Doubly weighted moving least squares and its application to structural reliability analysis

Jian Li[1]
Shanghai Jiao Tong University, Shanghai 200240, China

Nam H. Kim[2]
University of Florida, Gainesville, FL 32611, USA

and

Hai Wang [3]
Shanghai Jiao Tong University, Shanghai 200240, China

In this paper, we proposed a two-stage hybrid reliability analysis framework based on the surrogate model, which combines the first-order reliability method and Monte Carlo simulation with a doubly-weighted moving least squares (DWMLS) method. The first stage consists of constructing a surrogate model based on DWMLS. The weight system of DWMLS considers not only the normal weight factor of moving least squares, but also the distance from the most probable failure point (MPFP), which accounts for reliability problems. An adaptive experimental design scheme is proposed, during which the MPFP is progressively updated. The approximate values and sensitivity information of DWMLS are chosen to determine the number and location of the experimental design points in the next iteration, until a convergence criterion is satisfied. In the second stage, MCS on the surrogate model is then used to calculate the probability of failure. The proposed method is applied to four benchmark examples to validate its accuracy and efficiency. Results show that the proposed surrogate model with DWMLS can estimate the failure probability accurately, while requiring fewer original model simulations.

Nomenclature

n	=	number of input random variables
N	=	number of experimental points
$\mathbf{x}$	=	vector of input random variables
β	=	reliability index
β_{HL}	=	reliability index by Hasofer-Lind algorithm
μ	=	mean
σ	=	standard deviation
$g(\mathbf{X})$	=	limit state function
$\hat{g}(\mathbf{X})$	=	approximate limit state function/ response surface function
$\mathbf{X}_{add}$	=	new added experimental points in any iteration
MLS	=	moving least square
DWMLS	=	doubly weighted moving least square
SVR	=	support vector regression
ANN	=	artificial neural networks
MPFP	=	most probable failure point

[1] Graduate Student, School of Aeronautic and Astronautic, lijiannpu@gmail.com
[2] Associate Professor, Department of Mechanical and Aerospace Engineering, nkim@ufl.edu,
[3] Professor, School of Aeronautic and Astronautic, wanghai601@sjtu.edu.cn

MCS = Monte Carlo Simulation
FORM = first order reliability method
SORM = second order reliability method
H-L = Hasofer-Lind algorithm
RSM = response surface method
LHD = Latin hypercube design
FEA = finite element analysis
CFD = computational fluid dynamics
COV = coefficient of variation

I. Introduction

IT has been well recognized that uncertainties in engineering systems (e.g., applied loads, material properties and geometric tolerances) can result in catastrophic failure and should be managed appropriately. The traditional factor-of-safety approach to compensate for uncertainties often leads to either un-conservative or too-conservative designs. Reliability analysis takes into account these uncertainties in evaluating system's safety, which has become an important part of recent engineering design. There has been a growing interest in the use of reliability methods for structural design and safety assessment[1-4].

However, with the development of advanced numerical simulation methods, which commonly take several hours to perform a single evaluation, classical reliability methods can easily become impractical. Monte Carlo Simulation (MCS) and its variants demand tremendous computational resources that prohibit its practicality. On the other hand, approximation methods, such as first-order reliability method (FORM) and second-order reliability method (SORM), have issues with relatively poor performance in accuracy. Therefore, it seems reasonable to use a response surface or surrogate model to approximate the performance function and apply either MCS to calculate reliability.

However, reliability analysis is different from approximation problems, from which the surrogate model originated. In conventional approximation problems, the general criterion of a surrogate model is to minimize the error between the true function and the surrogate model in the entire domain of interest. In reliability analysis, however, it is important to identify a limit state, which is the boundary between safe and failed regions, especially near the most probable failure point (MPFP). In this paper, a doubly weighted moving least squares (DWMLS) method is proposed, which takes into account the characteristic feature of reliability analysis. The proposed DWMLS method consists of two weighting schemes—the first weight considers the distance between the sampling point and the prediction point, while the second considers the distance between the sampling points and the MPFP. A hybrid, two-stage reliability analysis framework is proposed, which takes full advantage of the FORM/SORM, MCS, adaptive experiment design and DWMLS to achieve both accuracy and efficiency.

The remainder of this paper is organized as follows. In Section II, a brief literature review of various surrogate models in reliability analysis is presented. In Section III, a general principle of the moving least squares method is stated, after which the newly added weighting system of DWMLS is detailed. In Section IV, a new adaptive experimental design procedure is illustrated in detail. A complete flowchart of the proposed hybrid reliability analysis framework is then presented in Section V. Four numerical examples are presented in Section VI, which highlight the capabilities of the proposed method and demonstrate its accuracy and efficiency, followed by a summary and conclusions in Section VII.

II. Review on surrogate models in reliability analysis

The increased complexity of simulations on real systems stimulates the development of surrogate models that approximate the behavior of complex systems, improve their validation process, and aid optimization of the system[7]. Surrogate models are developed in order to analyze experimental data and to build empirical models based on observations. These models were first introduced in design optimization and applied to reliability analysis and design because of their merits in efficiency.

Wong[5] first proposed a complete, quadratic form polynomial and applied it to reliability analysis. In his work, the number of polynomials and the required sampling points increase rapidly with the number of random variables. In order to reduce the number of sampling points, Bucher and Bourgund[3] proposed a two-iteration quadratic polynomial without cross-terms. Rajashekhar and Ellinwood[6], and Liu and Moses[42] improved this approach by updating the surrogate model parameters until a convergence criterion was satisfied. Kim and Na[8] proposed a sequential approach to the surrogate model where the gradient projection method is used to ensure that the sampling points are located near the failure surface. Zheng and Das[9] proposed an improved surrogate model and applied it to

reliability analysis of a stiffened plate structure. Guan and Melchers[10] evaluated the effect of surrogate model parameter variation on reliability. Kaymaz and McMahon[11] suggested a new surrogate model, in which a weighted regression method was applied instead of normal regression. Inspired by Kaymaz and McMahon's method, Nguyen and Sellier[12] improved the weighted regression method where the fitting points were weighted according to their distance from the true failure surface and the estimated design point.

However, when the limit state function is highly nonlinear, polynomial-based surrogate models can perform poorly because they try to approximate the performance function globally. Thus, advanced surrogate modeling methods were introduced to replace traditional global polynomial-based models. Choi and Grandhi[13], and Kim and Wang[14] introduced polynomial chaos expansion for reliability analysis and design. Papadrakakis et al. [15], and Hurtado and Alvarez[16] used Neural Networks to reliability analysis in conjunction with Monte Carlo Simulation. Gomes and Awruch[17] compared Neural Networks with FORM, MCS, and the Importance Sampling technique. Kaymaz[18], and Panda and Manohar[19] proposed the Kriging method to reliability analysis and compared it with the most common surrogate models. Echard and Gayton[20] presented a Kriging enhanced MCS method in reliability analysis and the test problems have demonstrated its efficiency and accuracy. Most[21] presented an efficient adaptive response surface approach for reliability analysis, where support vector machines were used to classify the failure and safe domain. Guo and Bai[22] proposed a least squares support vector machine for regression into reliability analysis and the results demonstrate excellent accuracy and smaller computational cost than the reliability method based on support vector machines.

