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Comparison of Methods for Calculating *B*-Basis Crack Growth Life Using Limited Tests

K. S. Bhachu,* R. T. Haftka,† and N. H. Kim‡
University of Florida, Gainesville, Florida 32611

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Large sampling uncertainty is generally introduced in the calculation of a low percentile of fatigue crack growth life due to a small number of coupon tests. It is often desirable to estimate a low percentile (for example, 10th percentile) with a certain coverage probability (for example, 95%) using the confidence bound approach. An equally competing objective is not to have overly conservative bounds.The performance of two bootstrap-based methods are investigated for calculating a one-sided 95% confidence bound on a low percentile of lognormal and gamma fatigue crack growth life distributions. A comparison is also made with the classical tolerance interval method and nonparametric (Hanson–Koopmans) method. These confidence bounds methods are tested to estimate *B*-basis fatigue crack growth design life using material properties estimated from coupon test samples ranging from 8 to 64.

Nomenclature

a	=	crack length, in.
B	=	number of bootstrap samples
C	=	Paris constant
c	=	effective crack length, in.
d	=	hole diameter, in.
K	=	stress intensity factor, ksi $\sqrt{\text{in.}}$
k	=	tolerance interval factor
N	=	crack growth design life, flight hours
N_B	=	*B*-basis design life, flight hours
n_{ct}	=	number of coupon tests
N_t	=	crack growth test life, cycles
$N_{10\%}$	=	10th percentile design life, flight hours
n	=	Paris exponent
p	=	coverage probability
R_{mm}	=	relative mean margin
z	=	standard normal z score
α	=	confidence level
β	=	shape parameter
γ	=	percentile
η	=	scale parameter
θ	=	parameter/statistic
λ	=	location parameter
μ	=	mean
ρ	=	correlation coefficient
σ	=	standard deviation

Subscripts

B	=	*B* basis
b	=	bootstrap estimate

Superscript

$*$	=	true/population value

I. Introduction

MATERIAL testing is a key task for estimating material properties (e.g., yield strength or fatigue crack growth behavior) needed for predicting failure and designing safe and low-weight aircraft structures [1–3]. Such material properties exhibit substantial variability, so it is customary to test multiple coupons in order to estimate the statistical distribution of variability. Design then usually proceeds on the basis of a low percentile of the material property distribution estimated with a high confidence level from the tests. For example, the *B* basis is a value that bounds the true (or population) 10th percentile value with 95% confidence.

The paper extends the concept of *B*-basis values to the fatigue crack growth (FCG) life calculation, which is not what aircraft companies practice today. This is mostly due to regulations by the Federal Aviation Administration (FAA) not requiring this kind of calculation. However, with the recent push within companies to account for the uncertainty and scatter in life due to FCG rate material properties, the concept of *B*-basis FCG life introduced in this paper might be useful. The FCG life of a structural component under design has substantial variability (aleatory uncertainty) due to randomness (or scatter) in FCG rate data acquired through coupon tests. Furthermore, because of high testing costs, substantial sampling (epistemic) uncertainty exists in the estimated lower percentiles (e.g., 10th percentile) due to smaller numbers (e.g., 8–64) of FCG tests. The sampling uncertainty is usually compensated by specifying confidence bounds, e.g., one-sided lower 95% confidence bound. Now, due to small sample size, different methods could differ substantially in estimating confidence bounds on a low percentile of a distribution. A classical approach for estimating confidence bounds on percentiles (or proportions) of a statistical distribution is the tolerance interval (TI) method. For some distribution types (e.g., normal, lognormal, and Weibull), TI factors are available (e.g., in MIL-HDBK-17-1F [4]) to calculate exact one-sided lower confidence bounds. By exact, we mean that 95% coverage probability is achieved. Various methods for computing *B*-basis material allowable from composites material data are discussed in [4,5].

When one is unwilling to assume normal, lognormal, and Weibull distributions due to inadequate fits to the data, it is recommended in [4] to use a nonparametric method (e.g., Hanson–Koopmans method). In addition, in such cases, bootstrap confidence bound procedures could prove beneficial. The bootstrap method is a well-established procedure in the statistical community for estimation of confidence bounds, but it is not a standard practice in the aerospace engineering community. This paper introduces and illustrates the concept of *B*-basis FCG life calculation using the bootstrap procedure.

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*Doctoral Candidate, Mechanical and Aerospace Engineering; kdsbhachu@gmail.com. Student Member AIAA.

†Distinguished Professor, Mechanical and Aerospace Engineering; haftka@ufl.edu. Fellow AIAA.

‡Associate Professor, Mechanical and Aerospace Engineering; nkim@ufl.edu. Associate Fellow AIAA.

Cross et al. [6] used the bootstrap method to infer the confidence intervals for the parameters of a random process (assuming lognormal process) crack growth model. The bootstrap resampling was performed on the FCG curves rather than on individual crack measurements. Similarly, Bhachu et al. [7] used the bootstrap method to combine aleatory and epistemic/sampling uncertainties in the Walker equation material constants by resampling the FCG rate curves for 7050-T7451 and 7475-T7351 aluminum alloys. The combined uncertainties were then propagated to estimate the total uncertainty in the predicted FCG life. Bigerelle et al. [8] and Bigerelle and Iost [9] used the bootstrap method to estimate the uncertainty in Paris law material constants due to measurement noise by using FCG rate data from a single specimen. McDonald et al. [10] used a jackknife resampling (the bootstrap method is an improvement on jackknife) technique to approximate the sampling uncertainty in the parameters of the Johnson probability distribution due to sparse data. Ravishankar et al. [11] found that bootstrap resampling provided reasonable estimates of the epistemic uncertainty in the separable Monte Carlo estimates of the probability of failure. Pieracci [12] considered a problem of estimating the parameters of a Weibull distribution from limited experimental data for durability analysis and used the bootstrap method to approximate the confidence intervals for the probability of crack exceedance at a given time.

We investigate the performance of two bootstrap confidence bound methods: bias-corrected normal approximation (BCNA), and bias-corrected accelerated percentile method (BC_a). These methods are used for calculating B-basis FCG life assuming lognormal and gamma distributions. The performance of the two methods is assessed by measuring their ability to achieve the desired confidence level of 95% (i.e., do B-basis values actually bound true 10th percentile life with exactly 95% probability) and how conservative these B-basis values are relative to the true 10th percentile life.

Romero et al. [13–15] performed similar studies to test the performances of the normal TI method, the Pradlwarter–Schuëller kernel density method, the Johnson method, and the nonparametric method. These methods were used to construct two-sided 90% confidence bounds on the probability density function (PDF), ranging between 0.025 and 0.975 percentiles for normal, triangular, and uniform distributions. They found that using normal TI factors had a noteworthy advantage over other methods, even if the underlying distribution was not normal. Therefore, we compare the performance of the two bootstrap confidence bound methods (BCNA and BC_a) against the normal TI method (i.e., using normal TI factors) and the nonparametric (Hanson–Koopmans) method. The comparisons are made by assuming that the FCG design life follows either a lognormal or a gamma distribution. The variability in the FCG rate (or material constants of a crack growth model) is estimated from a small number of coupon tests, i.e., $n_{\mathrm{ct}} = \{8, 16, 32, 64\}$.

The paper is organized in seven sections. Sections II and III discuss the concept of conservative material properties and briefly explain various sources of uncertainties. The paper then presents a brief introduction to the tolerance interval method, the nonparametric method, and the bootstrap confidence bound methods. Section IV presents a formal discussion on FCG coupon testing and how approximated material properties are used for prediction of FCG design life. This is followed by Sec. V, which poses a simulation procedure of fatigue crack growth coupon testing and prediction of FCG design life based on these test samples. It also presents the metrics that are used for measuring the performance of the various confidence bound method. Section VI explains the statistical modeling of lognormal and gamma FCG life distributions. The rest of the paper then presents comparisons between different confidence bound methods in estimating B-basis FCG life values for lognormal and gamma distributions.

II. Conservative Material Properties and Various Confidence Bound Methods

The safety of aircraft structures is achieved by designing them to operate reliably in the presence of uncertainties. To account for uncertainty in material properties, Federal Aviation Regulations (14 CFR 25.613) require the use of conservative material properties. The conservative material properties (A basis or B basis) used for designing structures for static strength have two layers of conservatism. The first layer is conservative with respect to the variability (aleatory uncertainty) or randomness in material properties. It requires that a high percentile (e.g., 90% for B basis) of the population of material properties be above the design strength. Statistically, this is expressed by requiring that the design strength is a low percentile of the distribution of material strength (e.g., 10th percentile for B basis). In terms of the cumulative density function (CDF), the B basis is the strength where the CDF is equal to 0.1. The FAA does not have similar requirements for fatigue growth properties, requiring instead a factor of two in fatigue life. However, we assume that that a designer may want to estimate the B-basis values in the initial design.

Since the CDF is necessarily estimated from finite coupon test data, there is epistemic uncertainty (lack of knowledge) in the value of the estimated percentile. Confidence bounds are usually employed as a second measure to compensate for this uncertainty. For example, the B basis includes the 10th percentile values from an estimated distribution with 95% confidence. Various confidence bound methods used in this study for comparison are presented as follows.

A. Tolerance Interval Method

The tolerance interval method [16] is widely used to estimate one-sided confidence bounds needed for obtaining B-basis values. It is applicable for distributions for which the confidence bounds can be calculated exactly, which include the normal, lognormal, and two-parameter Weibull distributions. For the normal distribution, the one-sided lower confidence bound is

$$x_B = \bar{x} - k(n_{\mathrm{ct}})\bar{s} \tag{1}$$

where k is the tolerance interval factor that is a function of number of coupon tests n_{ct}; $\bar{x}$ is the sample mean, and $\bar{s}$ is the sample standard deviation.

