possible true standard deviation

[e] Standard deviation of possible true mean

[f] Standard deviation of possible true standard deviation

**Table A2.** Estimated mean and standard deviation of failures stress using sampling approach

	10	30	50	70	90
$\mu_{c,test}$	1.1780	1.1472	1.1225	1.1146	1.1103
$\sigma_{c,test}$	0.0859	0.0940	0.0901	0.0852	0.0829
Std. $\mu_{c,test}$ [a]	0.0271	0.0172	0.0127	0.0102	0.0087
Std. $\sigma_{c,test}$ [b]	0.0199	0.0123	0.0091	0.0072	0.0062

[a] standard deviation of sample mean (standard deviation of Eq. (8) $\bar{\tau} / \sqrt{n_c}$)

[b] standard deviation of sample standard deviation (standard deviation of Eq. (9))

Table A1. and Table A2. show that there are little differences between the maximum likelihood mean and standard deviation in Table A1. and sample mean $\mu_{c,test}$ and sample standard deviation $\sigma_{c,test}$ in Table A2. So that it is interpreted that performances of estimating parameters are almost the same for both approaches. In comparison graph, little differences are observed between them and they are very close to the true distribution when the number of coupons is 1000.

Comparisons of two approaches will be given in Figure A2. The possible true distribution of failures stress using two approaches will be shown in one graph with true distribution. Eq. (A4) is used for Bayesian approach, and Eq. (A7) is used for sampling approach to estimate failure stress of coupon test. Gaussian integration was used with wide enough integration range.

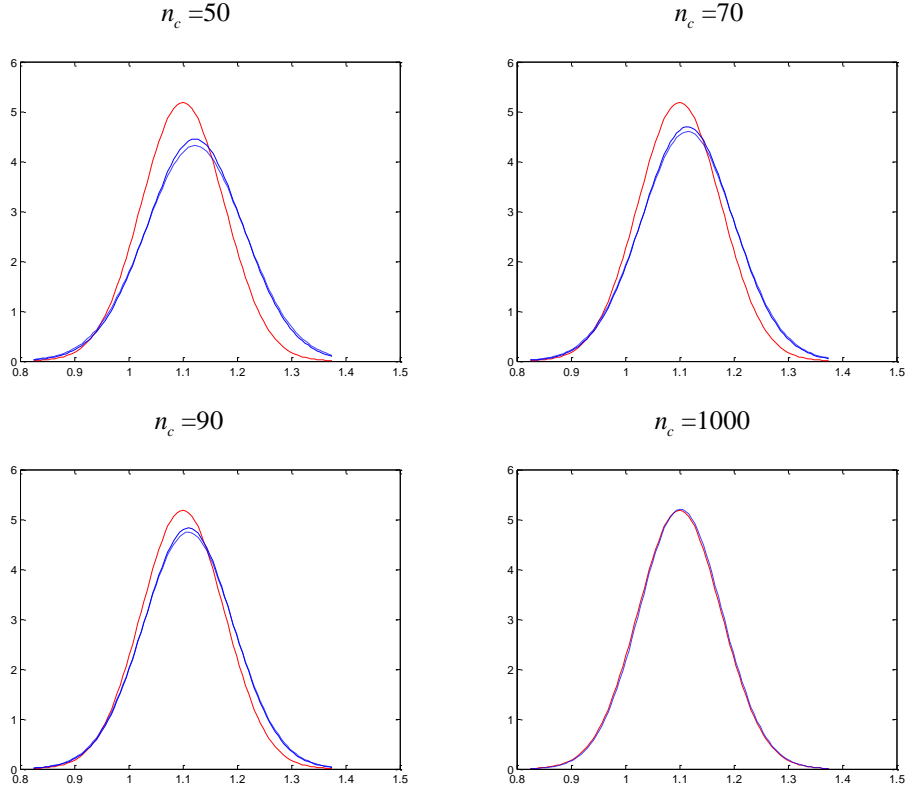$n_c = 10$　　　　　　　　　　　　　　$n_c = 30$
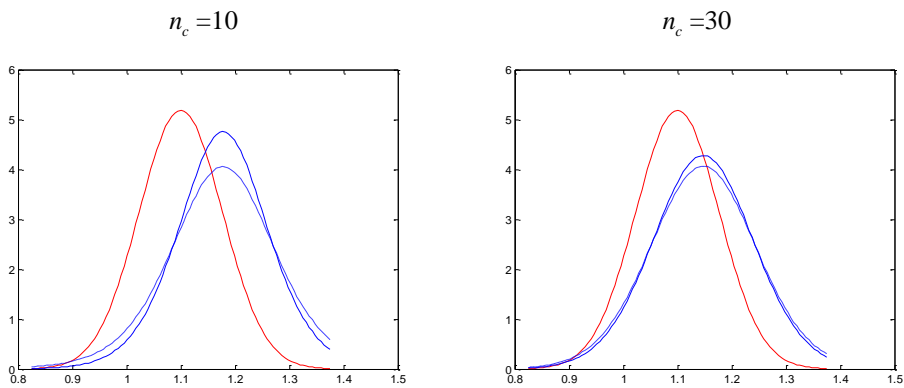
Figure. A2. Comparison between possible true distribution of failure stress (red line),
possible true distribution of failure stress from Bayesian approach (blue broken line),
and possible true distribution of failure stress from sampling approach (blue line).