Moving least squares (MLS) is a local weighted least squares method, originally introduced by Lancaster and Salkauskas[23] for smoothing and interpolating data. After that, it is widely used to obtain approximations in meshfree methods and structural optimization because of its 'localized' approximation property[24]. Unlike other surrogate models, the MLS has no expensive inner parameter optimization during the adaptive modeling process. Therefore, Krishnamurthy[25] compared the MLS method with other local methods, such as Kriging, and found it to be more accurate and computationally effective for the examples considered. Bucher[26] carried out research aimed at comparing the performance of these response surfaces and its application in reliability analysis. In Kang and Koh's research[27], MLS has made it possible to derive the approximation function closer to the limit state function and exhibited improved performance in terms of significant reduction of the number of structural analyses and sensitivity accuracy of the reliability index to the random variables. Youn and Choi[41], and Song and Lee[28] have applied the moving least squares method to reliability based design optimization and MLS performed well in uncertainty design.

III. Doubly weighted moving least squares (DWMLS)

The basic idea of surrogate modeling in reliability analysis is to replace the performance function with an approximate function, whose value can be computed easily. Compared with the global polynomial regression method, the moving least squares (MLS) method is a local regression and a relatively new surrogate modeling technique. The MLS method can be found extensively in the element-free Galerkin method and computer graphics. In this Section, the basic principle of the MLS method is presented first, followed by an introduction to a new weighting scheme that works better for reliability analysis.

A. Basic principle of moving least squares

We consider a performance function $g(\mathbf{x})$ in an n-dimensional space of random variables, in which the vector of random variables is defined as $\mathbf{x} = [x_1, x_2, ..., x_n]^T$. In the MLS method, the performance function is approximated by

$$\hat{g}(\mathbf{x}) = \mathbf{p}(\mathbf{x})^{\mathrm{T}} \mathbf{a}(\mathbf{x}) \tag{1}$$

where $\mathbf{p}(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), ..., p_m(\mathbf{x})]^{\mathrm{T}}$ is a vector of m polynomial basis functions, and $\mathrm{a}(\mathbf{x}) = [a_1(\mathbf{x}), a_2(\mathbf{x}), ..., a_m(\mathbf{x})]^{\mathrm{T}}$ is a vector of corresponding coefficients. It is noted that in global regression methods, the coefficients are constant, while they are functions of $\mathbf{x}$ in MLS. In this paper, the basis function $\mathbf{p}(\mathbf{x})$ is defined using polynomials up to the second order without cross terms, as

$$\mathbf{p}(\mathbf{x}) = [1, x_1, x_2, ..., x_n, x_1^2, x_2^2, ..., x_n^2]^{\mathrm{T}} \tag{2}$$

where the dimension of $\mathbf{p}(\mathbf{x})$ is $m = 2n + 1$. However, it is possible that the basis function can include cross-terms, as well as higher order terms.

The unknown coefficients in (1) can be calculated by minimizing the error between the performance function and its approximation at discrete points. In order to do that, N sample points are first selected from the input space; these samples are denoted by $\mathbf{x}_I, I = 1,\ldots, N$. Then, the performance functions, $g(\mathbf{x}_I)$, are calculated at these sample points. This process may involve numerical simulations, such as finite element analysis or computational fluid dynamics. At a given prediction point **x**, the MLS technique determines the unknown coefficients by minimizing the error between actual and approximated values of the performance function with weights, as

$$R(\mathbf{x}) = \sum_{I=1}^{N} w(\mathbf{x} - \mathbf{x}_I)[g(\mathbf{x}_I) - \mathbf{p}(\mathbf{x}_I)^{\mathrm{T}} \mathbf{a}(\mathbf{x})]^2 \tag{3}$$

The above formula can also be written in a matrix form as:

$$R(\mathbf{x}) = [\mathbf{P}\mathbf{a}(\mathbf{x}) - \mathbf{g}]^{\mathrm{T}} \mathbf{W}[\mathbf{P}\mathbf{a}(\mathbf{x}) - \mathbf{g}] \tag{4}$$

where **g**, **P** and **W** are defined as:

$$\mathbf{g} = [g(\mathbf{x}_1), g(\mathbf{x}_2), ..., g(\mathbf{x}_N)]^{\mathrm{T}} \tag{5}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}^{\mathrm{T}}(\mathbf{x}_1) \\ \mathbf{p}^{\mathrm{T}}(\mathbf{x}_2) \\ \vdots \\ \mathbf{p}^{\mathrm{T}}(\mathbf{x}_N) \end{bmatrix}_{N*m} \tag{6}$$

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} w(\mathbf{x} - \mathbf{x_1}) & 0 & \ldots & 0 \\ 0 & w(\mathbf{x} - \mathbf{x_2}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & w(\mathbf{x} - \mathbf{x_N}) \end{bmatrix}_{N*N} \tag{7}$$

In (7), the weight $w(\mathbf{x} - \mathbf{x}_I)$ takes the following Gaussian form:

$$w(\mathbf{x} - \mathbf{x_I}) = \begin{cases} \dfrac{(e^{-(\alpha \| \mathbf{x} - \mathbf{x_I} \| / D_I)^2} - e^{-\alpha^2})}{(1 - e^{-\alpha^2})} & \text{if } \| \mathbf{x} - \mathbf{x_I} \| \le D_I \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where the parameter α is used to control the weight function curve, D_I defines the domain of the influence of point $\mathbf{x}_I$, $\|\mathbf{x} - \mathbf{x}_I\|$ is the Euclid distance between sampling point $\mathbf{x}_I$ and prediction point x, and $r = \|\mathbf{x} - \mathbf{x}_I\| / D_I$ is a normalized distance. In Fig.1, the shape of Gaussian weight function w(*r*) with different values of α is demonstrated.

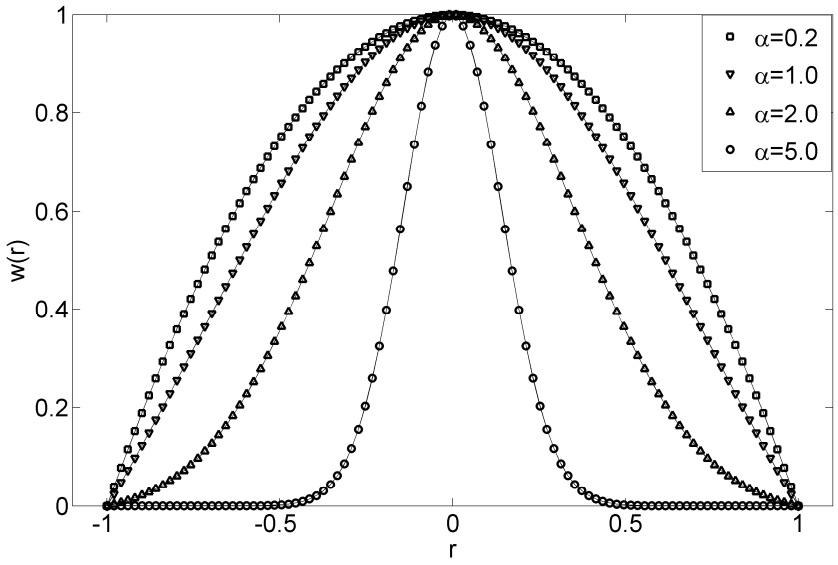

Figure.1. Shape of the Gaussian weight function with different control parameter

The minimum of the square error $R(\mathbf{x})$ can be achieved by vanishing the partial derivatives with respect to unknown coefficients, as

$$\frac{\partial R(\mathbf{x})}{\partial a_i} = 0, \qquad i = 1, \ldots, m \tag{9}$$

The conditions in Eq. (9) yield the following system of linear equations:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{b}(\mathbf{x}) \tag{10}$$

where $\mathbf{A}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ are defined by:

$$\mathbf{A}(\mathbf{x}) = \mathbf{P}^{\mathrm{T}}\mathbf{W}(\mathbf{x})\mathbf{P} \tag{11}$$

$$\mathbf{b}(\mathbf{x}) = \mathbf{P}^{\mathrm{T}}\mathbf{W}(\mathbf{x})\mathbf{g} \tag{12}$$

Once the unknown coefficients, a(x), are calculated by solving Eq. (10), Eq. (1) is used to approximate the performance function.