For lognormal distribution, the one-sided confidence bound is calculated by first taking the log transformation of the sample distribution, which transforms the sample distribution into the normal distribution. The B-basis values are then found by taking the exponential of Eq. (1):

$$x_B = \exp(\bar{x}_{\ln} - k(n_{\mathrm{ct}})\bar{s}_{\ln}) \tag{2}$$

where $\bar{x}_{\ln}$ is the sample mean, and $\bar{s}_{\ln}$ is the sample standard deviation after log transformation.

B. Nonparametric Method

MIL-HDBK-17-1F [4] recommends the use of a nonparametric method for calculating B-basis values if the underlying distribution is not adequately modeled by normal, lognormal, or Weibull probability models. The random data sample $x = \{x_1, x_2, \ldots, x_n\}$ is first ranked in ascending order. To calculate a B-basis value for $n_{\mathrm{ct}} > 28$, the value of rank r corresponding to the sample size n_{ct} is determined from a table given in [4]. The B-basis value is the rth observation in the dataset. For example, in a sample of size of $n_{\mathrm{ct}} = 30$, the lowest ($r = 1$) observation is the B-basis value. However, for sample size $n_{\mathrm{ct}} \leq 28$, the nonparametric method recommended for use is the Hanson–Koopmans (H-K) method [17]. The H-K method uses the following equation to calculate B-basis values:

$$x_B = x_r \left[\frac{x_1}{x_r}\right]^k \tag{3}$$

where the values of the rank r and tolerance factor k are determined from a separate table given in [4] corresponding to a given coupon test sample of size n_{ct}.

C. Bootstrap Sampling and Confidence Intervals

Like the nonparametric method, the bootstrap method is a data-based method, but it is based on sampling the data for statistical inference. The bootstrap method was introduced by Efron in 1979 as an improvement on the jackknife method for estimating the bias and variance of a statistic of interest [18]. A good introduction to bootstrap methods is given in [19]. The two most commonly used bootstrap sampling methods are nonparametric and parametric.

1. Nonparametric Bootstrap Sampling

The nonparametric method resamples (with replacement) directly from the available sample set without assuming any probability model. It treats the available data sample as a surrogate population $x = \{x_1, x_2, x_3, \ldots, x_{n_{\mathrm{ct}}}\}$ and draws n_{ct} samples (with replacement) B times, giving B sets of bootstrap samples of size n_{ct}, e.g., $x_{b1} = \{x_1, x_3, x_3, \ldots, x_{n_{\mathrm{ct}}}\}$ and $x_{b2} = \{x_2, x_2, x_3, \ldots, x_{n_{\mathrm{ct}}}\}$. It then computes a sample parameter $\hat{\theta}_b$ (e.g., γ percentile) directly from each bootstrap sample x_b. This gives the sampling distribution of $\hat{\theta}_b$ that can be used for defining bootstrap confidence bounds.

Nonparametric resampling is beneficial if one does not have enough confidence in the underlying parametric model. This is often thought to reduce errors in estimation due to model misspecification. This method cannot be used if one is interested in computing lower percentiles (e.g., 1 or 10%) from a small data sample.

2. Parametric Bootstrap Sampling

When there is enough confidence in the underlying probability model, simulations from the model are argued to give more accurate results than resampling from the empirical distribution. The parametric bootstrap method assumes a particular probability model for a sample and generates B sets of bootstrap samples of size n_{ct} using the approximated parameters of the probability model. The bootstrap samples are further used to compute a sample statistic or parameter $\hat{\theta}_b$ (e.g., mean), which gives the sampling distribution of $\hat{\theta}_b$ that is used for computing confidence bounds. The parametric bootstrap procedure is also sometimes referred to as Monte Carlo simulation.

3. Nonparametric Bootstrap for Parametric Inference

In this paper, we are particularly interested in calculating a low percentile value (10th percentile) from a small sample size. Due to relatively small sample sizes (8–64), the tails of the empirical distribution are not well approximated. Therefore, it would be more useful to compute a low percentile by assuming a parametric probability model. That is, bootstrap samples are generated nonparametrically, but low percentile values are computed by fitting a probability distribution to bootstrap samples. Meeker and Escobar [20] described and illustrated the applications of such nonparametric bootstrap sampling for parametric inference. This technique for computing the bootstrap sampling distribution of $\hat{\theta}_b$ was referred to as nonparametric bootstrap sampling for parametric models (NBSP) by Edwards et al. [21]. They suggested that the NBSP is a useful and more defensible approach when the sample size is small, provided there is some confidence in the underlying parametric model. More detail on this method is presented in Appendix B.

4. Bootstrap Confidence Bound Methods

The various methods of finding confidence bounds from bootstrap sampling distribution are the 1) percentile method, 2) bias-corrected accelerated method, 3) bias-corrected normal approximation (BCNA), 4) approximate bootstrap confidence (ABC) interval method, and 5) bootstrap-*t* method.

We use the BC_a method and BCNA method for calculating one-sided low 95% confidence bounds of the 10th percentile (B-basis FCG life) of lognormal and gamma distributions in this paper.

5. Percentile Method

The percentile method is the most basic bootstrap method that estimates confidence bounds by simply calculating the desired percentile from the sampling distribution of $\hat{\theta}_b$. For example, a one-sided lower $\alpha = 95\%$ confidence bound would be the fifth percentile point of the distribution of $\hat{\theta}_b$. If the number of samples for bootstrapping are not enough, then a confidence bound would not be large enough to bound the true value of interest θ^* exactly 95% of the time, i.e., coverage probability p would be less than $\alpha = 95\%$. The coverage probability p of a confidence bound is the probability that the interval/bound contains the true value of interest θ^*:

$$p = \mathrm{prob.}\{\theta_{b,\alpha} \leq \theta^*\} \tag{4}$$

If the method works perfectly, then the coverage probability p is equal to the confidence level (α, e.g., 95%).

6. Bias Correction

The bootstrap method offers a way to estimate the bias in statistical parameters (e.g., mean or percentile of the CDF). Such a bias may arise due to estimation of a parameter $\hat{\theta}$ from a small size of samples, which is different from the population parameter θ. The bootstrap method estimates this bias as

$$\hat{\theta}_{\mathrm{bias}} = E(\hat{\theta}_b) - \hat{\theta} \approx \hat{\theta} - \theta \tag{5}$$

where $\hat{\theta}$ is the parameter estimated from the original samples, and $E(\hat{\theta}_b)$ is the mean of the parameter $\hat{\theta}_b$ estimated from the bootstrap method. Therefore, the bias-corrected estimate of $\hat{\theta}$ can be obtained as

$$\hat{\theta}_{\mathrm{bc}} = \hat{\theta} - \hat{\theta}_{\mathrm{bias}} = 2\hat{\theta} - E(\hat{\theta}_b) \tag{6}$$

7. Bias-Corrected Normal Approximation

If one assumes that the distribution of the estimated parameter $\hat{\theta}_b$ from the bootstrap method is normal, then the (1-α)% one-sided lower confidence bound can be set to

$$\hat{\theta}_{b,\alpha} = \hat{\theta} - z_{1-\alpha}\hat{s}_{\hat{\theta}b} \tag{7}$$

where $z_{1\text{-}\alpha}$ is the z score from the standard normal distribution (e.g., $z_{0.95} = 1.65$), and $\hat{s}_{\hat{\theta}b}$ is the estimated standard deviation from the bootstrap sampling distribution of $\hat{\theta}_b$. The bias correction from Eq. (6) can be used to modify the lower confidence bound calculated from Eq. (7) as

$$\hat{\theta}_{\mathrm{bc},\alpha} = \hat{\theta}_b^{\mathrm{bc}} - z_{1-\alpha}\hat{s}_{\hat{\theta}b} = 2\hat{\theta} - E(\hat{\theta}_b) - z_{1-\alpha}\hat{s}_{\hat{\theta}b} \tag{8}$$

The bias correction could improve the coverage probability p of the confidence bounds.

8. Bias-Corrected Accelerated Percentile Method

A detailed introduction to this method was given in [22]. The one-sided lower confidence bound is found by using the following formula:

$$\hat{\theta}_{\mathrm{BC}_a,\alpha} = \hat{G}^{-1}\Phi\left(z_0 + \frac{z_0 + z_\alpha}{1 - A(z_0 + z_\alpha)}\right) \tag{9}$$

where $\hat{G}^{-1}$ is the inverse empirical CDF of the bootstrap sampling distribution of $\hat{\theta}_b$; z_α is the z score from the standard normal distribution; z_0 is the bias correction; A is the acceleration parameter, and Φ is the standard normal CDF. For $z_0 = 0$ and $A = 0$, Eq. (9) reduces to the basic percentile method

$$\hat{\theta}_{\mathrm{BC}_a,\alpha} = \hat{G}^{-1}(\alpha) \tag{10}$$

The purpose of z_0 is the same as the bias correction presented previously. The acceleration parameter A is another adjustment that is used to correct for the accelerating standard error on the normalized scale. A detailed discussion on the interpretation and calculation of these parameters is presented in [22].

9. *Approximate Bootstrap Confidence Interval Method*

The ABC method approximates the BC_a confidence bounds analytically. This reduces the computational burden, as Monte Carlo simulations are not required for estimation of confidence bounds. It works by approximating the bootstrap random sampling results by Taylor series expansions. It is reported to generate good approximations to BC_a intervals for most reasonably smooth statistics. Refer to [22] for details.

10. *Bootstrap-t Method*

This method is conceptually simpler than BC_a and it uses Student's t statistic for setting confidence bounds. The student t statistic is

$$T = \frac{\hat{\theta} - \theta}{\hat{\sigma}} \tag{11}$$

where $\hat{\sigma}$ is the estimate of standard deviation of $\hat{\theta}$. The α-level one-sided confidence interval for θ is

$$\hat{\theta} - \hat{\sigma} T^{1-\alpha} \tag{12}$$

where T^{α} is the percentile of t distribution that, in most cases, is unknown. The idea of the bootstrap-t method is to estimate the percentile of T by bootstrapping. This method is reported to have some numerical stability issues. For more details on this method, refer to [22].