In comparison graph, little differences are observed between them and they are very close to the true distribution when the number of coupons is 1000. The possible true distributions of failure stress from both approaches are almost the same distribution.

APPENDIX B: Chi-distribution

Let x_i be a random sample from $N(\mu,\sigma)$ where both μ and σ are true mean and standard deviation. Define two statistics sample mean and sample standard deviation as

Sample mean:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{B1}$$

Sample standard deviation:

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} \tag{B2}$$

From [9],

$$\frac{n-1}{\sigma^2}s^2 = \chi_{n-1}^2 \tag{B3}$$

In the same manner [10],

$$\frac{\sqrt{n-1}}{\sigma}s = \chi_{n-1} \tag{B4}$$

where χ_{n-1} follows the chi-distribution with n-1 degrees of freedom. The probability density function of chi is defined [11]

$$f_\chi(\chi_{n-1} \mid n-1) = \frac{2^{1-(n-1)/2} \chi_{n-1}^{n-2} e^{-\chi_{n-1}^2/2}}{\Gamma(n-1/2)}$$ (B5)

where Γ is the Gamma function. Mean of the chi-distribution is

$$\mu_\chi = \sqrt{2}\,\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}$$ (B6)

Standard deviation of the chi-distribution is σ_χ.

$$\sigma_\chi = \sqrt{(n-1) - \mu_\chi^2} = \sqrt{2\left(\frac{n-1}{2} - \frac{\Gamma^2(n/2)}{\Gamma^2((n-1)/2)}\right)}.$$ (B7)

The chi-distribution is a function of χ_{n-1}, if the function is changed as a PDF $f_s(s \mid n-1)$ with respect to s,

$$f_s(s \mid n-1) = \frac{2^{1-(n-1)/2}\left(\dfrac{\sqrt{n-1}}{\sigma}\right)^{n-1} s^{n-2} e^{-\left(\frac{\sqrt{n-1}}{\sigma}s\right)^2/2}}{\Gamma(n-1/2)}$$ (B8)

Mean of the $f_s(s \mid n-1)$ is

$$\mu_s = \frac{\sqrt{2}\sigma}{\sqrt{n-1}}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}$$ (B9)

Standard deviation of the sample standard deviation is σ_s.

$$\sigma_s = \sqrt{\sigma^2 - 2\frac{\sigma^2}{(n-1)}\frac{\Gamma^2(n/2)}{\Gamma^2((n-1)/2)}}$$ (B10)

Eq. (B10) is uncertainty of sample standard deviation with regard to the number of samples.