As can be seen from the weight scheme in Eq. (8), the weight w_I, which is associated with sampling point $\mathbf{x}_I$, decreases as x moves away from $\mathbf{x}_I$. The contribution of those points whose distance from $\mathbf{x}$ is greater than D_I vanishes, and thus, there is no need to include them in the regression process. Therefore, the dimension of the matrices in the MLS process is much smaller than the total number of sample points N. It is important to note that at a given prediction point $\mathbf{x}$, there must be enough sample points $\mathbf{x}_I$ such that the coefficient matrix $\mathbf{A}(\mathbf{x})$ should be non-singular. Also, it should be noted that the coefficients, a(x), must be calculated at every prediction point.

B. Doubly weighted moving least squares

In MLS, it is rational to impose a heavy weight to the points that are close to the prediction point and a light weight for more distant points in order to better approximate the performance function. In reliability analysis, however, the most important region is around the MPFP, because it contributes most to the probability of failure. Thus, we introduce an additional weighting scheme into MLS where the distances of sampling points to the prediction point $\mathbf{x}$ and to the MPFP are considered simultaneously. As discussed in the previous section, the first weighting scheme is based on the distance of sampling points $\mathbf{x}_I$ to the prediction point $\mathbf{x}$:

$$w_I(\mathbf{x}, \mathbf{x}_I) = w(\|\mathbf{x} - \mathbf{x}_I\|) \tag{13}$$

The second weighting scheme takes into account the distance between the sampling points and the current MPFP. It aims to penalize points located far from the current MPFP. The second weight factor is expressed as:

$$w_{II}(\mathbf{x}^*, \mathbf{x}_I) = \exp(-d_I{}^2) \tag{14}$$

where d_I is the distance between the *I-th* sampling point and the current MPFP. It should be noted that since there is no information on the position of MPFP in the first stage, the conventional MLS is applied to the first surrogate model.

Then, through the advantage of the above-mentioned two weighting schemes, we find the following expression suitable to obtain the weight for each sampling point:

$$w(\mathbf{x}, \mathbf{x}_I) = w_I(\mathbf{x}, \mathbf{x}_I) * w_{II}(\mathbf{x}_I) \tag{15}$$

In this paper, the weight matrix in (4) is replaced with this double weight matrix and the new surrogate model considers both the distance from the prediction point and the MPFP.

IV. Adaptive Design of Experiments (DOE)

A. Discussion on DOE in reliability analysis

The quality of probability estimates using a surrogate model depends on not only the surrogate model itself, but also the location of the points chosen to build the surrogate model (design of experiments). In the context of reliability analysis, there exist two kinds of DOE strategy: single design and adaptive design. For the former, Wong[5] and Faravelli[29] employed single factorial experimental design containing 2n points to fit a quadratic function and to estimate the failure probability. Cheng[30] proposed a surrogate model based on the artificial neural network and using uniform experimental design in predicting failure probability. However, in the case of a black-box computer model, single experimental design cannot guarantee the accuracy of approximation, especially in domain around the limit state surface. Hence, it would be better to start with an initial set of samples and gradually add more samples based on the information provided by previous samples. Such a process is commonly referred to as the adaptive design of experiments and has received more attentions than the former strategy.

In reliability analysis, Bucher and Bourgund[3] first applied a surrogate model in which experimental points are chosen around the mean values of random variables, which form a matrix called the design matrix. Quadratic polynomials without cross terms are then used to fit these experimental points. Among various sampling methods, a common approach is to evaluate $g(\mathbf{X})$ at a $2n+1$ combination of μ_i and $\mu_i \pm f\sigma_i$, where μ_i and σ_i are the mean and standard deviation of random variables, X_i, and f is a factor that defines the sampling range. The number of unknown coefficients in the performance function is $2n+1$, given as

$$\hat{y}(\mathbf{x}) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} a_{ii} x_i{}^2 \tag{16}$$

A FORM algorithm was applied to estimate the MPFP based on the above quadratic polynomial function, and the next design matrix was constructed on the new center point determined by the following expression:

$$\mathbf{x}_m = \boldsymbol{\mu} - g(\boldsymbol{\mu}) \frac{\boldsymbol{\mu} - \mathbf{x}_D}{g(\boldsymbol{\mu}) - g(\mathbf{x}_D)} \tag{17}$$

where $\mathbf{x}_m$ and $\mathbf{x}_D$ indicate the new center point and the interim MPFP obtained in the previous stage, respectively. This second design matrix is used to form another quadratic polynomial just as the first matrix and the final failure probability was estimated by it. Thus, this procedure requires a $4n+3$ evaluation of $g(\mathbf{X})$. Rajashekhar and Ellingwood[6] questioned if a single cycle of updating is adequate, and they proposed to improve it by using more iterations until a convergence criterion is satisfied.

The above surrogate model in structural reliability analysis has some disadvantages: (1) with the increase of random variables, the total $g(\mathbf{X})$ evaluation number increases excessively fast, particularly while the convergence process is slow; (2) the result obtained by the above procedures has been shown to be sensitive to the

parameter f and may not always give an acceptable approximation to the true failure probability (it is possible that the sequential center points during iteration may oscillate in the domain around the true design point and not converge); and (3) at different stages of surrogate modeling, only a part of available information on all previous $g(\mathbf{X})$ is directly used. Thus, it is considered that the accuracy of above iterative algorithm depends mainly on the characteristics of the nonlinear performance function, thereby limiting its application.

Simpson[31] concluded that a recommended experimental design should have a space-filling property. In the current study, an optimal uniform Latin hypercube design is utilized as the initial experimental design. In the following adaptive experiment design process, the distance from the center point to the new sample point is determined not by the constant parameter f, but by the location of limit state estimated using all the previous information of $g(\mathbf{X})$.