We have used MATLAB's "*bootci*" function to calculate one-sided 95% lower confidence bounds on the 10th percentile of FCG life distribution.

III. Bootstrap and Probability Model Selection

The normal TI factors would give exact confidence bounds after log transformation of samples from a lognormal distribution. However, since TI factors specific to a gamma distribution are not available, it would be interesting to evaluate the performance of normal TI factors for approximating B-basis values. The unavailability of TI factors for a gamma distribution makes bootstrap methods attractive. The class of bootstrap resampling technique implemented in this paper is "nonparametric bootstrap for parametric inference." That is, resampling is done nonparametrically from a small number of samples without any assumption of probability distribution but 10th percentile values are computed from bootstrap samples (or from resamples) by assuming a particular probability model. An assumption of a probability distribution is necessary for approximation of low percentiles by extrapolating the empirical CDF into a tail region (see Appendix B). In addition, with a small sample size, one could also select a wrong probability model that could lead to erroneous inference. Therefore, we use the two bootstrap methods to calculate B-basis values for the following scenarios:

1) If a sample belongs to a lognormal distribution (i.e., underlying parent distribution), the following may occur:

a) The underlying distribution is detected correctly, i.e., a lognormal distribution is chosen (see Fig. B1a in Appendix B).

b) The underlying distribution type is detected incorrectly, i.e., a gamma distribution is chosen.

2) If a sample belongs to a gamma distribution (i.e., underlying parent distribution), the following may occur:

a) The underlying distribution is detected correctly, i.e., a gamma distribution is chosen (see Fig. B1b in Appendix B).

b) The underlying distribution type is detected incorrectly, i.e., a lognormal distribution is chosen.

IV. Fatigue Crack Growth Testing and Life Prediction

The FCG behavior of a material is estimated through coupon testing and is used for predicting FCG design life for a structural component. The crack length a vs load cycle N_t data obtained via testing are converted to a FCG rate $(\mathrm{d}a/\mathrm{d}N_t)$ vs stress intensity ΔK data [23]. ΔK defines the intensity of stress at a crack tip that controls the rate of crack growth, i.e., higher stress intensity values lead to larger crack growth, and vice versa. The relationship between $\mathrm{d}a/\mathrm{d}N_t$ vs ΔK data measures FCG behavior and can be treated as a material property. Many analytical crack growth models (i.e., nonlinear regression equations) [24,25] have been proposed to express the relationship between $\mathrm{d}a/\mathrm{d}N_t$ vs ΔK data. For illustration, the Paris law given in Eq. (13) is considered in this paper for modeling the FCG rate data and for the calculation of FCG life. The Paris law is typically only applicable to model the linear region of the macrocrack growth and does not take the stress ratio R effect into account. In reality, models that take the R effect into account are used for calculation of FCG life under variable-amplitude loading, e.g., the Walker equation and Nasgro equation. Such equations involve more fitting parameters that may need additional statistical modeling:

$$\frac{\mathrm{d}a}{\mathrm{d}N_t} = C(\Delta K)^n \tag{13}$$

where C is the Paris constant (intercept), and n is the Paris exponent (slope). These material constants are random, as each coupon test (under same testing conditions) gives somewhat different $\mathrm{d}a/\mathrm{d}N_t$ vs ΔK data, which depict variability due to material randomness. The variability in these material constants can be modeled with a joint probability distribution with a strong negative correlation ρ. So, the purpose of FCG coupon testing is to estimate the parameters of a joint probability distribution. With a small number of tests, the sampling uncertainty in the estimated joint distribution parameters would be substantial, which further causes considerable sampling uncertainty in the parameters of the predicted FCG design life distribution (e.g., 10th percentile FCG life is uncertain). To achieve safe designs, it may be desirable to calculate the B-basis FCG life values to compensate for the sampling uncertainty arising from a small number of tests. This is done by calculating the lower one-sided 95% confidence bound from different methods. The FCG life for a structure under design (e.g., for the geometry shown in Fig. 1 and loading conditions given in Table 1) can be predicted by evaluating the integral form of the Paris law:

$$N = \frac{1}{C} \int_{a_i}^{a_f} (\Delta K)^{-n}\, \mathrm{d}a \tag{14}$$

where N is the design life measure in flight hours (FHs), a_i is the initial crack length, and a_f is the failure/critical crack length. The ΔK solution for the through-edge crack at a hole is given in Appendix A. The loading type (Table 1) assumed in this paper is the constant-amplitude loading with $R = 0$, which is not the case in the reality where variable-amplitude loading is a typical scenario. The value of

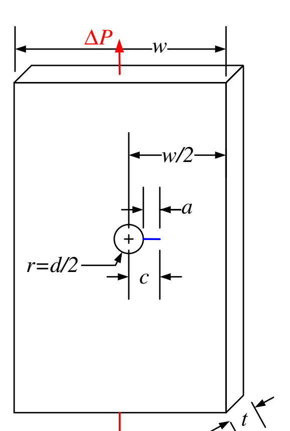

Fig. 1 Through-edge crack at centered hole in a plate under a constant-amplitude cyclic tension load.

Table 1 Dimensions and loads for the geometry shown in the Fig. 1

Dimension	Unit	Value
Width w	in.	2
Diameter d	in.	0.20
Thickness t	in.	0.4275
Initial crack length a_i	in.	0.05
Failure crack length a_f	in.	0.90
Load amplitude; $\Delta P = P_{\max} - P_{\min}$	kip	1
Stress ratio; $R = P_{\min}/P_{\max}$	— —	0

constant-amplitude loading is chosen such that life is representative of a typical goal for the wing root of a business jet [26]. Also, one FH is assumed to be equal to 1000 cycles that are based on inputs from a business jet manufacturer.

V. Simulation of FCG Testing and Life Prediction

The B-basis FCG life for a given geometry (e.g., Fig. 1) and load conditions (e.g., Table 1) is a function of available test data ($\mathrm{d}a/\mathrm{d}N_t$ vs ΔK data). That is, different testing laboratories would get different sets of coupon test data for a given number of coupon tests and materials. This gives different values of the parameters of a joint probability distribution of the material constant and that of the corresponding FCG life distribution, which would further lead to different B-basis values for the same geometry and loads. We simulate this scenario by repeatedly generating correlated random samples of C and n (of size n_{ct}) from their true joint probability distribution, given in Tables 2 and 3. Each sample of C and n represents a single FCG coupon test. The parameters of the true joint probability distribution are only known if one can perform an infinite (very large) number of tests. For our simulation, we assume that the parameters of a true joint probability distribution of C and n are known (e.g., given in Tables 2 and 3). The corresponding true distribution of FCG design life could be estimated by Monte Carlo simulation. That is, for the geometry shown in Fig. 1 (Appendix A) and loading conditions given in Table 1 (Appendix A), 20 million samples of C and n generated from the true joint distribution would approximate the true distribution of FCG life by evaluating Eq. (14). The geometry is sized such that the true 10th percentile $N^*_{10\%}$ of the resulting true FCG design life distribution is equal to 24,000 flight hours. This is done by increasing the thickness of the geometry to 0.4275 in. while keeping other dimensions fixed. The width ($w = 2$ in.) is about the size of a half-spar cap for a midsize business jet's wing, and the diameter corresponds to a no. 6 fastener. These dimensions are representative of the center wing spar shown in [26]. We use the procedure outlined in Table 4 to evaluate the performance of different methods to calculate the B-basis FCG life. Note that the bootstrapping method is performed using samples of fatigue crack growth life that are not correlated.

The performance of different methods is measured by computing their ability to achieve the desired coverage probability p but, at the same time, provide the least conservative B-basis values as follows:

1) The coverage probability p of a confidence bound (or interval) is the probability that the confidence bound contains the true value of interest. The coverage probability p and $1-\alpha$ confidence level are the same before the confidence bounds are actually calculated. If a method is working perfectly, then $p = 1-\alpha$ (e.g., $p = 1-\alpha = 0.95$ for 95% confidence bounds). The value of p corresponding to a particular sample size n_{ct} is estimated by calculating the number of B-basis life values that bound a true 10th percentile FCG life:

Table 2 True parameters of the bivariate normal distribution of material constants

Material constant	Distribution	True location λ^*	True scale η^*	True correlation ρ^*
$\log(C^*)$	Normal	−18.20	0.328	−0.982
n^*	Normal	2.872	0.165	— —

Table 3 True parameters of the joint extreme value and normal distribution of material constants

Material constant	Distribution	True location λ^*	True scale η^*	True correlation ρ^*
C^*	Extreme value	-1.223×10^{-8}	3×10^{-9}	−0.952
n^*	Normal	2.872	0.165	— —

Table 4 Simulation procedure to estimate the performance of different confidence bound methods[a]

Step	Action
1	Generate correlated random samples of C and n of size n_{ct} from a true joint probability distribution: for example, by using functions available within MATLAB that automatically takes correlation into account, e.g., *mvnrnd*.
2	Use Eq. (14) to calculate n_{ct} samples of FCG life corresponding to each sample of C and n.
3	Estimate the B-basis value of FCG life N_B using different methods, i.e., BC_a bootstrap, BCNA bootstrap, normal TI method, and nonparametric (H-K) method.
4	Check if B-basis value contains the true 10th percentile life, i.e., $N_B \le N^*_{10\%}$.
5	Repeat steps 1 through 4 (e.g., $n_{\mathrm{rep}} = 10{,}000$ times) to estimate the coverage probability, $p = \mathrm{prob.}\{N_B \le N^*_{10\%}\}$. The 10,000 repetitions are enough to clearly see the trends in performance of methods as a function of the number of tests. That is, trends will not change if the simulation is repeated.
6	Repeat steps 1 through 5 for $\{n_{\mathrm{ct}} = 8, 16, 32, 64\}$.
7	Repeat steps 1 through 6 for underlying parent lognormal and gamma FCG life distributions.
8	Repeat steps 1 through 7 for the bootstrap method if the underlying distribution is detected as lognormal and gamma.

