

Uncertainty Identification of Damage Growth Parameters using Health Monitoring Data and Nonlinear Regression

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ABSTRACT

When it comes to identifying model parameters such as damage growth parameters in Paris law for example, Bayesian inference is a popular method. However, it involves substantial computational cost, especially with increasing number of parameters. When the prior distribution for the parameters is not narrow, non-linear regression may provide almost all the benefits of Bayesian updating at a small fraction of the computational cost. In this paper we apply this approach to the identification of damage growth parameters. As a first step we simplify the problem to a single parameter in order to compare it with the same problem solved using Bayesian inference. We then discuss the issues related to uncertainty quantification in the case of a highly non-linear problem.

1 INTRODUCTION

Aircraft structures are designed under the damage tolerance concept in which small damage are allowed to exist as long as they are stable and grow slowly (Gallagher *et al.*, 1984). In this design concept, it is important to predict when damage become unstable and require maintenance. In the conventional preventive maintenance (Thomas, 1986), it has been practiced that a very small size of damage threshold, as small as 0.1", is used because inspection cannot be performed frequently, and there are many uncertainties involved in the process, including uncertainty in damage growth parameters, initial damage size, operating conditions, environment, etc. Due to the overly conservative replacement schedule, many airplanes are inspected even if they do not have any damage, and many insignificant damages are removed.

Recently, it has been shown that structural health monitoring (SHM) systems can be used instead of manual inspection to detect damage (Giurgiutiu, 2008). SHM-based maintenance is more effect as only those airplanes that are in danger will be sent for maintenance (condition-based maintenance). SHM may have significant impacts on increasing safety as well as reducing the operating and maintenance costs of structures by providing an accurate quantification of degradation and damage at an early stage to reduce or eliminate malfunctions. Furthermore, Coppe et al. (Coppe, 2010) showed that SHM can not only provide damage diagnosis but also predict the remaining useful life (RUL) by identifying damage growth parameters. They used Bayesian inference (Coppe, 2009) to reduce uncertainty in the damage growth parameters using measured damage size information.

Bayesian inference is a powerful method of quantifying uncertainty in the model parameters. It can take into account the prior knowledge on the unknown parameters and improve it using experimental observations. However in the case of SHM the advantage of the prior information can be overpowered by the amount of data available. That is, the effect of prior information becomes insignificant when numerous SHM data are used in Bayesian inference. In addition, when many parameters are updated simultaneously, Bayesian inference becomes computationally expensive due to multi-dimensional integration.

On the other hand, the traditional linear least square method (Lawson and Hanson, 1995) can be used to identify deterministic parameters when the model is a linear function of the parameters. This method is in particular powerful when many data are available, which is the case for SHM data. Different from Bayesian inference, this method does not require prior information. By assuming that the noise in the experimental data is Gaussian, it is possible to estimate the uncertainty in the identified parameters, which is also Gaussian.

When the physical model is a nonlinear function of model parameters, it is not straightforward to apply for uncertainty quantification in the linear least square method. As will be shown in the number examples, the damage growth in aircraft structures is governed by a nonlinear equation whose parameters need to be identified. In this paper, we proposed a linear perturbation concept to quantify uncertainty in the nonlinear least square method. First, nonlinear mathematical programming problem is solved to find the model parameters that minimize error between the model and experiment. Then, the nonlinear model is linearized with the identified parameter values, from which the uncertainty quantification in the linear least square method can be used. This approach can introduce two errors into the estimate of the uncertainty: (1) linearization error and (2) error associated with assumption of Gaussian noise. In addition, it is assumed that noises at different experiments are uncorrelated.

This paper presents a nonlinear least square method to identify damage growth parameters using a through-thickness crack in an aircraft fuselage panel which grows through cycles of pressurization. A simple damage growth model, with two damage growth parameters is utilized. We present here two least square fits identifying one or three variables. The objective is to examine the accuracy of uncertainty quantification using nonlinear least squares. The uncertainty is derived analytically and compared to a Monte Carlo estimate to examine its accuracy for both problems presented. For the one-variable problem it is then compared to the uncertainty obtained using Bayesian inference.