B. Adaptive experimental design

1. Initial experimental design--Latin hypercube design (LHD)

The statistical method of the Latin hypercube design (LHD) was developed to generate a set of samples from a multidimensional distribution. The technique was first described by McKay et al.[32], and further elaborated by Iman and Conover[33]. LHD is popular in design and analysis of computer experiments. The location of LHD points is determined through a random procedure and a complete theory can be found in Forrester's work[34]. The goal of a good LHD is to make the selected sampling points as uniform as possible to cover the entire design space. In this paper, the φ_p-criterion is selected as a uniformity measure and the translational propagation algorithm proposed by Viana[35] is used to obtain the optimal uniform LHD. In this algorithm, sampling points are determined by minimizing the following criterion:

$$\varphi_p = \left[\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} d_{ij}^{-p} \right]^{1/p} \tag{18}$$

where d_{ij} is the distance between two sample points, $\mathbf{x}_i$ and $\mathbf{x}_j$,

$$d_{ij} = d(\mathbf{x}_i, \mathbf{x}_j) = \left[\sum_{k=1}^{n} \left| x_{ik} - x_{jk} \right|^t \right]^{1/t} \tag{19}$$

and $p = 50$ and $t = 1$ were recommended by Jin[36]. The space-filling property of optimal uniform LHD ensures that no two sampling points are too close to each other; uniformly distributed experimental points would enhance the approximation capacity of surrogate models. In this paper, LHD is used as an initial set of experimental designs.

2. Adaptive experimental design process

Based on the developed DWMLS model, the FORM algorithm can determine an interim MPFP, which is presumably located close to the true MPFP. The iterative process adds new experimental points to improve the accuracy of the surrogate model near the MPFP. In this paper, an additional sampling point is added in the location of the limit state surface in each variable direction, starting from the current MPFP.

Let us assume that $\mathbf{x}_k^*$ is the MPFP at k th DWMLS model. Then, the performance function $g(\mathbf{X})$ is evaluated at $\mathbf{x}_k^*$, and additional sampling points will be added. Firstly, it is necessary to check the magnitude of $g(\mathbf{x}_k^*)$, which helps judge whether $\mathbf{x}_k^*$ is close enough to the true MPFP. A ratio factor C_r is introduced to measure the closeness of $\mathbf{x}_k^*$ to limit state surface, as

$$C_r = \frac{g(\mathbf{x}_k^*)}{g(\boldsymbol{\mu})} \tag{20}$$

where $\boldsymbol{\mu}$ is the mean of random variables. If C_r is smaller than the threshold C_r^0 (always set as 0.05), then $\mathbf{x}_k^*$ is considered to be close enough to the limit state surface, and the following single point is added to the existing sampling points:

$$\mathbf{x}_{add}^{k} = \boldsymbol{\mu} - g(\boldsymbol{\mu})\frac{\boldsymbol{\mu} - \mathbf{x}_k^*}{g(\boldsymbol{\mu}) - g(\mathbf{x}_k^*)} \tag{21}$$

where $\mathbf{x}_{add}^{k}$ is the newly added experimental design in $k+1$th iteration.

On the other hand, when C_r is larger than the threshold, it means that $\mathbf{x}_k^*$ is relatively far away from the limit state surface. In such a case, sampling points are added, starting from $\mathbf{x}_k^*$, close to the limit state surface in the direction of each variable (see Fig. 4). By doing this, the accuracy of the surrogate model is improved locally near the current MPFP and the limit state surface. Unlike other surrogate models, the MLS has no expensive inner parameter optimization during the adaptive modeling process, so a temporary DWMLS model $\tilde{g}_{N_k+1}(\mathbf{x})$ can be established quickly based on existing N_k+1 points. The distance between $\mathbf{x}_k^*$ and the limit state surface in the i-th variable direction, Δ_i, is calculated using the first-order Taylor series expansion, as

$$\Delta_i = -\frac{g(\mathbf{x}_k^*)}{\left.\dfrac{\partial \tilde{g}_{N_k+1}(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}_k^*}} \tag{22}$$

Therefore, the following n+1 samples are added at the $k+1$th experimental design:

$$\begin{gathered} \mathbf{x}_k^* \\ \mathbf{x}_k^* + [\Delta_1, 0, \ldots, 0] \\ \mathbf{x}_k^* + [0, \Delta_2, \ldots, 0] \\ \vdots \\ \mathbf{x}_k^* + [0, 0, \ldots, \Delta_n] \end{gathered} \tag{23}$$

The first-order approximation in (22) can have a large error when the performance function is highly nonlinear and the partial derivative becomes too small. Therefore, in order to prevent Δ_i from being too large, a threshold $\Delta_i^c = t \times \sigma_i$ is proposed, where σ_i is the standard deviation of i-th random variable and t is a constant between 2.0 and 3.0. Hence the adjusted sampling points become:

$$\mathbf{x}_{add_i}^{k} = \begin{cases} \mathbf{x}_k^* + [0, \ldots, \Delta_i, \ldots, 0] & \Delta_i \le \Delta_i^c \\ \mathbf{x}_k^* + [0, \ldots, \mathrm{sgn}(\Delta_i)\Delta_i^c, \ldots, 0] & \Delta_i > \Delta_i^c \end{cases} \tag{24}$$

After n+1 additional experimental design points are determined, the performance functions are evaluated at these locations. After updating the MWMLS with more points, FORM is used to determine a new $\mathbf{x}_{k+1}^*$.

The adaptive experimental design process repeats until the limit state surface can be approximated accurately, particularly in the region near the MPFP. The following two convergence criteria are used:

$$\begin{cases} \left|\beta_k - \beta_{k+1}\right| \le \varepsilon_\beta \\ \left\|\mathbf{x}_k^* - \mathbf{x}_{k+1}^*\right\| \le \varepsilon_{MPFP} \end{cases} \tag{25}$$

where the two tolerances, ε_β and ε_{MPFP}, are fixed at 10^{-3}.

V. Procedure of the proposed method

The proposed two-stage hybrid reliability analysis method can be divided into (1) DWMLS enhanced FORM iterations and (2) MCS to calculate the failure probability. In the first stage, the DWMLS surrogate model is constructed and updated by adaptively adding sampling points near the limit state surface. In the second stage, MCS is performed on the final DWMLS to calculate the probability of failure. Fig. 2 summarizes the step-by-step procedures of the proposed algorithm.

(1). Determine the initial experimental designs, $\mathbf{X}_{\text{initial}}$:

The initial number of samples is determined by $N_1 = 3n$ where n is the number of random variables. The initial optimal LHD is generated in standard normal space U of random variables, so it is necessary to define the sampling range in U space. The sampling range on each dimension is selected by $[\mu - f\sigma_i , \mu + f\sigma_i]$ where f is fixed at 4.0. Then, an optimal LHD U_{initial} is generated through the SURROGATES Toolbox[37]. The independent experimental points in U_{initial} are transferred into the physical space of mutually correlated non-normal random variables by Nataf's rule, and the initial experimental design $\mathbf{X}_{\text{initial}}$ is determined. Since the mean point $\overline{\mathbf{x}}$ is not included in $\mathbf{X}_{\text{initial}}$, $\overline{\mathbf{x}}$ is added to $\mathbf{X}_{\text{initial}}$; thus, the number of initial samples is $N_1 + 1$.

(2). Compute the value of the performance function at each point of $\mathbf{X}_{\text{initial}}$:

$$g_k = g(\mathbf{x}_k), \quad k = \{1, 2, ..., N_1 + 1\} \tag{26}$$

(3). Calculate weight factors assigned to each point according to Eq. (8), and fit the first conventional MLS. Apply FORM based on the first surrogate model and determine the first reliability index β_1 and MPFP $\mathbf{x}_1^*$.

(4). Calculate the values of the performance function at the current MPFP and check the closeness ratio factor C_r by Eq. (20). If C_r is smaller than the threshold value, obtain the new adding point $\mathbf{X}_{add}$ using Eq. (21) and skip to step (7).

