0.1. Failure Theories

In the previous section, we introduced the concept of stress, strain and the relationship between stresses and strains. We also discussed failure of materials under uniaxial state of stress. Failure of engineering materials can be broadly classified into ductile and brittle failure. Most metals are ductile and fail due to yielding. Hence, the yield strength characterizes their failure. Ceramics and some polymers are brittle and rupture or fracture when the stress exceeds certain maximum value. Their stress–strain behavior is linear up to the point of failure and they fail abruptly.

The stress required to break the atomic bond and separate the atoms is called the theoretical strength of the material. It can be shown that the theoretical strength is approximately equal to $E/3$ where, E is Young's modulus.[1] However, most materials fail at a stress about one–hundredth or even one–thousandth of the theoretical strength. For example, the theoretical strength of aluminum is about 22 GPa. However, the yield strength of aluminum is in the order of 100 MPa, which is 1/220th of the theoretical strength. This enormous discrepancy could be explained as follows.

In ductile material yielding occurs not due to separation of atoms but due to sliding of atoms (movement of dislocations) as depicted in Figure 1.1. Thus, the stress or energy required for yielding is much less than that required for separating the atomic planes. Hence, in a ductile material the maximum shear stress causes yielding of the material.

In brittle materials, the failure or rupture still occurs due to separation of atomic planes. However, the high value of stress required is provided locally by stress concentration caused by small pre-existing cracks or flaws in the material. The stress concentration factors can be in the order of 100 to 1,000. That is, the applied stress is amplified by enormous amount due to the presence of cracks and it is sufficient to separate the atoms. When this process becomes unstable, the material separates over a large area causing brittle failure of the material.

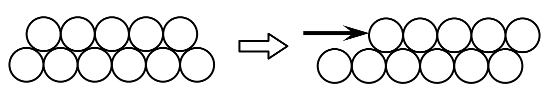

Figure 1.1: Material failure due to relative sliding of atomic planes

Although research is underway not only to explain but also quantify the strength of materials in terms of its atomic structure and properties, it is still not practical to design machines and structures based on such atomistic models. Hence, we resort to phenomenological failure theories, which are based on observations and testing over a period of time. The purpose of failure theories is to extend the strength values obtained

[1] T.L. Anderson, *Fracture Mechanics – Fundamentals and Applications*, Third Edition, CRC Press, Boca Raton, FL, 2006.

from uniaxial tests to multi-axial states of stress that exists in practical structures. It is not practical to test a material under all possible combinations of stress states. In the following, we describe some well-established phenomenological failure theories for both ductile and brittle materials.

Strain Energy

When a force is applied to a solid, it deforms. Then, we can say that work is done on the solid, which is proportional to the force and deformation. The work done by applied force is stored in the solid as potential energy, which is called the *strain energy*. The strain energy in the solid may not be distributed uniformly through out the solid. We introduce the concept of strain energy density, which is strain energy per unit volume, and we denote it by U_0. Then the strain energy in the body can be obtained by integration as follows:

$$U = \iiint_V U_o(x, y, z)dV \tag{0.1}$$

where the integration is performed over the volume V of the solid. In the case of uniaxial stress state stain energy density is equal to the area under the stress–strain curve (see Figure 1.2). Thus, it can be written as

$$U_0 = \frac{1}{2}\sigma\varepsilon \tag{0.2}$$

For the general 3-D case the stain energy density is expressed as

$$U_0 = \frac{1}{2}\ \sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} + \tau_{xy}\gamma_{xy} \tag{0.3}$$

If the material is elastic, then the strain energy can be completely recovered by unloading the body.