[a]For consistency, each method computes confidence bounds from same set of data that minimizes noise due to sampling.

$$p = \frac{\# N_B \le N^*_{10\%}}{n_{\mathrm{rep}}} \approx \mathrm{Prob.}\{N_B \le N^*_{10\%}\} \tag{15}$$

where N_B is the B-basis design life, n_{rep} is the number of repetitions, and $N^*_{10\%}$ is the true 10th percentile design life.

2) The second objective is to calculate B-basis values with the least conservatism. This is measured by calculating the mean (or expected value) of margin between the B-basis life and true 10th percentile FCG life, relative to the true 10th percentile,

$$R_{\mathrm{mm}} = E(N^*_{10\%} - N_B)/N^*_{10\%} \tag{16}$$

A positive value of R_{mm} (relative mean margin) indicates a conservative estimate of FCG life.

VI. Fatigue Crack Growth Life Distributions

To test the performance of different confidence bound methods, as per the procedure listed in Table 4, we consider lognormal and gamma FCG life distributions. In general, lifetime failure data are generally modeled with distributions having a positive skew; i.e., a long right tail. Accordingly, [27–29] modeled lifetime data with a lognormal distribution. The type of FCG life distribution in our case depends on the marginal distributions of C and n. Annis [30] and Akkaram et al. [31] modeled C with a lognormal distribution and n with a normal distribution based on the test data for a 2024-T3 aluminum alloy from Virkler et al. [32]. Millwater and Wieland [33] found that an extreme value distribution was good fit to C for 7075-T735 aluminum alloy. Based on this, we consider two cases: first, when the marginal distributions of C and n are lognormal and normal, respectively, leading to lognormal FCG life distribution; and second, when the marginal distributions of C and n are the extreme value and normal, respectively, leading to gamma distribution of FCG life.

A. True Lognormal FCG Life Distribution

The true lognormal distribution of FCG life N is obtained by assuming C with a lognormal distribution [or $\log(C)$ as normal] and n with a normal distribution. The parameters describing the true joint normal distribution are listed in Table 2 and are taken from [30], which are derived from the 68 coupon tests performed by Virkler et al. [32]. The marginal true distributions of C and n along with the correlated random samples generated from their joint distribution are shown in Fig. 2. Note that each correlated random sample represents a single FCG coupon test.

The true lognormal FCG design life distribution (for the geometry and load conditions given in Appendix A) is obtained by evaluating Eq. (14) for 20 million correlated random samples of C and n generated from the bivariate normal distribution given in Table 2. The lognormal fit to these 20 million data samples of FCG design life is shown in Figs. 3a and 3b. The probability plot in Fig. 3b indicates that the lognormal distribution is a good fit to the FCG life data. The true 10th percentile FCG life value is $N^*_{10\%} = 24{,}000$ FHs. The true parameters of the lognormal PDF given in Eq. (17) are $\lambda^*(\text{location}) = 10.48$, and $\eta^*(\text{scale}) = 0.314$:

$$f_{\text{Logn}}(N) = \frac{1}{N\eta\sqrt{2\pi}} \exp\left(-\frac{(\ell n(N) - \lambda)^2}{2\eta^2}\right) \tag{17}$$

B. True Gamma FCG Life Distribution

The true gamma distribution of FCG life N is obtained by modeling C with type-1 extreme value distribution (Gumbel) and n with the normal distribution. That is, C is assumed to follow an extreme value distribution, as it has a longer right tail similar to lognormal distribution and it gives a test case where life follows gamma distribution. So, the bootstrap method is a possible option to calculate B-basis values, as tolerance interval factors are not available for gamma distribution. The parameters describing the true joint distribution are listed in Table 3. The marginal true distributions of C and n along with the correlated random samples generated from their joint distribution are shown in Fig. 4. The true parameters of the marginal distribution of C are selected so that the true 10th percentile FCG life is 24,000 flight hours, whereas the parameters of n are kept the same, as given in Table 2. Transforming the samples from a joint standard normal distribution with a correlation coefficient of −0.982 (given in Table 2) to parameters of C and n (given in Table 3) changes the correlation coefficient from −0.982 to −0.952. The PDF of the extreme value distribution is

$$f_{\text{extval}}(C) = \frac{1}{\eta} \exp\left(\frac{C-\lambda}{\eta}\right) \exp\left(-\exp\left(\frac{C-\lambda}{\eta}\right)\right) \tag{18}$$

where λ is the location parameter, and η is the scale parameter.

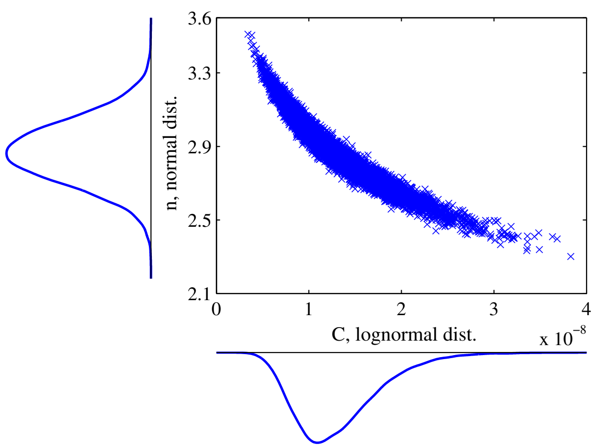

Fig. 2 Correlated random samples generated from bivariate normal distribution of $\log(C)$ and n. The marginal distribution (dist.) of C, is lognormal and n is normal.

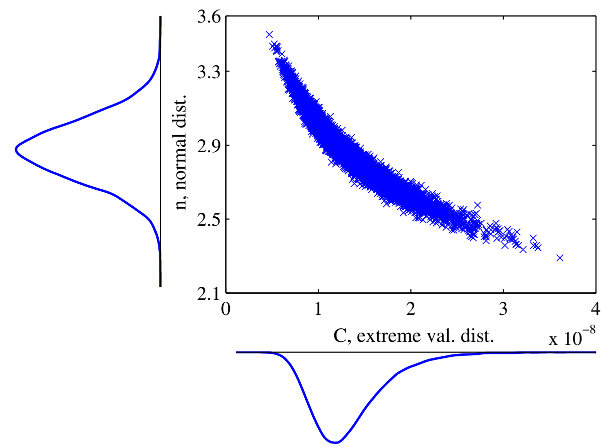

Fig. 4 Correlated random samples generated from bivariate distribution of C and n. The marginal distribution of C is extreme value (val.), and n is normal.

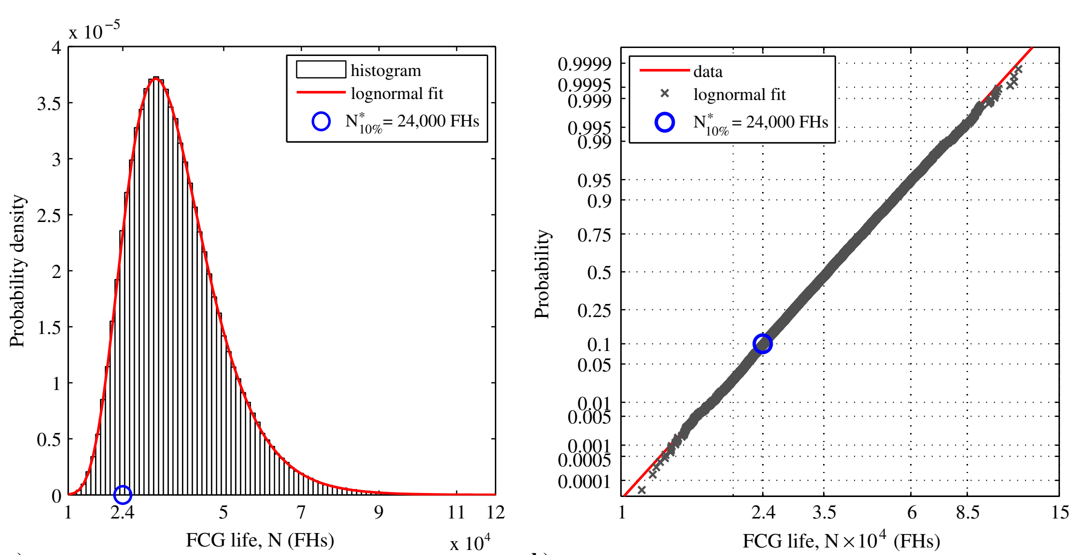

Fig. 3 Representations of a) lognormal fit to the 20 million data samples of FCG life, and b) probability plot showing lognormal distribution is a good fit to the data.

The true gamma FCG design life distribution is also obtained via Monte Carlo simulation in the same way as true lognormal distribution. The gamma fit to these 20 million data samples of FCG life is shown in Figs. 5a and 5b. The probability plot in Fig. 5b indicates that the gamma distribution is a good fit to the FCG life data. The true 10th percentile FCG life value is $N^*_{10\%} = 24,000$ FHs. The parameters of the true (fully sampled) gamma PDF given in Eq. (19) are β^*(shape parameter) $= 16.70$ and η^*(scale parameter) $= 2047$:

$$f_{\text{gamma}}(N) = \left(\frac{N}{\eta}\right)^{\beta-1} \exp\left(-\frac{N}{\eta}\right)/\eta\Gamma(\beta) \tag{19}$$

VII. Performance Comparison of Methods

The performance of the two bootstrap methods (i.e., BC_a and BCNA) is compared with the normal TI and nonparametric (H-K) methods. Both bootstrap methods assume two cases per distribution: first, when the underlying probability distributions type is identified correctly; and second, when the underlying probability distribution type is not identified correctly. The implementation of nonparametric bootstrap sampling for parametric inference is illustrated in Appendix B.