(5). Add the new MPFP observation evaluated from the above step into existing sampling points and constructing a temporary DWMLS to estimate the partial derivatives to each random variables on MPFP.

(6). Calculate the length Δ_i by Eq. (22) on each direction, compare with Δ_i^c and then determine $\mathbf{x}_{add_i}$ based on Eq. (24), and add to $\mathbf{X}_{add}$.

(7). Calculate the values of the performance function at given $\mathbf{X}_{add}$ and add these points into the existing sampling points.

(8). Calculate the weight factors assigned to each point according to Eq. (15) and build a DWMLS model.

(9). Apply the FORM algorithm to the DWMLS model and determine the reliability index β_{HL} and MPFP.

(10). Repeat step (4) ~ step (9) until the convergence criteria in Eq. (25) is satisfied.

(11). Perform MCS using DWMLS to calculate the probability of failure.

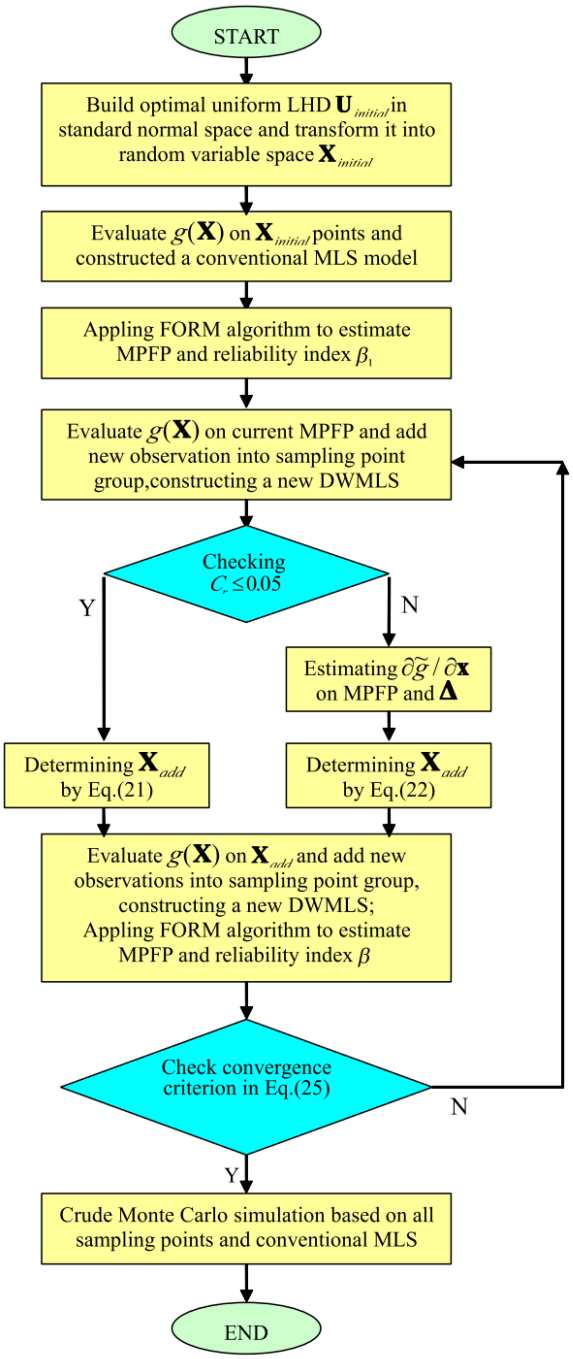

Figure 2. Flow chart for the proposed doubly weighted moving least square method for reliability analysis

VI. Numerical examples

In this section, four examples involving explicit and implicit functions from structural applications are presented to illustrate the efficiency and accuracy of the proposed DWMLS method. Since it is assumed that numerical simulation is far more expensive than developing surrogate models, the number of numerical simulations N_s is used as a measure of efficiency. The proposed DWMLS method is a hybrid reliability analysis approach where first, adaptive experimental design combined with FORM algorithm is mainly aimed at positioning the MPFP, and then, subsequent Monte Carlo simulation is used to estimate the failure probability p_f. Therefore, the accuracy of the

proposed method would not only depend on MCS for p_f, but also the interim FORM. In numerical examples, comparisons have been made with the other reliability analysis methods presented in literature to evaluate the performance of the proposed method.

A. A nonlinear limit state function

In the first numerical example, the following two-dimensional performance function is used:

$$g(\mathbf{x}) = \exp[0.4(x_1 + 2) + 6.2] - \exp[0.3x_2 + 5] - 200 \tag{27}$$

where x_1 and x_2 are independent, standard normal random variables. This example is widely used in reliability analysis methods[11,12,27,38].

Figure. 3 ~ Fig. 5 illustrate the process of the DWMLS algorithm. Figure. 3 shows the initial six LHD samples along with the true limit state surface. Figure. 4 shows the initially estimated MPFP1 along with two additional sampling points. Figure. 5 shows two more iterations with MPFP2 and MPFP3 in a partially enlarged drawing. It is clear that the estimated MPFP converges to the true MPFP.

The iteration history of the MPFP search using the proposed method is shown in Table 1. The performance function is evaluated twelve times during the three iteration cycles before arriving at the final convergence. Table 2 compares the results from the proposed method with that of various methods from the literature. It should be noted that the exact p_f is estimated by the Monte Carlo method with one million samples and the exact solution of β_{HL} and MPFP is estimated by the Hasofer-Lind algorithm. The comparison of P_f aims at accuracy of the proposed method, while the comparison of β_{HL} and MPFP indicate the performance of DWMLS in positioning and converging to the true MPFP.

As it can be seen, the results of the reliability analysis are close to each other for all methods under consideration. This is mainly because the nonlinearity of the problem in the vicinity of the design point is relatively low. The proposed method predicts very accurate estimates of the reliability index and design point when compared with the FORM method. However, a slight difference in the failure probability prediction is observed. The coefficient of variation (COV) δ of the estimated failure probability P_f by direct MCS with sample size of N_s is

$$\delta = \sqrt{\frac{1 - P_f}{N_s P_f}} \tag{28}$$

When one million samples are used, the COV of P_f is 0.017. The same number of samples is used to calculate the failure probability in DWMLS. The difference in P_f between MCS and the proposed DWMLS is within the range of COV. Compared with other approaches, the hybrid DWMLS method has demonstrated not only the accuracy in positioning the MPFP and estimating reliability index β_{HL}, but also the efficiency as only twelve function evaluations are needed in achieving it.

Table 1: Iteration history of DWMLS method -Example #1

Iteration	β_{HL}	**MPFP**	*Ns*
1	2.229	[-2.116, 0.699]	7
2	2.701	[-2.522, 0.968]	3
3	2.710	[-2.547, 0.924]	2

Table 2: Summary of results-Example #1

Method	β_{HL}	**MPFP**	p_f	*Ns*
Monte Carlo simulation	2.685	Unavailable	**3.68E-3 (1.7%)***	1E6
FORM (H-L algorithm)	**2.710***	**[-2.540,0.945]***	3.37E-3	27
RSM in Ref. 27	2.710	[-2.541,0.942]	3.36E-3	12
RSM in Ref. 12	2.707	[-2.567,0.860]	3.39E-3	12
RSM in Ref. 18	2.686	[-2.558,0.820]	3.62E-3	8
RSM in Ref. 38	2.710	[-2.538,0.951]	3.36E-3	21
DWMLS + MCS	2.710	[-2.547,0.924]	3.61E-3	12

* The results with bold font are regarded as exact results to compare with.