The paper is organized as follows. In Section 2, the derivation of the uncertainty quantification for the general least square methods, linear and non linear is presented. In Section 3 the one variable identification problem is discussed, showing the derivation of the analytical uncertainty quantification, as well as results comparing that uncertainty to the one obtained using Monte Carlo simulations. In Section 4 the three variable problem is discussed and used to introduce the issues related to the uncertainty estimation resulting from potentially correlated variable and highly non-linear problems. Concluding remarks are presented in Section 5.

2 UNCERTAINTY QUANTIFICATION IN NON-LINEAR LEAST SQUARE

The least square method is commonly used for identifying unknown parameters of a physical model using experimental observations, which normally include random noise. Thus, if the experiment is repeated, it is likely that different values of the

parameters may be identified. In this section, a method of calculating the distribution of the identified parameters in the nonlinear least square method will be introduced. In order to make the presentation easy to understand, estimation of parameter uncertainty in the linear least square method is discussed first, followed by that of the nonlinear least square method.

2.1 Uncertainty in the Linear Least Square Method

In regression, the response function $y(t)$ is approximated by a function $\hat{y}(t, \boldsymbol{\beta})$ with vector of parameters $\boldsymbol{\beta}$ whose dimension is $n_{\boldsymbol{\beta}} = \dim(\boldsymbol{\beta})$:

$$y(t) = \hat{y}(t, \boldsymbol{\beta}) + \varepsilon \quad (1)$$

where ε is the approximation error. The objective of regression is to estimate the parameters $\boldsymbol{\beta}$ based on n_y data, which are given in the form of $(t_i, y_i), i = 1, \dots, n_y$, that may contain measurement noise. In regression the parameters are estimated by minimizing the sum of the squares of the discrepancies between the measurements and $\hat{y}(t, \boldsymbol{\beta})$. The regression model is called linear when the approximate function is a linear function of $\boldsymbol{\beta}$, as

$$\hat{y}(t, \boldsymbol{\beta}) = \sum_{i=1}^{n_{\boldsymbol{\beta}}} \beta_i \xi_i(t) \quad (2)$$

where $\xi_i(t)$ are basis functions. In general, the exact values of $\boldsymbol{\beta}$ can only be found when the number of experimental data is infinity. With finite n_y , the values are only estimate, which will be denoted by \mathbf{b} in this paper.

The vector of errors (discrepancies) can be written as

$$\begin{Bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n_y} \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_y} \end{Bmatrix} - \begin{bmatrix} \xi_1(t_1) & \xi_2(t_1) & \cdots & \xi_{n_{\boldsymbol{\beta}}}(t_1) \\ \xi_1(t_2) & \xi_2(t_2) & \cdots & \xi_{n_{\boldsymbol{\beta}}}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1(t_{n_y}) & \xi_2(t_{n_y}) & \cdots & \xi_{n_{\boldsymbol{\beta}}}(t_{n_y}) \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_{\boldsymbol{\beta}}} \end{Bmatrix} \quad (3)$$

Or, symbolically,

$$\mathbf{e} = \mathbf{y} - \mathbf{X} \cdot \mathbf{b} \quad (4)$$

The parameters \mathbf{b} are estimated by minimizing the root-mean-square error defined as

$$e_{\text{RMS}} = \sqrt{\frac{1}{n_y} \mathbf{e}^T \mathbf{e}} \quad (5)$$

After substituting Eq. (4) into Eq. (5) and minimizing the root-mean-square error, the following linear regression equation is obtained:

$$\mathbf{X}^T \mathbf{X} \cdot \mathbf{b} = \mathbf{X}^T \mathbf{y} \quad (6)$$

which can be solved for the estimate \mathbf{b} of parameters.

Because the experimental data includes random noise, the estimated parameters will be different for different sets of experimental data. The objective is to estimate the uncertainty in the estimated parameters due to the random noise in the experimental data.