A. Lognormal Distribution Identified Correctly

If the parent lognormal distribution is identified correctly, then both BC_a and BCNA use lognormal distribution for computing 10th percentile values from bootstrap samples (see Appendix B). The performance comparisons discussed here are based on the empirical CDFs of B-basis values (obtained from the simulation outlined in Table 4) shown in Fig. C1 in Appendix C. Note that each method computes B-basis values from the same set of data to ensure fair comparison. The coverage probability p and relative mean margin R_{mm} results are shown graphically in Fig. 6, and the corresponding data are presented in Table 5 (the values out of parentheses are coverage probabilities, and the values in parentheses are relative mean margins). Table 3 and Fig. 6, however, also show the cases where the lognormal data are identified as gamma (curves marked with squares and plus signs in Fig. 6, and second and third columns in Table 3). These are discussed in the next section.

It can be noticed that the "normal TI" method achieves $p = 95\%$ (to the precision of the Monte Carlo simulation from 10,000 repetitions) in the estimated B-basis FCG life values for all test coupon sizes. This is no surprise, as normal TI factors are known to give exact B-basis values for lognormal distribution through log transformation. It could be further noticed from Table 5 (for all methods) that the coverage

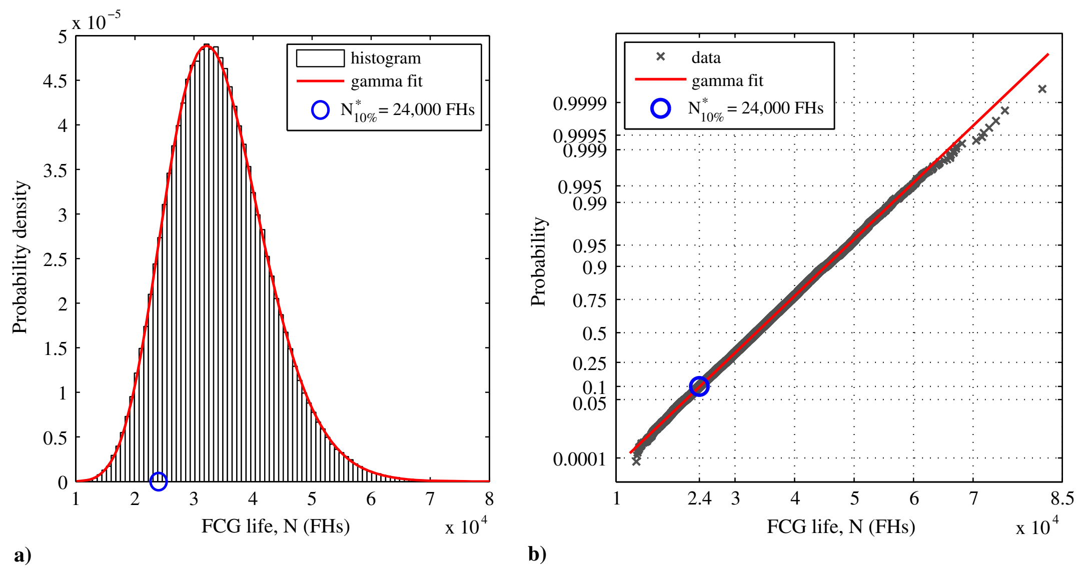

Fig. 5 Representations of a) gamma fit to the 20 million data samples of FCG life, and b) probability plot showing gamma distribution is a good fit to the data.

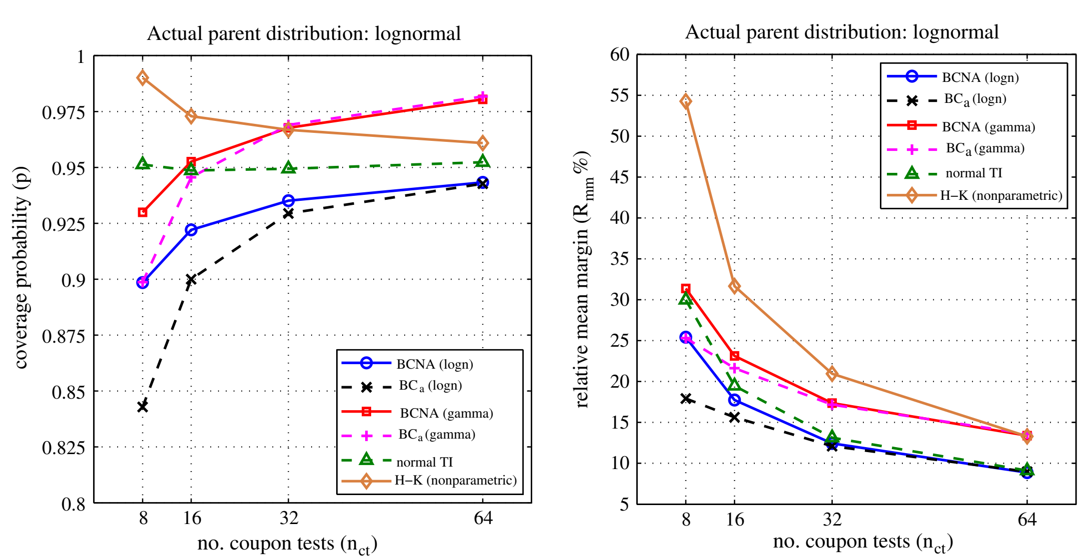

Fig. 6 Representations of a) coverage probability p, and b) relative mean margin R_{mm} % as a function of n_{ct} for different confidence bound methods, when data come from lognormal distribution.

Table 5 Coverage probability 100p% and relative mean margin R_{mm}% for data from lognormal distribution[a]

n_{ct}	BCNA, gamma	BC_a, gamma	BCNA, logn	BC_a, logn	Normal TI	Nonparametric, H-K
8	93.0 (31.4)	89.9 (25.3)	89.9 (25.4)	84.3 (17.9)	95.1 (30.0)	99.0 (54.3)
16	95.3 (23.1)	94.6 (21.6)	92.2 (17.7)	90.0 (15.6)	94.9 (19.5)	97.3 (31.7)
32	96.8 (17.4)	96.9 (17.2)	93.5 (12.4)	92.9 (12.1)	94.9 (13.1)	96.7 (20.9)
64	98.0 (13.4)	98.2 (13.6)	94.3 (8.9)	94.3 (9.0)	95.2 (9.1)	96.1 (13.3)

[a]These numbers are based on 10,000 repetitions, so there is about a 0.1–0.5% error in these proportions.

probability and relative mean margin have strong correlation, i.e., increasing p increases R_{mm} and vice versa. That is, increased coverage (more than 95%) is obtained at the cost of more conservativeness. However, for the same coverage, the conservative margin required decreases substantially with sample size. This indicates that, with more samples, a lighter structure may be designed.

In comparison, the bootstrap and nonparametric methods have substantial errors in coverage, with the nonparametric method converging from above and the bootstrap methods converging from below. However, it is clear that, if the distribution is known to be lognormal, the method of choice is the normal TI method.

B. Lognormal Distribution Identified as Gamma

Now, if the actual parent distribution (i.e., lognormal) is misidentified as gamma, then both bootstrap methods use gamma distribution for computing 10th percentile values from bootstrap samples (see Appendix B). The results of the coverage probability p and relative mean margin R_{mm} are shown in Fig. 6, and corresponding data are presented in Table 5. It can be noticed that both bootstrap methods (BC_a gamma and BCNA-gamma) obtain $p < 95\%$ for $n_{\mathrm{ct}} = 8$, and $p \geq 95\%$ for $n_{\mathrm{ct}} > 16$. Also, the values of p are safer than if distribution is identified correctly, which is also reflected in the higher R_{mm} values in comparison to BC_a-logn and BCNA-logn. The coverage probability is higher because of the bias that exists (i.e., underlying sample is from lognormal, but we are treating it as gamma). So, confidence bounds are wider that give higher coverage probability than if the distribution is correctly identified as lognormal. The bias remains even if $n_{\mathrm{ct}} = 64$, i.e., 98% coverage instead of 95%. Also, for $n_{\mathrm{ct}} > 8$, the bootstrap methods appear to provide conservative coverage more efficiently (that is, with smaller margins) than the nonparametric method. The important conclusions from the comparison of methods for estimating B-basis values for lognormal distribution are as follows:

1) The normal TI method gives the most accurate coverage probabilities and the least relative mean margins for all sample sizes. However, this is true if one has enough confidence that the underlying distribution is lognormal.

2) With incorrect identification (lognormal as gamma), the bootstrap methods obtain higher coverage probabilities (for $n_{\mathrm{ct}} > 16$) but, in terms of relative mean margin, are still better than the nonparametric method.

3) With correct identification (lognormal as lognormal), bootstrap methods are more efficient in the relative mean margin (i.e., less conservative) in comparison with incorrect identification (lognormal as gamma).

4) For large sample sizes, the nonparametric method provides good results that do not depend on identifying the right distribution.

5) There are substantial reductions in margins, and hence in design weight as the number of samples increase.

C. Gamma Distribution Identified Correctly

The following discussion presents comparisons if FCG life is modeled as a gamma distribution (as was shown in Sec. VI). The results of the coverage probability p and relative mean margin R_{mm} are shown in Fig. 6, and corresponding data are presented in Table 6. These results are derived from the empirical CDFs of B-basis values (obtained after performing the simulation outlined in Table 4) shown in Fig. C2 in Appendix C. Also, as was alluded to earlier, tolerance interval factors for gamma distribution are not available to give exact confidence bounds. So, the normal TI method might also obtain less than 95% coverage probabilities.