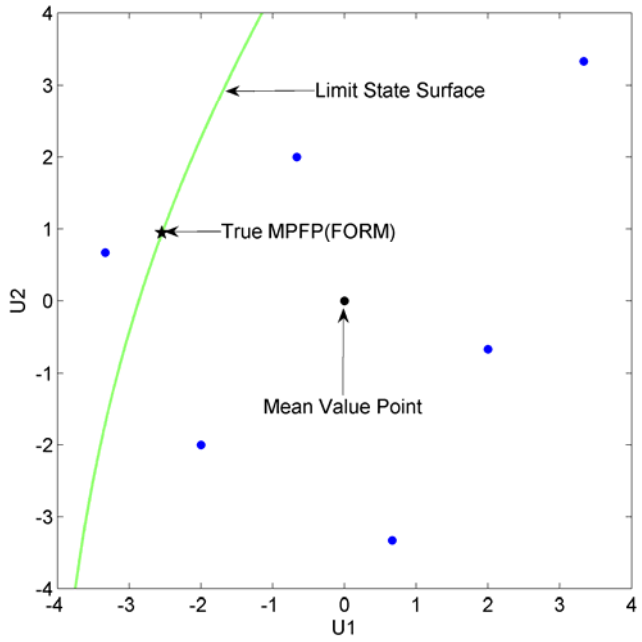

Figure 3. Initial LHD and mean value point for example 1 (Step 1)

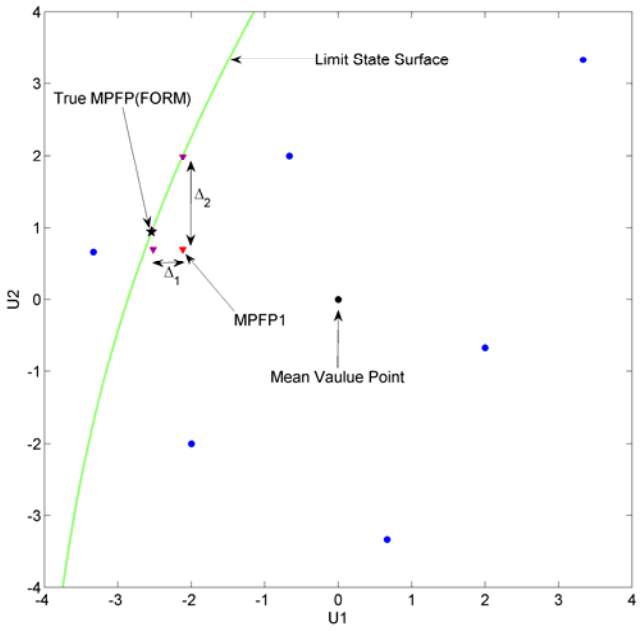

Figure 4. Illustration of adding *n* new experimental points based on previous MPFP for example 1(Step 6)

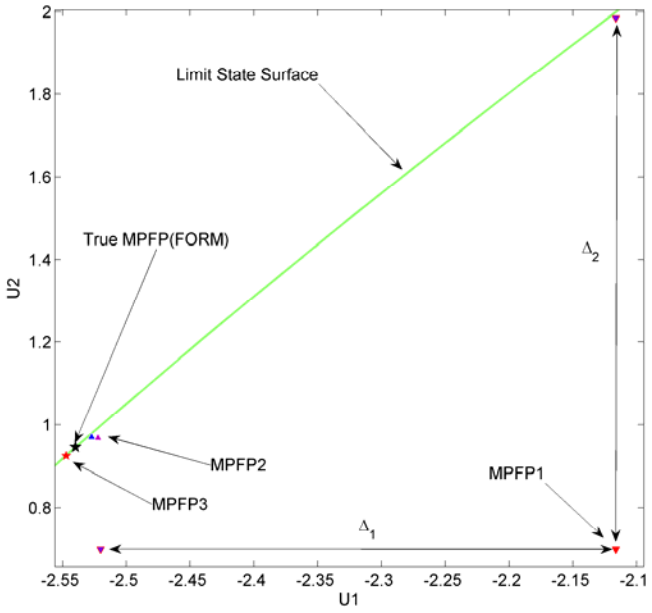

Figure 5. Illustration of adding one new experimental point based on previous MPFP for example 1(Step 4)

B. Example 2: Dynamic response of a nonlinear oscillator

In order to investigate the performance of the proposed method in complex problems with more random variables and greater non-linearity, the second example deals with a nonlinear undamped single degree of freedom system as presented in Fig. 6. This example is also used in several other studies[6,20,39]. The nonlinear oscillator with random system parameters subjected to a rectangular pulse load with random duration and amplitude is presented, and the performance function is defined by:

$$\begin{aligned} g\left(c_1, c_2, m, r, t_1, F_1\right) &= 3r - \left|z_{\max}\right| \\ &= 3r - \left|\frac{2F_1}{mw_0^2}\sin\left(\frac{w_0 t_1}{2}\right)\right| \end{aligned} \tag{29}$$

where $w_0 = \sqrt{(c_1 + c_2)/m}$. The random parameters of six basic variables are listed in Table 3.

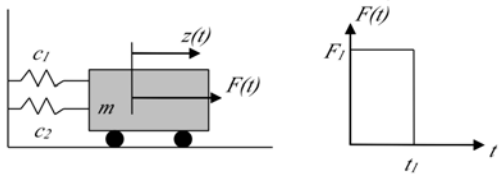

Figure 6. Non-linear oscillator - system definition and applied load

Table 3: Probabilistic distribution of random variables - Example 2

Random variable	Distribution Type	Mean Value	Standard Deviation
c_1	Normal	1.0	0.10
c_2	Normal	0.1	0.01
m	Normal	1.0	0.05
r	Normal	0.5	0.05
t_1	Normal	1.0	0.20
F_1	Normal	1.0	0.20

Table 4 shows the reliability analysis results from different methods in the literature along with the proposed DWMLS method. The reported results come from directional sampling (DS) and importance sampling (IS) methods combined with typical surrogate models (e.g., polynomials, splines, and neural network), and an active learning method combining Kriging and Monte Carlo simulation (AK-MCS). It can be seen that the DWMLS method can get competitive accuracy with the least amount of simulations, $Ns = 43$. It is noted that the results reported by Schueremans and Gemert [39] and by Echard and Gayton[20] are based on interpreting the last column of Table 3 as a standard deviation, not as a COV.

Table 4: Summary of reliability results-Example #2

Method	Ns	p_f	β_{HL}
Monte Carlo Method	1E6	3.89E-2(1.9%)	1.76
FORM	84	4.29E-2	1.72
Directional Sampling(DS)*	1281	3.5E-2	1.81
DS+Polynomial*	62	3.4E-2	1.83
DS+Spline*	76	3.4E-2	1.83
DS+Neural Network*	86	2.8E-2	1.91
Importance Sampling(IS)*	6144	2.7E-2	1.93
IS+Polynomial*	109	2.5E-2	1.96
IS+Spline*	67	2.7E-2	1.93
IS+Neural Network*	68	3.1E-2	1.87
MCS**	7E4	2.83E-2(2.2%)	1.91
AK-MCS+U**	58	2.83E-2	1.91
AK-MCS+EFF**	45	2.85E-2	1.90
DWMLS+MCS	43	3.94E-2	1.73

* the results come from Ref. 39
** the results come from Ref. 20

C. Example 3: A cantilever beam

The above two examples demonstrated the performance of the proposed method in estimating the failure probability and locating the MPFP. The proposed method can also be applied to multiple performance functions; i.e., calculating the system probability of failure. In such a case, adaptive experimental design is applied to each performance function .