APPENDIX C: Using Bayesian network instead of MCS

Bayesian network can be used to obtain a convolution function of PDFs. Here is a distribution of distribution; a normal distribution has two parameters which are random variables. Simply MCS is used to obtain a PDF of the distribution of distribution. However obtaining a PDF through MCS is not desirable because of calculation for curve fitting after MCS and uncertainty of MCS itself so that the PDF though MCS cause undesirable computation cost for curve fitting and uncertainty from MCS. Here, Bayesian network is used as an alternative way of MCS.

For the distribution of distribution, two random variable's distributions and final distribution which uses two random variables as parameters are known; the Bayesian network can be used to calculate the PDF of the distribution of distribution. Fig. C1 shows structure of the distribution of distribution.
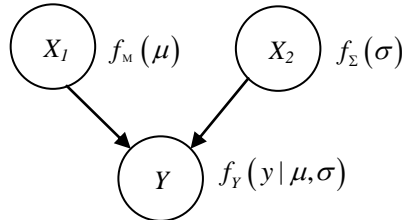

Figure C1. Bayesian network example: Distribution of distribution

A Bayesian network containing continuous random variables and the Bayesian network is given as Fig. C1, PDF of y is defined as

$$f_Y(y) = \int_\mu \int_\sigma f_Y(y \mid \mu, \sigma) f_M(\mu) f_\Sigma(\sigma)\, d\sigma d\mu$$ (C1)

The possible true distribution with sampling in Appendix A is exactly the same Bayesian network with Fig. C1. The possible true distribution is defined as

$$f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\mid\mu_{c,Pture},\sigma_{c,Pture}\right)$$
$$f_{\mu_{c,Ptrue}}\left(\mu_{c,Ptrue}\right)f_{\sigma_{c,Ptrue}}\left(\sigma_{c,Ptrue}\right)\mathrm{d}\mu_{c,Ptrue}\mathrm{d}\sigma_{c,Ptrue} \tag{C2}$$

Conditional PDF of calculated failure stress of coupon test is

$$f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\mid\mu_{c,Pture},\sigma_{c,Pure}\right)=N\left(\tau_{c,Ptrue}\mid\mu_{c,Pture},\sigma_{c,Pture}\right) \tag{C3}$$

Eq. (C3) is a conditional probability density function for given mean and standard deviation. Distribution of the sampling mean is given as

$$f_{\mu_{c,Ptrue}}\left(\mu_{c,Ptrue}\right)=N\left(\mu_{c,Ptrue}\mid\mu_{c,test},\frac{\sigma_{c,test}}{\sqrt{n_c}}\right) \tag{C4}$$

Distribution of the calculated standard deviation is defined as

$$f_{\sigma_{c,Ptrue}}\left(\sigma_{c,Ptrue}\right)=\chi\left(x\mid n_c-1\right)\frac{\sqrt{n_c-1}}{\sigma_{c,test}}\text{ and }x=\frac{\sqrt{n_c-1}}{\sigma_{c,test}}\sigma_{c,Ptrue} \tag{C5}$$

Distribution of the calculated failure stress can be rewritten as

$$f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\right)=\int_{\mu_{c,Ptrue}}\int_{\sigma_{c,Ptrue}}N\left(\tau_{c,Ptrue}\mid\mu_{c,Pture},\sigma_{c,Pture}\right)N\left(\mu_{c,Ptrue}\mid\mu_{c,test},\frac{\sigma_{c,test}}{\sqrt{n_c}}\right)$$
$$\chi\left(\frac{\sqrt{n_c-1}}{\sigma_{c,test}}\sigma_{c,Ptrue}\mid n_c-1\right)\frac{\sqrt{n_c-1}}{\sigma_{c,test}}\mathrm{d}\mu_{c,Ptrue}\mathrm{d}\sigma_{c,Ptrue} \tag{C6}$$