Once again, the nonparametric (Hanson–Koopmans) method obtains $p > 95\%$ for all values of n_{ct} but at the cost of higher R_{mm} values (or by having more conservative B-basis values; see Fig. C2 in Appendix C). In comparison, using a normal TI gives $p < 95\%$ for all values of n_{ct}, and in-fact performance declines with the increase in sample size, i.e., $p = 93\%$ for $n_{\mathrm{ct}} = 16$, and $p = 92\%$ for $n_{\mathrm{ct}} = 64$. Although, for $n_{\mathrm{ct}} \leq 16$, coverage probabilities p from the normal TI are not that bad from a practical standpoint (about 93%).

Also, both bootstrap methods (BC_a-gamma and BCNA-gamma) obtain $p < 95\%$ for $n_{\mathrm{ct}} < 64$, and $p \approx 95\%$ for $n_{\mathrm{ct}} = 64$. Further, BCNA-gamma performs significantly better than BC_a-gamma for $n_{\mathrm{ct}} \leq 16$ in terms of getting coverage probability p much closer to 95%, and both methods have similar performance afterward. An important observation for $n_{\mathrm{ct}} = 8$–16 is that the normal TI method gave more accurate coverage probabilities than both bootstrap methods. On the other hand, bootstrap methods gave better coverage probabilities for $n_{\mathrm{ct}} = 32$–64 (see Fig. 7a) than the normal TI. For this case, even if one identifies the distribution correctly, the nonparametric distribution is the only safe alternative.

D. Gamma Distribution Identified as Lognormal

Now, if the actual parent distribution (gamma) is identified "incorrectly" as lognormal, both bootstrap methods (BC_a-logn and BCNA-logn) obtain $p < 95\%$ for all values of n_{ct}. In fact, with the increase in the number of tests, coverage probabilities seem to converge to 92.5% instead to 95%. Furthermore, it is noticed that p from the normal TI method is similar to bootstrap-logn methods for $n_{\mathrm{ct}} = 64$. The important observations from the analysis of different methods for calculating B-basis values for gamma distribution are as follows:

1) The normal TI method (even though the underlying distribution is not normal or lognormal) gave better coverage probabilities than the bootstrap methods (i.e., about 93%) for smaller sample sizes (8–16). So, the normal TI method may be a good choice if one is willing to trade off slight underestimation of coverage probability (i.e., 93% instead of 95%) with significantly better B-basis values or lower relative mean margins in comparison to nonparametric method (e.g., about 24–15% for normal TI vs 47–29% for nonparametric).

Table 6 Coverage probability 100p% and relative mean margin R_{mm}% for data from gamma distribution[a]

n_{ct}	BCNA, gamma	BC_a, gamma	BCNA, logn	BC_a, logn	Normal TI	Nonparametric, H-K
8	89.3 (21.9)	86.4 (18.8)	86.7 (19.5)	81.3 (14.2)	93.0 (23.8)	98.4 (47.2)
16	92.4 (15.9)	91.7 (15.5)	89.7 (14.2)	88.0 (13.0)	93.1 (15.2)	97.0 (28.5)
32	93.7 (11.3)	94.1 (11.6)	91.2 (10.1)	91.1 (10.1)	92.4 (10.1)	96.7 (19.4)
64	94.2 (8.0)	94.7 (8.2)	91.6 (6.9)	92.2 (7.2)	91.8 (6.7)	96.4 (11.9)

[a]These numbers are based on 10,000 repetitions, so there is about a 0.1–1% error in these proportions.

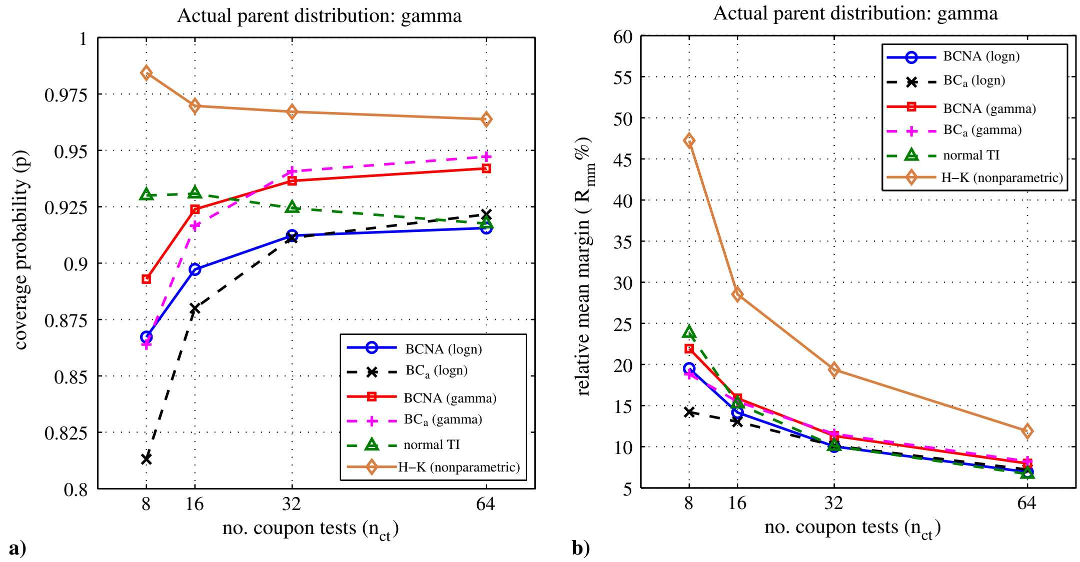

Fig. 7 Representations of a) coverage probability p, and b) relative mean margin R_{mm} % as a function of n_{ct} for different confidence bound methods, when data come from gamma distribution.

2) If underlying distribution is identified correctly (gamma as gamma), then both bootstrap methods compete with the nonparametric method and normal TI method for larger sample sizes. However, practically, the bootstrap method may not lead to a substantial reduction in weight penalty in comparison if the nonparametric method is used for calculating B-basis design life values for those sample sizes. Also, if the underlying distribution is misidentified (gamma as lognormal), then the bootstrap methods do not perform well.

3) With correct identification (gamma as gamma), bootstrap methods are more efficient in the relative mean margin (i.e., less conservative) in comparison with incorrect identification (gamma as lognormal).

4) Finally, for large sample sizes, the nonparametric method provides good results that do not depend on identifying the right distribution.

5) As for the lognormal case, when life is governed by the gamma distribution, there are substantial margin reductions, hence weight savings, with increasing sample sizes.

VIII. Conclusions

From the foregoing analysis, the following important observations are made:

1) If one wants to be safe, the nonparametric method is the best method. It is always conservative, it appears to converge to the right coverage with increasing sample size, and it does not depend on identifying the distribution correctly. However, this is done by calculating significantly more conservative confidence bounds, which would increase design weight.

2) If there is enough confidence in the identified distribution and tolerance interval factors are available (e.g., for normal, lognormal, and Weibull) for that distribution type, then the tolerance interval method is the best method.

3) If there is not enough confidence in the identified distribution and tolerance interval factors are not available (e.g., for gamma distribution), then the normal TI method (i.e., using a tolerance interval factor specified for normal distribution) could be used for smaller sample sizes (less than 32). This is an attractive option if one is comfortable with trading off a slightly smaller coverage probability (e.g., 93% instead of 95%) with a smaller relative mean margin (almost twice as small) in comparison to the nonparametric method.

4) If there is enough confidence in the identified distribution and tolerance interval factors are not available (e.g., for gamma distribution), then the bootstrap methods provide an attractive alternative for sample sizes greater than 16. The confidence bounds are less conservative in comparison to the nonparametric method. However, one needs to look at the benefit (e.g., in terms of design weight) of having less conservative bounds with the bootstrap method.

5) It is a well-known fact that uncertainty (conservative margin is greatly reduced) in B-basis prediction reduces with an increase in the number of tests. However, we reemphasize that one must do a larger number of tests, which companies do not do today due to absence of regulations from the FAA.

Even though this paper concludes that the TI method or nonparametric method might work well depending on the data situation (i.e., available sample size or availability of TI factors for the underlying distribution type), a situation may arise in practice where the bootstrap method is the only viable option to define the B-basis values (depending upon how one wants to use the concept of B-basis values). This paper defined the B-basis values on the FCG life, but one may want to define the B-basis value on the crack model parameters themselves. In that case, bootstrapping might be the only available option. In such cases, some of the conclusions of this study might be very useful for practitioners. For example, it was shown that the BCNA bootstrap method would be more useful than the BC_a bootstrap method for small sample sizes (e.g., for $n = 8$ to 16), as it gives higher coverage probability.