The third example considers system reliability with two limit state surfaces. A cantilever beam, as shown in Fig. 7, is subjected to a tip load of 200.0 N. Two failure criteria are considered: (i) the displacement at the tip of the beam should be less than 0.005m, as expressed in Eq. (30), and (ii) maximum stress in the beam should be less than 33 MPa, as expressed in Eq. (31).

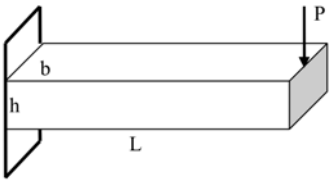

Figure 7. Simply supported beam

$$\text{Displacement limit state}: \; g_1(X) = 0.005 - \frac{4PL^3}{Ebh^3} \le 0 \tag{30}$$

$$\text{Stress limit state}: \; g_2(X) = 33.0 \times 10^6 - \frac{12PL}{bh^2} \le 0 \tag{31}$$

In the above equations, L, b, h indicate the length, width and height of the beam, whose probability distribution properties are shown in Table 5. The modulus of elasticity of the beam was taken to be 70.0 GPa. The system probability of failure is defined by:

$$p_f = 1 - P[g_1(\mathbf{X}) \geq 0 \cup g_2(\mathbf{X}) \geq 0] \tag{32}$$

Table 5: Probabilistic characteristics of the basic random variables of the cantilever beam

Random Variable	Mean(m)	Standard Deviation	Distribution type
L	0.90	0.090	Normal
b	0.08	0.008	Normal
h	0.04	0.004	Normal

As the closed form expressions of performance functions are available, it is possible to estimate system reliability with Monte Carlo simulations. Table 6 listed the results of MCS with a million samples and the result estimated by the proposed DWMLS method. It should be noted that 98 samples in the DWMLS method include all the points used to locate two limit state surfaces and corresponding two MPFPs until convergence. In order to inspect the accuracy and efficiency of the DWMLS method on finding each MPFP, the FORM algorithm is used for each limit state function to estimate an individual reliability index β for comparison in Table 7. It can be concluded that the highly nonlinear limit state functions, particularly the first, can be positioned accurately without a large number of samples.

Table 6: Comparison of failure probability for the cantilever beam system

Method	p_f	Ns
Monte Carlo simulation	8.33E-3(1.09%)	1E6
DWMLS + MCS	8.98E-3	98

Table 7: Comparison of reliability index for the cantilever beam system

Method	**g_1**		**g_2**	
	Ns	β_{HL}	Ns	β_{HL}
FORM	803	2.499	24	2.516
DWMLS + MCS	--	2.549	--	2.544

D. Example 4: Ten-bar truss structure

As a practical example using finite element analysis, a ten-bar truss structure (as shown in Fig. 8) is considered. The ten-bar truss structure is a classical structural analysis problem and widely studied[27,40]. The structure is simply supported at nodes 1 and 4, and is subjected to two concentrated loads P = 10^5lb at nodes 2 and 3. The truss members, which have random cross-sectional areas $A_i, i = 1,2,...,10$, are made of an aluminum alloy with Young's modulus E=10^7psi. The input random variables $\mathbf{X} = \{A_1, A_2, ..., A_{10}\}^T$ follow normal distribution and have a mean $\mu = 2.5\ in^2$ and standard deviation $\sigma = 0.5\ in^2$. The maximum vertical displacement $v(\mathbf{X})$, which occurs at node 3, is limited to $v_0 = 18\ in$. Therefore, the performance function is defined as:

$$g = v_0 - v_{\max}(\mathbf{X}) = 18.0 - v_{\max}(\mathbf{X}) \tag{33}$$

The structural analysis was done in MSC/NASTRAN, a commercial finite element analysis program. The estimation results are reported in Table 8.

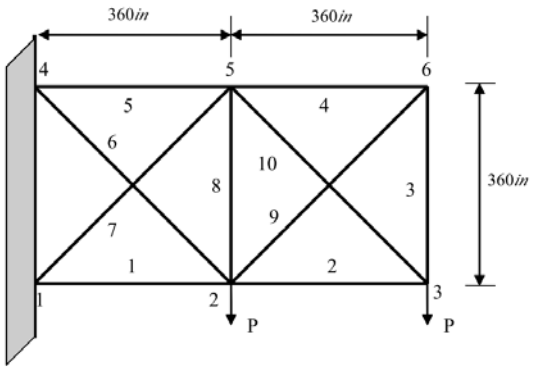

Figure 8. A ten-bar truss structure

Table 8: Failure probability of ten bar truss

Method	*Ns*	p_f	β_{HL}
Monte Carlo method	10E6	0.139(0.25%)	1.083
FORM(HL algorithm)	127	0.086	1.364
SORM(Breitung)	506	0.129	1.142
DWMLS + MCS	98	0.146	1.369

Apparently, the performance of the DWMLS method performs well in this high dimensional reliability analysis problem. The failure probability of the DWMLS method is comparable with other methods and demands the least number of simulations.

VII. Conclusions

In this paper, a doubly weighted moving least squares and a two-stage hybrid reliability analysis scheme are proposed to improve the surrogate model for reliability analysis. The proposed method provides a larger weight to the point near the MPFP and adds more samples closer to the limit state surface. From the first two benchmark examples, it was shown that, in comparison to classical MCS and FORM with various surrogate models, DWMLS improves the convergence speed, and locates the limit state function more accurately with a less number of sampling points. In the cantilever beam example, the proposed two-stage reliability analysis scheme was able to calculate system reliability by identifying multiple MPFPs.

However, it should be noted that the proposed method is not intended as a replacement of existing surrogate models in reliability analysis, but as a possible complement and improvement to these methods. Furthermore, more studies are needed to extend the proposed method to reliability based design optimization of complex systems.

Acknowledgements

The study reported in this paper was funded by China Scholarship Council under support number 2010623114. The authors also thank Prof. Raphael T. Haftka and Dr Felipe A. C. Viana for their valuable comments.

References

[1] Ditlevsen, O., Madsen, H., *Structural reliability methods*. Wiley, 1996, Hoboken.

[2] Hurtado, J. E., *Structural reliability: Statistical learning perspectives*. Springer, 2004, Heidelberg.

[3] Bucher, C. G., Bourgund, U., "A fast and efficient response surface approach for structural reliability problems," *Structural Safety,* Vol. 7, 1991, pp.57-66.

[4] Rachwitz, R., "Reliability analysis-a review and some perspectives," *Structural Safety,* Vol. 23, 2001, pp. 365-395.

[5] Wong, F. S., "Slope reliability and response surface method,". *Journal of Geotechnical Engineering,* Vol. 111, 1985, pp. 32-53.

[6] Rajashkhar, M., Ellingwood, B., "A new look at the response approach for reliability analysis". *Structural Safety,* Vol. 12, 1993, pp. 205-20.