The strain energy density in Eq. (0.3) can be further simplified. Consider a coordinate system that is parallel to the principal stress directions. In this coordinate system, no shear components exist. Extending Eq. (0.3) to this stress states yields

$$U_0 = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3) \tag{0.4}$$

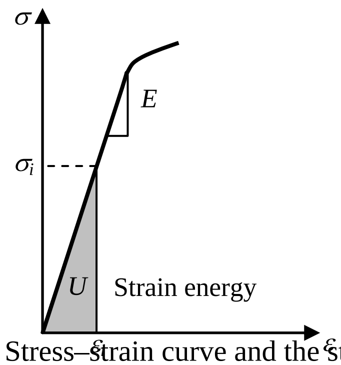

Figure 1.2: Stress–strain curve and the strain energy

From Section 1.3, we know that stresses and strains are related through the linear elastic relations. For example, in case of principal stresses and strains,

$$\begin{cases} \varepsilon_1 = \dfrac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) \\ \varepsilon_2 = \dfrac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3) \\ \varepsilon_3 = \dfrac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2) \end{cases} \tag{0.5}$$

Substituting from Eq. (0.5) into Eq. (0.4), we can write the strain energy density in terms of principal stresses as

$$U_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right] \tag{0.6}$$

The strain energy density can be thought of as consisting of two components: one due to dilation or change in volume and the other due to distortion or change in shape. The former is called dilatational strain energy and the latter distortional energy. Many experiments have shown that ductile materials can be hydrostatically stressed to levels beyond their ultimate strength in compression without failure. This is because the hydrostatic state of stress reduces the volume of the specimen without changing its shape.

Decomposition of Strain Energy

The strain energy density at a point in a solid can be divided into two parts: dilatational strain energy density U_h that is due to change in volume, and distortional strain energy density, U_d, that is responsible for change in shape. In order to compute these components, we divide the stress matrix also into similar components, dilatational stress matrix, σ_h, and deviatoric stress matrix, σ_d. For convenience we will refer the stresses to the principal stress coordinates. Then the aforementioned stress components can be derived as follows:

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_h & 0 & 0 \\ 0 & \sigma_h & 0 \\ 0 & 0 & \sigma_h \end{bmatrix} + \begin{bmatrix} \sigma_{1d} & 0 & 0 \\ 0 & \sigma_{2d} & 0 \\ 0 & 0 & \sigma_{3d} \end{bmatrix} \tag{0.7}$$

The dilatational component σ_h is defined as

$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \tag{0.8}$$

which is also called the volumetric stress. Note that $3\sigma_h$ is nothing but the first invariant I_1 of the stress matrix in Eq.. Thus, it is independent of coordinate system. One can note that σ_h is a state of hydrostatic stress and hence the subscript h is used to denote the dilatational stress component as well as dilatational energy density

The dilatational energy density can be obtained by substituting the stress components of the hydrostatic stress state in Eq. (0.8) into the expression for strain energy density in Eq. (0.6),

$$\begin{aligned} U_h &= \frac{1}{2E} \ \sigma_h^2 + \sigma_h^2 + \sigma_h^2 - 2\nu(\sigma_h\sigma_h + \sigma_h\sigma_h + \sigma_h\sigma_h) \\ &= \frac{3}{2}\frac{(1-2\nu)}{E}\sigma_h^2 \end{aligned} \tag{0.9}$$

and using the relation in Eq. (0.8),

$$\begin{aligned} U_h &= \frac{3}{2}\frac{(1-2\nu)}{E} \ \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \ ^2 \\ &= \frac{1-2\nu}{6E} \ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) \end{aligned} \tag{0.10}$$

Distortion Energy

The distortion part of the strain energy is now found by subtracting Eq. (0.10) from Eq. (0.6), as

$$\begin{aligned} U_d &= U_0 - U_h \\ &= \frac{1+\nu}{3E} \ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3 \\ &= \frac{1+\nu}{3E}\frac{\ \sigma_1 - \sigma_2 \ ^2 + \ \sigma_2 - \sigma_3 \ ^2 + \ \sigma_3 - \sigma_1 \ ^2}{2} \end{aligned} \tag{0.11}$$