Appendix A: Crack Geometry, Load Conditions, and Stress Intensity Solution

The FCG life equation given in Eq. (14) can be expanded as follows:

$$N = t^n \left[C^{-1} (\Delta P \sqrt{\pi}/w)^{-n} \int_{a_i}^{a_f} (G_{cf} \sqrt{a})\, da \right] \tag{A1}$$

$$G_{cf} = F_{wc} G_{cf}^{\infty} = \left(0.7071 + 0.7548 \left(\frac{r}{r+a} \right) + 0.3415 \left(\frac{r}{r+a} \right)^2 + 0.642 \left(\frac{r}{r+a} \right)^3 + 0.9196 \left(\frac{r}{r+a} \right)^4 \right) \times \sqrt{ \sec\left(\frac{\pi r}{w} \right) \sec\left(\frac{\pi (r + a/2)}{w - a} \right) } \tag{A2}$$

where F_{wc} is the finite width correction, and G_{cf}^{∞} is the infinite plate solution. The integrand in Eq. (A1) requires numerical integration that is computationally expensive. It is approximated by the surrogate

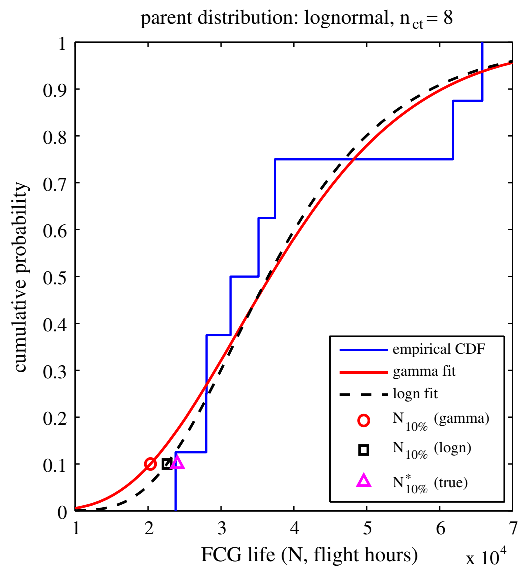

Fig. B1 Representations of a) lognormal fit and gamma fit to a sample generated from lognormal distribution, and b) lognormal and gamma fit to a sample generate from gamma distribution.

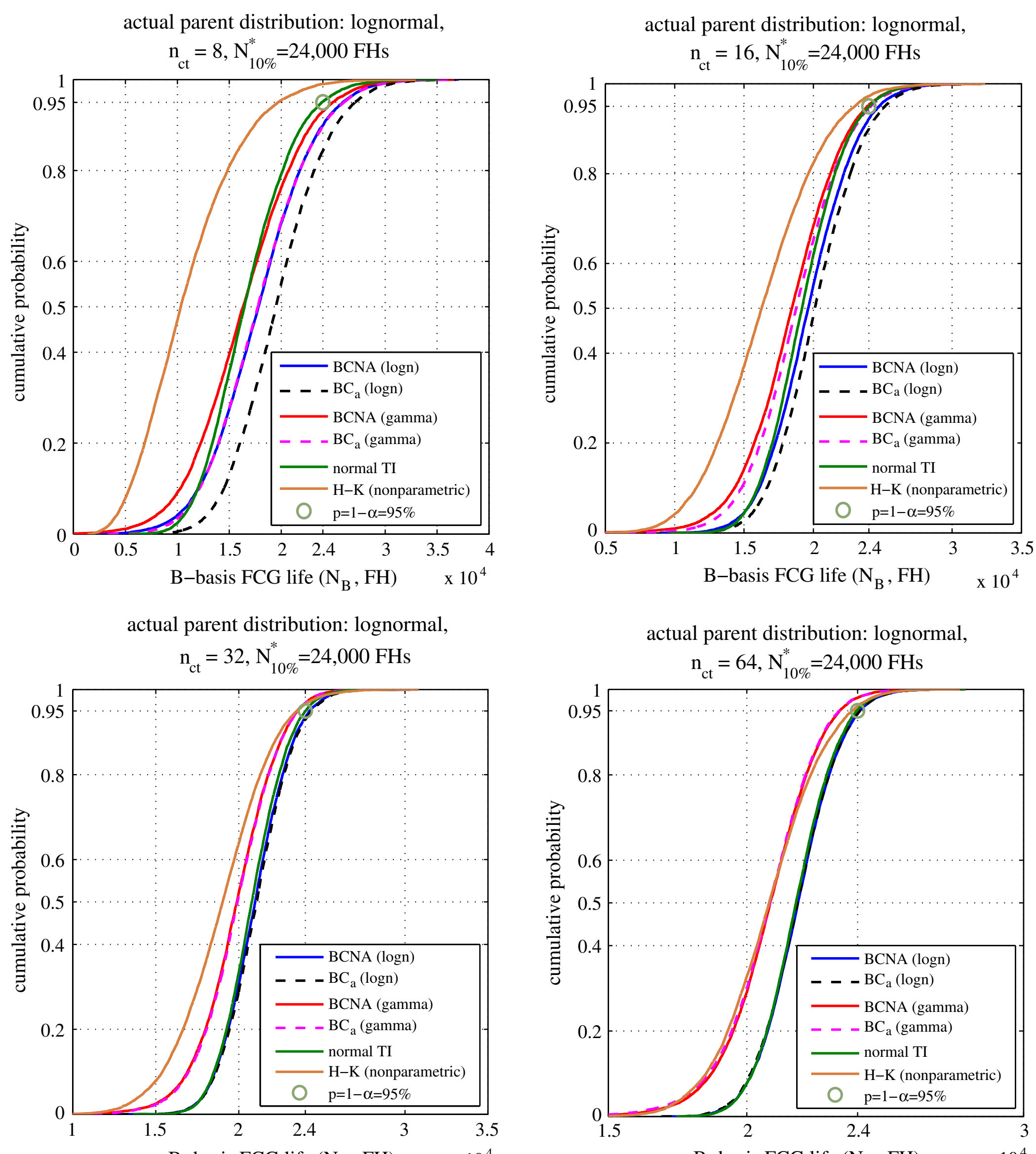

Fig. C1 *B*-basis FCG life empirical CDFs (for lognormal parent distribution) obtained via different confidence bound methods by simulation shown in Table 4.

given in Eq. (A3) that is valid for the width, diameter, and crack lengths given in Table 1:

$$N = t^n [C^{-1} (\Delta P \sqrt{\pi}/w)^{-n} (1.64 - 1.22n + 1.14n^2 - 0.3n^3 + 0.047n^4)] \tag{A3}$$

Appendix B: Nonparametric Bootstrap for Lower Percentiles

For the original FCG life sample of size n_{ct}, the bootstrap method resamples data from it nonparametrically (with replacement) to create B (5000) bootstrap samples of size n_{ct}. We then compute the 10th percentile values from each bootstrap sample by approximating parameters of the assumed probability model to create its sampling distribution. If the FCG life data are treated as a lognormal distribution, then parameters are estimated by a method of moments:

$$\lambda = \mu_{\ell n}, \qquad \eta = \sigma_{\ell n} \tag{B1}$$

where $\mu_{\ell n}$ and $\sigma_{\ell n}$ are the mean and standard deviation of the log-transformed FCG life. If the FCG life data are treated as a gamma distribution, then the parameters are estimated by a method of moments:

$$\hat{\beta} = \left(\frac{\mu}{\sigma}\right)^2, \qquad \hat{\eta} = \frac{\sigma^2}{\mu} \tag{B2}$$

where μ and σ are the mean and standard deviation of the FCG life sample. The resulting sampling distributions are then used to set confidence bounds, as per the bootstrap methods discussed in this paper.

Appendix C: Empirical CDFs of *B*-Basis FCG Life from Different Confidence Bound Methods

The round marker in the legends of Figs. C1 and C2 indicates points where the coverage probability p and desired confidence level α are equal, i.e., 95%. If a CDF passes through this point, then it indicates that a desired confidence level is achieved. On the other hand, if the CDF passes above or below this point, then it indicates

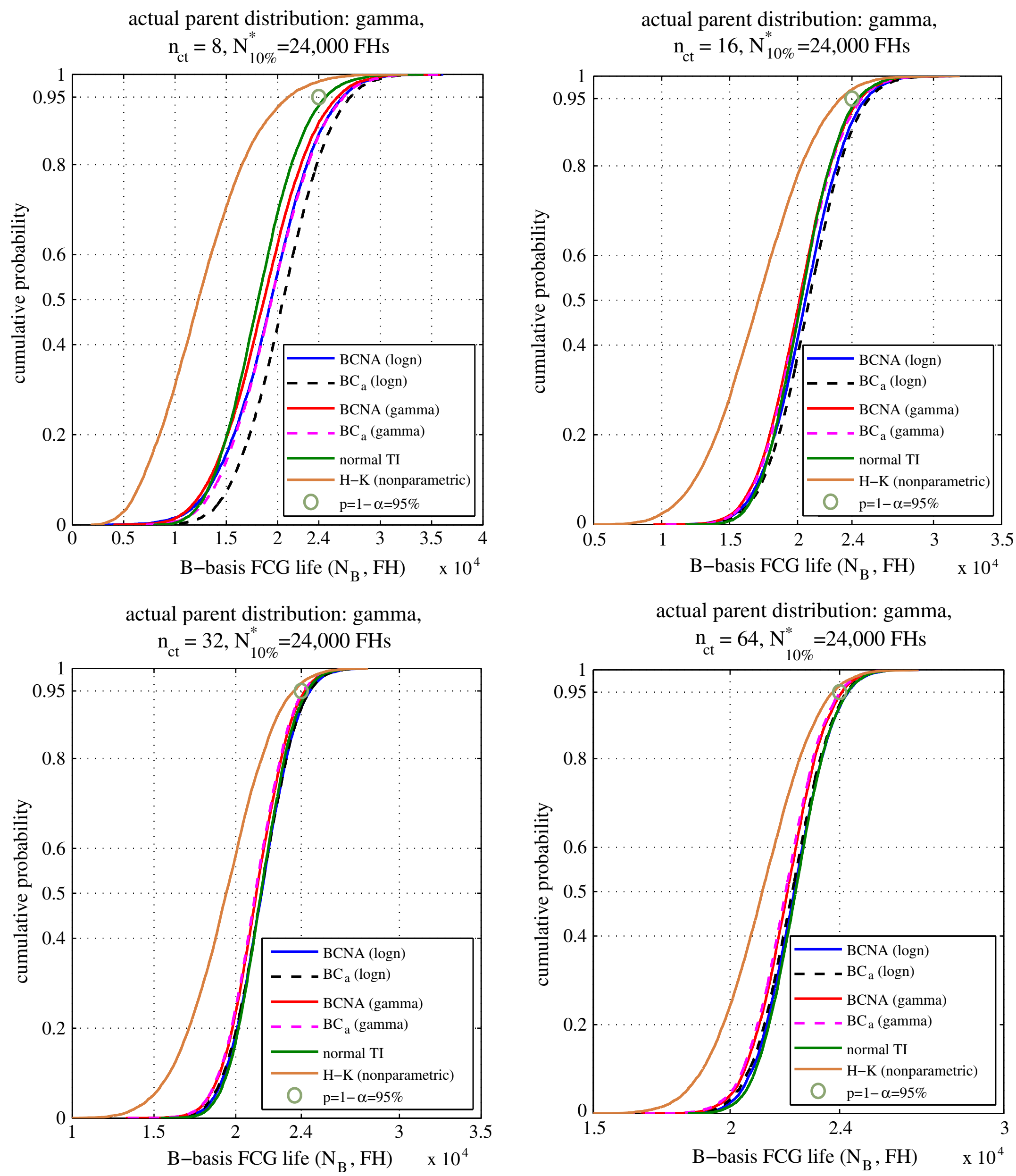

Fig. C2 ***B*-basis FCG life empirical CDFs (for gamma parent distribution) obtained via different confidence bound methods by simulation shown in Table 4.**

overestimation or underestimation of the desired confidence level of 95%.