The uncertainty in the parameters can be found assuming that the random noise has a Gaussian distribution with standard deviation (STD) of σ ; i.e., $v \sim N(0, \sigma^2)$, and that the noise at different measurements is uncorrelated. An unbiased estimate of the standard deviation can be obtained from

$$\hat{\sigma}^2 = \frac{\mathbf{e}^T \mathbf{e}}{n_y - n_p} \quad (7)$$

The sensitivity of estimated parameters with respect to small differences in data can be calculated using the covariance matrix of \mathbf{b} defined as

$$\Sigma_b = [\mathbf{b} - E(\mathbf{b})][\mathbf{b} - E(\mathbf{b})]^T \quad (8)$$

where $E(\mathbf{b})$ is the expected value of \mathbf{b} . Using Eq. (6), the covariance matrix can be obtained as

$$\Sigma_b = \sigma^2 [\mathbf{X}^T \mathbf{X}]^{-1} \quad (9)$$

The diagonal components of Σ_b is the square of standard deviation of \mathbf{b} , which represents a measure of the sensitivity of estimated parameters with respect to the noise. Since the standard deviation of the noise is unknown in advance, its estimate in Eq. (7) can be used. Thus, the standard error (SE) of parameter b_i can be obtained by

$$SE(b_i) = \hat{\sigma} \sqrt{[\mathbf{X}^T \mathbf{X}]^{-1}_{ii}} \quad (10)$$

The above standard error is indeed the estimate of the standard deviation of b_i .

2.2 Uncertainty in the Nonlinear Least Square Method

Different from a linear regression problem, the physical model cannot be represented as a linear combination of unknown parameters as in Eq. (2). Thus, instead of solving a linear regression Eq. (6), a nonlinear optimization problem is solved to minimize the root-mean-square error in Eq. (5). In this paper, Matlab lsqnonlin function is used to solve the nonlinear regression problems. The identified parameters are denoted by \mathbf{b}^* . In the following, the nonlinear equation will be linearized with respect to the identified

parameters in order to estimate the uncertainty in the parameters.

In order to linearize the problem, the first-order Taylor series expansion method can be used for $\hat{y}(t, \mathbf{b})$ where $\mathbf{b} = \mathbf{b}^* + \Delta \mathbf{b}$. By ignoring higher-order terms, we have

$$\hat{y}(t, \mathbf{b}) \approx \hat{y}(t, \mathbf{b}^*) + \sum_i \frac{\partial \hat{y}}{\partial b_i}(t, \mathbf{b}^*) \Delta b_i \quad (11)$$

By moving $\hat{y}(t, \mathbf{b}^*)$ to the left-hand side, the equation for residual can be defined as

$$r = \hat{y} - \hat{y}(t, \mathbf{b}^*) = \sum_i \frac{\partial \hat{y}}{\partial b_i}(t, \mathbf{b}^*) \Delta b_i \quad (12)$$

Equation (12) can be considered as a linear least square problem with unknown parameters $\Delta \mathbf{b}$, and the gradients $\partial \hat{y} / \partial b_i$ becomes the basis vector ξ_i in Eq. (2). Thus, the uncertainty in parameters $\Delta \mathbf{b}$ can be calculated using the same procedure described in Section 2.1. For that purpose, the vector of basis functions can be written as

$$\mathbf{X} = \begin{bmatrix} \frac{\partial \hat{y}_1}{\partial b_1} & \frac{\partial \hat{y}_1}{\partial b_2} & \dots & \frac{\partial \hat{y}_1}{\partial b_{n_p}} \\ \frac{\partial \hat{y}_2}{\partial b_1} & \frac{\partial \hat{y}_2}{\partial b_2} & \dots & \frac{\partial \hat{y}_2}{\partial b_{n_p}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \hat{y}_{n_y}}{\partial b_1} & \frac{\partial \hat{y}_{n_y}}{\partial b_2} & \dots & \frac{\partial \hat{y}_{n_y}}{\partial b_{n_p}} \end{bmatrix} \quad (13)$$

Then, Eq. (10) can be used to estimate the standard error of $\Delta \mathbf{b}$, which can also be considered as the standard error of \mathbf{b}^* if the problem is linear.