It is customary to write U_d in terms of an equivalent stress called von Mises stress σ_{VM} as

$$U_d = \frac{1+\nu}{3E}\sigma_{VM}^2 \tag{0.12}$$

The von Mises stress is defined in terms of principal stresses as

$$\sigma_{VM} = \sqrt{\frac{\sigma_1 - \sigma_2{}^2 + \ \sigma_2 - \sigma_3{}^2 + \ \sigma_3 - \sigma_1{}^2}{2}} \tag{0.13}$$

Distortion Energy Theory (Von Mises)

According to the von Mises's theory, a ductile solid will yield when the distortion energy density reaches a critical value for that material. Since this should be true for uniaxial stress state also, the critical value of the distortional energy can be estimated from the uniaxial test. At the instance of yielding in a uniaxial tensile test, the state of stress in terms of principal stress is given by: $\sigma_1 = \sigma_Y$ (yield stress) and $\sigma_2 = \sigma_3 = 0$. The distortion energy density associated with yielding is

$$U_d = \frac{1+\nu}{3E}\sigma_Y^2 \tag{0.14}$$

Thus, the energy density given in Eq. (0.14) is the critical value of the distortional energy density for the material. Then according to von Mises's failure criterion, the material under multi-axial loading will yield when the distortional energy is equal to or greater than the critical value for the material:

$$\frac{1+\nu}{3E}\sigma_{VM}^2 \geq \frac{1+\nu}{3E}\sigma_Y^2$$

$$\therefore \quad \sigma_{VM} \geq \sigma_Y \tag{0.15}$$

Thus, the distortion energy theory can be stated that material yields when the von Mises stress exceeds the yield stress obtained in a uniaxial tensile test.

The von Mises stress in Eq. (0.11) can be rewritten in terms of stress components as

$$\sigma_{VM} = \sqrt{\frac{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}} \tag{0.16}$$

For a two–dimensional plane stress state, $\sigma_3 = 0$, the von Mises stress can be defined in terms of principal stresses as

$$\sigma_{VM} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} \tag{0.17}$$

and in terms of general stress components as

$$\sigma_{VM} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2} \tag{0.18}$$

The two–dimensional distortion energy equation (0.18) describes an ellipse, which when plotted on the σ_1–σ_2 plane as shown in Figure 1.3. The interior of this ellipse defines the region of combined biaxial stress where the material is safe against yielding under static loading.

Consider a situation in which only a shear stress exists, such that $\sigma_x = \sigma_y = 0$, and $\tau_{xy} = \tau$. For this stress state, the principal stresses are $\sigma_1 = -\sigma_2 = \tau$ and $\sigma_3 = 0$. On the σ_1–σ_2 plane this pure shear state is represented as a straight line through the origin at –45° as shown in Figure 1.3. The line intersects the von Mises failure envelope at two points, *A* and *B*. The magnitude of σ_1 and σ_2 at these points can be found from Eq. (0.17) as

$$\sigma_Y^2 = \sigma_1^2 + \sigma_1\sigma_1 + \sigma_1^2 = 3\sigma_1^2 = 3\tau_{\max}^2$$

$$\tau_{\max} = \sigma_1 = \frac{\sigma_Y}{\sqrt{3}} = 0.577\sigma_Y \tag{0.19}$$

Thus, in pure shear stress state, the material yields when the shear stress reaches 0.577 of σ_Y. This value will be compared to the maximum shear stress theory described below.

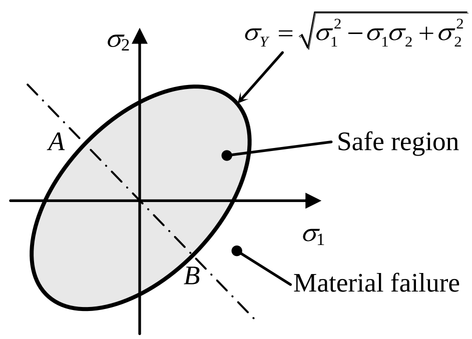

Figure 1.3: Failure envelope of the distortion energy theory

The Maximum Shear Stress Theory (Tresca)