References

[1] Acar, E., Haftka, R. T., and Kim, N.-H., "Effects of Structural Tests on Aircraft Safety," *AIAA Journal*, Vol. 48, No. 10, 2010, pp. 2235–2248. doi:10.2514/1.J050202

[2] Acar, E., Kale, A., and Haftka, R., "Comparing Effectiveness of Measures That Improve Aircraft Structural Safety," *Journal of Aerospace Engineering*, Vol. 20, No. 3, 2007, pp. 186–199. doi:10.1061/(ASCE)0893-1321(2007)20:3(186)

[3] Park, C., Kim, N. H., and Haftka, R. T., "How Coupon and Element Tests Reduce Conservativeness in Element Failure Prediction," *Reliability Engineering and System Safety*, Vol. 123, March 2014, pp. 123–136. doi:10.1016/j.ress.2013.10.012

[4] *Composites Materials Handbook*, Vol. 1, U.S. Department of Defence MIL-HDBK-17-1F, 2002, Chap. 8.

[5] Rust, S. W., Todt, F. R., Harris, B., Neal, D., and Vangel, M., "Statistical Methods for Calculating Material Allowables for MIL-HDBK-17," *Test Methods for Design Allowables for Fibrous Composites*, American Soc. for Testing and Materials Paper STP 1003, West Conshohocken, PA, 1989, pp. 136–149.

[6] Cross, R. J., Makeev, A., and Armanios, E., "A Comparison of Predictions from Probabilistic Crack Growth Models Inferred from Virckler's Data," *Journal of ASTM International*, Vol. 3, No. 10, 2006, pp. 1–11.

[7] Bhachu, K. S., Haftka, R. T., Kim, N. H., and Hurst, C., "Performance of Different Alloys by Assessing Epistemic and Aleatory Uncertainty in the Crack Growth Rates," *16th AIAA Non-Deterministic Approaches Conference*, AIAA Paper 2014-1009, 2014.

[8] Bigerelle, M., Najjar, D., Fournier, B., Rupin, N., and Iost, A., "Application of Lambda Distributions and Bootstrap Analysis to the Prediction of Fatigue Lifetime and Confidence Intervals," *International Journal of Fatigue*, Vol. 28, No. 3, 2006, pp. 223–236. doi:10.1016/j.ijfatigue.2005.06.033

[9] Bigerelle, M., and Iost, A., "Bootstrap Analysis of FCGR Application to the Paris Relationship and to Lifetime Prediction," *International Journal of Fatigue*, Vol. 21, No. 4, 1999, pp. 299–307. doi:10.1016/S0142-1123(98)00076-0

[10] McDonald, M., Zaman, K., and Mahadevan, S., "Probabilistic Analysis with Sparse Data," *AIAA Journal*, Vol. 51, No. 2, 2013, pp. 281–290. doi:10.2514/1.J050337

[11] Ravishankar, B., Smarslok, B. P., Haftka, R. T., and Sankar, B. V., "Error Estimation and Error Reduction in Separable Monte-Carlo Method," *AIAA Journal*, Vol. 48, No. 11, 2010, pp. 2624–2630. doi:10.2514/1.J050439

[12] Pieracci, A., "Parameter Estimation for Weibull Probability Distribution Function of Initial Fatigue Quality," *AIAA Journal*, Vol. 33, No. 9, 1995, pp. 1574–1581. doi:10.2514/3.12858

[13] Romero, V., Swiler, L., Urbina, A., and Mullins, J., "SAND2013-4561, A Comparison of Methods for Representing Sparsely Sampled Random Quantities," 2013.

[14] Romero, V., Mullins, J., Swiler, L., and Urbina, A., "A Comparison of Methods for Representing and Aggregating Uncertainties Involving Sparsely Sampled Random Variables—More Results," *SAE International Journal of Materials and Manufacturing*, 2013, pp. 447–473.

[15] Romero, V., Swiler, L., and Urbina, A., "An Initial Comparison of Methods for Representing and Aggregating Experimental Uncertainties Involving Sparse Data," *13th AIAA Non-Deterministic Approaches Conference*, AIAA Paper 2011-1705, 2011.

[16] Montgomery, D. C., and Runger, G. C., "Analysis of Statistical Intervals," *Applied Statistics and Probability for Engineers*, Wiley, Hoboken, NJ, 2010, Chap. 8.

[17] Hanson, D. L., and Koopmans, L. H., "Tolerance Limits for the Class of Distributions with Increasing Hazard Rates," *Annals of Mathematical Statistics*, Vol. 35, No. 4, 1964, pp. 1561–1570. doi:10.1214/aoms/1177700380

[18] Efron, B., "Bootstrap Methods: Another Look at the Jackknife," *Annals of Statistics*, Vol. 7, No. 1, 1979, pp. 1–26. doi:10.1214/aos/1176344552

[19] Efron, B., and Tibshirani, R. J., *An Introduction to the Bootstrap*, CRC Press, Boca Raton, FL, 1994, pp. 178–199.

[20] Meeker, W. Q., and Escobar, L. A., *Statistical Methods for Reliability Data*, Wiley, New York, 1998, pp. 204–226.

[21] Edwards, D. J., Guess, F. M., and Young, T. M., "Improved Estimation of the Lower Percentiles of Material Properties," *Wood Science Technology*, Vol. 45, No. 3, 2011, pp. 533–546. doi:10.1007/s00226-010-0346-2

[22] DiCiccio, T. J., and Efron, B., "Bootstrap Confidence Intervals," *Statistical Science*, Vol. 11, No. 3, 1996, pp. 189–228.

[23] "Standard Test Method for Measurement of Fatigue Crack Growth Rates," American Soc. for Testing and Materials International Rept. ASTM E647-13ae1, West Conshohocken, PA, 2013.

[24] Toor, P. M., "A Review of Some Damage Tolerance Design Approaches for Aircraft Structures," *Engineering Fracture Mechanics*, Vol. 5, No. 4, Dec. 1973, pp. 837–880. doi:10.1016/0013-7944(73)90054-4

[25] Miller, M. S., and Gallagher, J. P., "An Analysis of Several Fatigue Crack Growth Rate (FCGR) Descriptions," *Fatigue Crack Growth Measurement and Testing, ASTM STP 738*, American Soc. for Testing and Materials, West Conshohocken, PA, 1981, pp. 205–251.

[26] Bhachu, K. S., Haftka, R. T., Waycaster, G., Kim, N.-H., and Hurst, C., "Probabilistic Manufacturing Tolerance Optimization of Damage-Tolerant Aircraft Structures Using Measured Data," *Journal of Aircraft*, Vol. 52, No. 5, 2015, pp. 1412–1421. doi:10.2514/1.C032945

[27] Gope, P. C., "Determination of Sample Size for Estimation of Fatigue Life Using Weibull or Lognormal Distribution," *International Journal of Fatigue*, Vol. 21, No. 8, 1999, pp. 745–752. doi:10.1016/S0142-1123(99)00048-1

[28] Yang, J. N., and Manning, S. D., "Aircraft Fleet Maintenance Based on Structural Reliability Analysis," *Journal of Aircraft*, Vol. 31, No. 2, 1994, pp. 419–425. doi:10.2514/3.46502

[29] Yang, J. N., and Manning, S. D., "Stochastic Crack Growth Analysis Methodologies for Metallic Structures," *Engineering Fracture Mechanics*, Vol. 37, No. 5, 1990, pp. 1105–1124. doi:10.1016/0013-7944(90)90032-C

[30] Annis, C., "Probabilistic Life Prediction Isn't as Easy as It Looks," *Journal of ASTM International*, Vol. 1, No. 2, 2004.

[31] Akkaram, S., Loghin, A., and Khan, G., "A Framework for Probabilistic Assessment of Fracture Mechanics Life, U.S. Air Force Research Lab. Rept. AFRL-RX-WP-TP-2011-4390," Niskayuna, NY, 2011.

[32] Akkaram, S., Loghin, A., and Khan, G., "The Statistical Nature of Fatigue Crack Propagation," *Air Force Flight Dynamics Laboratory*, Tech. Rept. AFFDL-TR-78-43, April 1978.

[33] Millwater, H. R., and Wieland, D. H., "Probabilistic Sensitivity Based Ranking of Damage Tolerance Analysis Elements," *Journal of Aircraft*, Vol. 47, No. 1, 2010, pp. 161–171. doi:10.2514/1.44498

R. K. Kapania
Associate Editor