Here is a simple example. A sample mean of coupon test $\mu_{c,test}=1.1780$, and a sample standard deviation of coupon test $\sigma_{c,test}=0.0858$. The number of coupon tests $n_c=10$. With the given values, PDF $f_{\tau_{c,Ptrue}}\left(\tau_{c,Ptrue}\right)$ can be simply calculated using Gaussian integration and it is compared with MCS of 10^7 samples. For the MCS, 1000 random variables are generated for each sample mean and sample standard deviation and generated 10000 random variables from 1000 normal distributions which were constructed with the generated 1000 means and standard deviations.
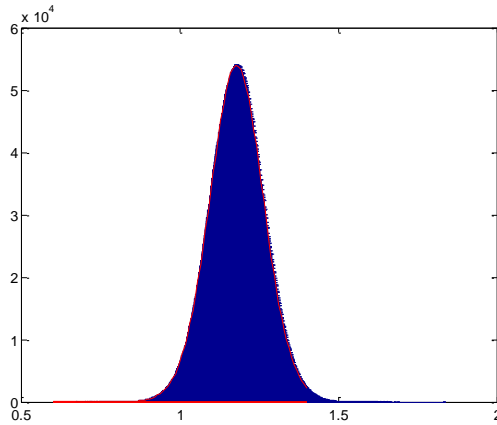

Figure C1. Comparison between the estimated failure stress using Bayesian network (red solid line), and the estimated failure stress using MCS (blue shaded area)

References

[1] Dhillon, B.S., Pashandi A.R.M., and Liu, K.L., (2002) "Robot systems reliability and safety: a review," *Journal of Quality Maintenance Engineering,* 8 (3): 170-212

[2] Kale, A.A., Haftka, R.T., and Sankar, B.V., (2008) "Efficient Reliability Based Design and Inspection of Stiffened Panels against Fatigue," *Journal of Aircraft*, **45** (1): 86-97

[3] Y. Garbatov and C. Guedes Soares (2001), Cost and reliability based strategies for fatigue maintenance planning of floating structures, Reliability Engineering and System Safety 73:294-301.

[4] S. S. Kulkarni and J. D. Achenbach (2008) Structural health monitoring and damage prognosis in fatigue, Structural Health Monitoring 7: 37-49.

[5] M. E. Johnson, G. L. Tietjen, and R. J. Beckman (1980) A New Family of Probability Distributions With Applications to Monte Carlo Studies, Journal of the American Statistical Association, Vol. 75, No. 370. 276-279

[6] M. McDonald and S. Mahadevan, Uncertainty Quantification and Propagation in Multidisciplinary Analysis and Optimization, AIAA-2008-6038, 12th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, 2008

[7] P. M. Stanfield, J. Wilson, G. A. Mirka, Multivariate input modeling with Johnson distributions, Proceedings of the 1996 Winter Simulation Conference, 1996

[8] E.J. Dudewicz and Karian, Z.A, The Extended Generalized Lambda Distribution (EGLD) for Fitting Distribution with Moments, (1996) American Journal of Mathematical and Management Science 16: 271-332.

[9] E. L. Lehmann and G. Casella, *Theory of point prediction*, Springer-Verlag, New York, 1998.

[10] M. Evans, N. Hastings, and B. Peacock, *Statistical distributions*, Wiley, New York, 1993.

[11] An, A., Acar, A., Haftka, R.T., Kim, N.H., Ifju, P.G, and Johnson, T.F., (2008) "Being Conservative with a Limited Number of Test Results," *Journal of Aircraft*, 45(6), 1969-1975.

[12] Composite Materials Handbook MIL-HDBK-17, "Guidelines for Property Testing of Composites," ASTM Publications 2002

[13] R. E. Neapolitan, *Learning Bayesian Network*, Pearson Education, New Jersey, 2004.