[7] Queipo, N., Haftka, R.T., Shyy, W., Goel, T., Vaidyanathan, R., Tucker, P., "Surrogate-based analysis and optimization". *Progress in Aerospace Sciences,* Vol. 41, 2005, pp.1-28.

[8] Kim, S, Na, S., "Response surface method using vector projected sampling points". *Structural Safety*, Vol. 19, 1997, pp. 3-19.

[9] Das, P., Zheng, Y., "Cumulative formation of response surface and its use in reliability analysis". *Probabilistic Engineering Mechanics,* Vol. 15, 2000, pp. 309-15.

[10] Guan, X.L., Melchers, R.E., "Effect of response surface parameter variation on structural reliability estimates". *Structural Safety,* Vol. 23, 2001, pp.429-44.

[11] Kaymaz, I., McMahon, C. "A response surface method based on weighed regression for structural reliability analysis". *Probabilistic Engineering Mechanics,* Vol.20, 2005, pp.11-17.

[12] Nguyen, X.S., Sellier, A., Duprate, F., Pons, G., "Adaptive response surface method based on a double weighted regression technique". *Probabilistic Engineering Mechanics,* Vol. 24, 2009, pp.135-143.

[13] Choi, S.K., Grandhi, R., Canfield, R., "Reliability Analysis of a Large Computational Model Using Polynomial Chaos Expansion". *SAE 2003 World Congress*, Detroit, Michigan, 2003.

[14] Kim, N.H., Wang, H., Queipo, N.V., "Adaptive reduction of random variables using global sensitivity in reliability-based optimisation", *Int. J. Reliab. Safety,* Vol. 1(1/2), 2006, pp. 102-115..

[15] Papadrakakis, M., Papadopoulos, V., Lagaros, N.D., "Structural reliability analysis of elastic-plastic structures using neural networks and Monte Carlo simulation" . *Comput. Methods Appl. Mech. Engrg,* Vol. 136, 1996, pp. 145-163.

[16] Hurtado, J.E., Alvarez, D.A., "Neural-network-based reliability analysis: A comparative study". *Comput. Methods Appl. Mech. Engrg,* Vol.191, 2001, pp. 113-132.

[17] Gomes, H.M., Awruch, A.M., "Comparison of response surface and neural network with other methods for structural reliability analysis". *Structural Safety,* Vol. 26, 2004, pp. 49-67.

[18] Kaymaz, I., "Application of Kriging method to structural reliability problems". *Structural Safety,* Vol. 27, 2005, pp.133-151.

[19] Panda, S.S., Manohar, C.S., "Applications of meta-models in finite element based reliability analysis of engineering structures". *Computer Modeling in Engineering Sciences,* Vol. 28, 2008, pp. 161-184.

[20] Echard, B., Gayton, N., Lemaire, M., "AK-MCS: An active learning reliability method combining Kriging and Monte Carlo Simulation". *Structural Safety,* Vol. 33, 2011, pp. 145-154.

[21] Most, T., "An Adaptive Response Surface Approach for Structural Reliability Analyses based on Support Vector Machines". *Proceedings of the Eleventh International Conference on Civil, Structural and Environmental Engineering Computing*, B.H.V. Topping, 2007.

[22] Guo, Z., Bai, G., "Application of Least Squares Support Vector Machine for Regression to Reliability Analysis". *Chinese Journal of Aeronautics,* Vol. 22, 2009, pp.160-166.

[23] Lancaster, P., Salkauskas, K., "Surfaces generated by moving least squares methods". *Math Comp,* Vol. 37, 1981, pp.141-158.

[24] Zadeh, P.M., Toropov, V.V., Wood, A.S., "Use of Moving least squares method in collaborative optimization". *6th World Congresses of Structural and Multidisciplinary Optimization*, Rio de Janeiro Brazil,2005.

[25] Krishnamurthy, T., "Comparison of Response Surface Construction Methods for Derivative Estimation Using Moving Least Squares, Kriging and Radial Basis Functions". *46th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*. Austin, Texas, 2005, pp. 1-28.

[26] Bucher, C., Most, T., "A comparison of approximate response functions in structural reliability analysis". *Probabilistic Engineering Mechanics,* Vol. 23, 2008, pp. 154-163.

[27] Kang, S.C., Koh, H.M., Choo, J.F. "An efficient response surface method using moving least squares approximation for structural reliability analysis". *Probabilistic Engineering Mechanics,* Vol. 25, 2010, pp. 365-371.

[28] Song, C.Y., Lee, J., Choung, J.M. "Reliability-based design optimization of an FPSO riser support using moving least squares response surface meta-models". *Ocean Engineering,* Vol. 38, 2011, pp. 304-318.

[29] Faravelli, L. "Response□Surface Approach for Reliability Analysis". *Journal of Engineering Mechanics,* Vol. 115, 1989, pp. 2763-2781.

[30] Cheng, J., Li, Q., Xiao, R., "A new artificial neural network based response surface method for structural reliability analysis". *Probabilistic Engineering Mechanics,* Vol. 23, 2008, pp. 51-63.

[31] Simpson, T.W., Poplinski, J.D., Koch, P.N., Allen, J.K., "Metamodels for computer based Engineering Design: Survey and recommendations". *Engineering with Computers,* Vol. 17, 2001, pp.129-150.

[32] McKay, M.D., Bechman, R.J., Conover, W.J., "A comparison of three methods for selecting values of input variables from a computer code". *Technometrics,* Vol. 21, 1979, pp. 239-245.

[33] Iman, R.L., Conover, W.J., "Small sample sensitivity analysis techniques for computer models, with an application to risk assessment". *Communications in Statistics, Part A. Theory and Methods,* Vol. 17, 1980, pp. 1749-1842.

[34] Forrester, A., Sobester, A., Keane, A., *Engineering Design via Surrogate Modelling: A Practical Guide*. Wiley, 1990, Chichester.

[35] Viana, F.A.C., Venter, G., Balabanov, V., "An algorithm for fast optimal Latin hypercube design of experiments". *Int J Numer Methods Eng,* Vol. 82, No. 2, 2010, pp.135-156.

[36] Jin, R., Chen, W., Sudjianto, A., "An efficient algorithm for constructing optimal design of computer experiments". *Journal of Statistical Planning and Inference,* Vol.134, 2005, pp. 268-287.

[37] Viana, F.A.C., SURROGATES Toolbox User's Guide, Version 2.1, 2001 .

[38] Duprat, F., Sellier, A., "Probabilistic approach to corrosion risk due to carbonation via an adaptive response surface method". *Probabilistic Engineering Mechanics,* Vol. 21, 2006, pp. 207-216.

[39] Schueremans, L., Gemert, D. V., "Benefit of splines and neural networks in simulation based structural reliability analysis". *Structural Safety,* Vol.27, 2005, pp. 246-261.

[40] Wei, D., Rahman, S., "Structural reliability analysis by univariate decomposition and numerical integration". *Probabilistic Engineering Mechanics,* Vol.22, 2007, pp. 27-38.

[41] Youn, B.D., Choi, K.K., "A new response surface methodology for reliability-based design optimization". *Computers and Structures,* Vol.82, 2004, pp.241-256.

[42] Liu, Y., Moses, F., "A sequential response surface method and its application in the reliability analysis of aircraft structural system". *Structural Safety*, Vol. 16, 1994, pp.39-46.