Due to the nonlinearity, the standard error of $\Delta \mathbf{b}$ will be different from that of \mathbf{b}^* . However, if the nonlinearity is small, or if the uncertainty in \mathbf{b}^* is small, then the difference between them will be small.

In order to verify the proposed uncertainty quantification method of nonlinear least square method, Monte Carlo simulation can be used to estimate the uncertainty in the identified parameters. In this approach, it is assumed that the experiments are repeated many times, and the parameters are identified for each experiment, from which the distribution of identified parameters can be estimated.

In the following two sections we respectively illustrate how the uncertainty resulting from least square fit compares to the one obtained using Bayesian inference using a one variable problem and how the

uncertainty quantification behaves in the case of a multi-variable, highly non-linear problem.

3 IDENTIFICATION OF SINGLE PARAMETER

The problem we are looking into here is identifying a damage growth parameter using damage size estimation. We want to fit a damage growth law, in this case Paris' law (Paris, 1999) to a set of damage size measurement, the variable to be identified is the Paris' law exponent m .

$$\frac{da}{dN} = C(\Delta K)^m \quad (14)$$

where a is half the damage size, N the number of cycles, C Paris law parameter and ΔK is the range of stress intensity factor. Using Paris law (14) the damage size at the i^{th} cycle can be derived as

$$a_i^{\text{model}} = \left[iC \left(1 - \frac{m}{2} \right) (\sigma\sqrt{\pi})^m + a_0^{1-\frac{m}{2}} \right]^{\frac{2}{2-m}} \quad (15)$$

In the above damage growth model, several unknown parameters are involved. First, the damage growth parameters, C and m , need to be identified. In addition, the initial damage size, a_0 , is often unknown, and needs to be identified too. Coppe et al. (Coppe, 2009) used Bayesian inference to identify the unknown parameters with measure damage growth. However, due to computational challenge in Bayesian inference, they only identify the unknown damage growth parameter, m , by assuming all other parameters are known. In this section, uncertainty quantification of nonlinear least square method is used to identify the uncertainty in m , and compared it with that of Bayesian inference.

If we define a^{meas} as the measured data, the least square fit problem can then be stated as minimizing the L2-norm of r ,

$$r = a_i^{\text{meas}} - a_i^{\text{model}} \quad (16)$$

If we define m^* as the fitted value of m , the standard error in m can then be obtained (see Eqs. 7 and 10) as

$$SE(m) = \frac{\hat{\sigma}}{\sqrt{\left(\frac{da^{\text{model}}}{dm^*} \right)^2}} \quad (17)$$

with

$$\hat{\sigma}^2 = \sum_{i=1}^{n_y} \frac{e_i^2}{N-1} \quad (18)$$

and

$$e_i = r_i - \left(\frac{da^{\text{model}}}{dm^*} \right)_i \Delta m \quad (19)$$

The measured data, a^{meas} , are actually simulated in this paper by applying an error model to the modeled damage size, a^{model} . The error model includes the effect of bias, b , and noise, v , of the sensor measurement. The former is deterministic and represents a calibration error, while the latter is random and represents a white noise. The measured half damage size after N cycles can then be defined as

$$a_N^{\text{meas}} = a_N^{\text{model}} + b + v \quad (20)$$

where b is the deterministic bias and v is a random noise defined as

$$v \sim \text{Uniform}(-1, 1) \text{ mm} \quad (21)$$

It has to be noted here that when fitting only m , we assume the bias to be zero and that the initial damage size is known accurately.

Since the uncertainty in identifying m results from the noise in the data we can also quantify it using Monte Carlo Simulation (MCS) by simulating 1,000 sets of data, perform a fit for each of them and then calculate the standard deviation for those data. We can afford to do this here because we are simulating the measured data using an error model. **Figure 1** shows the comparison between the standard error and the standard deviation from the MCS.