According to the maximum shear stress theory, the material yields when the maximum shear stress at a point equals the critical shear stress value for that material. Since this should be true for uniaxial stress state, we can use the results from uniaxial tension test to determine the maximum allowable shear stress. The stress state in a tensile specimen at the point of yielding is given by: $\sigma_1 = \sigma_Y$, $\sigma_2 = \sigma_3 = 0$. The maximum shear stress is calculated as

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} \geq \tau_Y = \frac{\sigma_Y}{2} \tag{0.20}$$

This value of maximum shear stress is also called the yield shear stress of the material and is denoted by τ_Y. Note that $\tau_Y = \sigma_Y/2$. Thus, the Tresca's yield criterion is that yielding will occur in a material when the maximum shear stress equals the yield shear strength, τ_Y, of the material.

The hexagon in Figure 1.4 represents the two–dimensional failure envelope according to maximum shear stress theory. The ellipse corresponding to von Mises's theory is also shown in the same figure. The hexagon is inscribed within the ellipse and contacts it at six vertices. Combinations of principal stresses σ_1 and σ_2 that lie within this hexagon are considered safe based on the maximum shear stress theory, and failure is considered to occur when the combined stress state reaches the hexagonal boundary. This is obviously more conservative failure theory than distortion energy theory as it is contained within the latter. In the pure shear stress state, the shear stress at the points C and D correspond to $0.5\sigma_Y$, which is smaller than $0.577\sigma_Y$ according to the distortion energy theory.

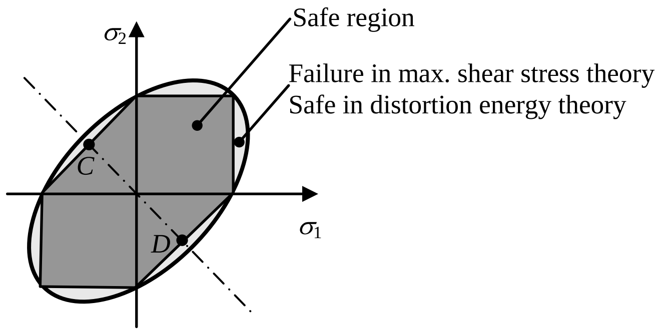

Figure 1.4: Failure envelope of the maximum shear stress theory

Maximum Principal Stress Theory (Rankine)

According to the maximum principal stress theory, a brittle material ruptures when the maximum principal stress in the specimen reaches some limiting value for the material. Again, this critical value can be inferred as the tensile strength measured using a uniaxial tension test. In practice, this theory is simple, but can only be used for brittle materials. Some practitioners have modified this theory for ductile materials as

$$\sigma_1 \geq \sigma_U \tag{0.21}$$

where σ_1 is the maximum principal stress and σ_U the ultimate strength described in. Figure 1.5 shows the failure envelope based on the maximum principal stress theory. Note that the failure envelopes in the first and third quadrants are coincident with that of the maximum shear stress theory and contained within the distortion energy theory. However, the envelopes in the second and fourth quadrants are well outside of other two theories. Hence, the maximum principal stress theory is not considered suitable for ductile materials. However, it can be used to predict failure in brittle materials.

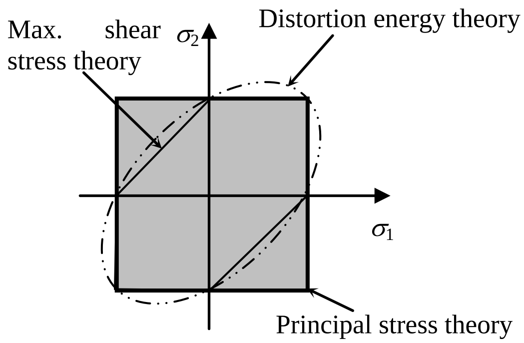

Figure 1.5: Failure envelope of the maximum principal stress theory