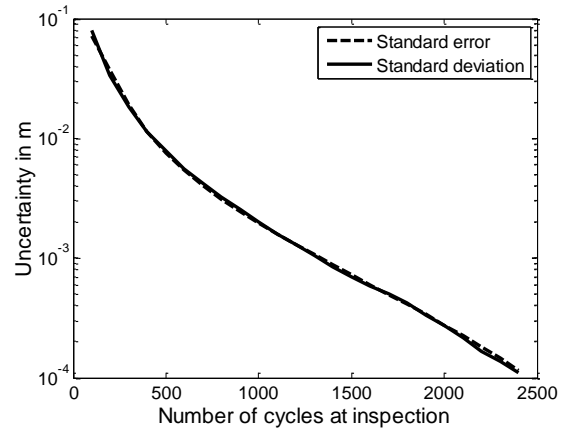


Figure 1: Comparison of the derived standard error with the simulated standard error

It can be observed that the derived standard error (solid line) fits very well the estimated standard deviation (dashed line), the plots can hardly be distinguished.

The next step is then to compare that standard error to the standard deviation of the distribution of m ,

identified using Bayesian inference. In order to do that we use the Bayesian inference method developed and presented by Coppe (Coppe, 2009). **Figure 2** illustrates the comparison between the standard deviation resulting from least square fit identification and the one resulting from Bayesian inference.

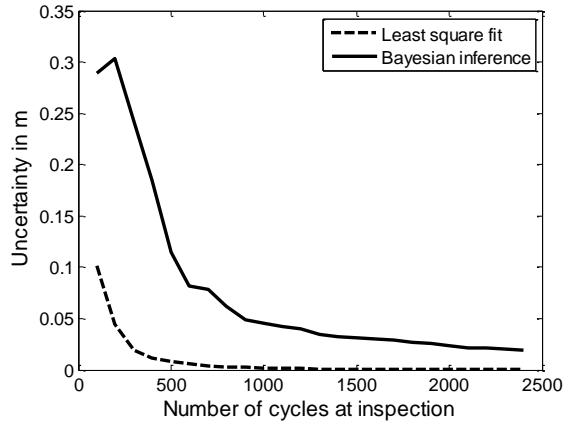


Figure 2: Comparison of the standard error obtained using least square fit and Bayesian inference

It can be observed that least square fit is able to identify m much more accurately than Bayesian inference at least 3 times more accurately. This can be explained by the fact that at a given inspection performed every 100 cycles, least square fit uses data at every cycle. Bayesian on the other hand used the measurement at every 100 cycles, if the same was done with least square fit similar standard deviation would be achieved, just slightly lower than Bayesian inference.

4 IDENTIFICATION OF MULTIPLE PARAMETERS

As mentioned before, Bayesian inference becomes computationally expensive when multiple parameters are identified simultaneously, for that reason we are not presenting a comparison with Bayesian inference results. However, the proposed method is straightforward for quantifying uncertainty of multiple parameters. Unlike the idealized situation of the previous section, we now assume that the initial crack a_0 and the bias b also need to be identified. Then the three-variable problem can then be defined as

$$R = a^{meas} - \hat{y}(N, \beta) \text{ with } \beta = (m, a_0, b) \quad (22)$$

with

$$\hat{y} = a_i^{model} + b \quad (23)$$

The derivation of the standard error is the same as previously and it leads to

$$SE(m) = \hat{\sigma} \sqrt{[X^T X]_{11}^{-1}} \quad (24)$$

$$SE(a_0) = \hat{\sigma} \sqrt{[X^T X]_{22}^{-1}} \quad (25)$$

$$SE(b) = \hat{\sigma} \sqrt{[X^T X]_{33}^{-1}} \quad (26)$$

With

$$X = \begin{bmatrix} \frac{\partial \hat{y}}{\partial m}(\beta^*) & \frac{\partial \hat{y}}{\partial a_0}(\beta^*) & \frac{\partial \hat{y}}{\partial b}(\beta^*) \end{bmatrix} \quad (27)$$

And

$$\hat{\sigma}^2 = \sum_i \frac{e_i^2}{N-3} \quad (28)$$

As described in the previous section we compare the analytical estimate of SE to the simulated STD obtained using MCS. This can be found in **Figure 3**.

It can be observed that in this case the derived standard error does not match the simulated standard deviation very well for the first 1,000 cycles. There are many explanations for this discrepancy. First, the least square method predicts a larger standard error because not many data are available at the early stage. The nonlinearity of the nonlinear least square problem can also contribute to the discrepancy. Another aspect is the close relationship between a_0 and b , where these two parameters compensate each other and can lead to an ill conditioned $X^T X$ matrix. As the damage grows, the effect of a_0 and b become more independent.

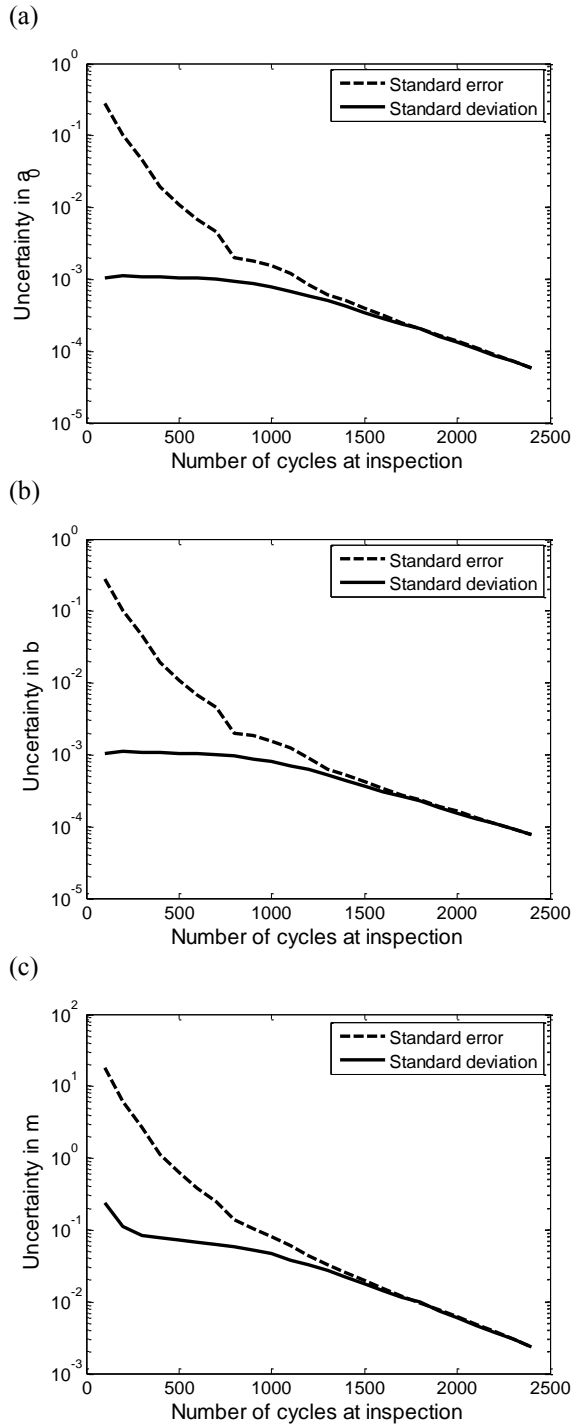


Figure 3: Uncertainty in a_0 (a), b (b) and m (c) using least square fit with three variables

5 CONCLUSION

This paper presents a derivation of standard error to quantify uncertainty for nonlinear least square models. It has been shown that the method gives very good

results when compared to MCS estimation for a one variable case.

For a multiple variable case the interaction between the variable can cause problems if they are not enough variables available. In this case MCS can be used to quantify the uncertainty.

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NOMENCLATURE

a	half damage size
a_0	initial half damage size
a^{meas}	half measured damage size
a^{model}	half modeled damage size
b	bias in the measurements
C	Paris law parameter
e_i	Error in the fit
f	generic functions
m	Paris law exponent
n	number of parameters
N	number of cycles at inspection
r	residual after least square optimization
R^2	least square objective function
S	overall standard error
SE	standard error
x, β	vector of parameters
X	generic matrix
y	data points
\hat{y}	approximated function
ΔK	range of stress intensity factor
ε	approximation error
σ	applied stress

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