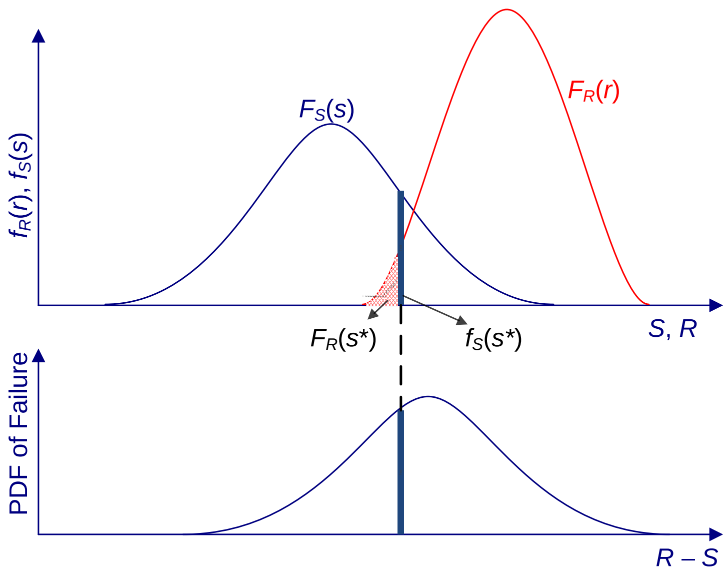
Structural Reliability

- Structural limit states are often defined as a difference between Strength (*R*) and Load (*S*):
- Probability of failure can be defined as

$$P_F = P(R - S \leq 0) = \int_{-\infty}^{\infty} F_R(s) f_S(s) ds$$

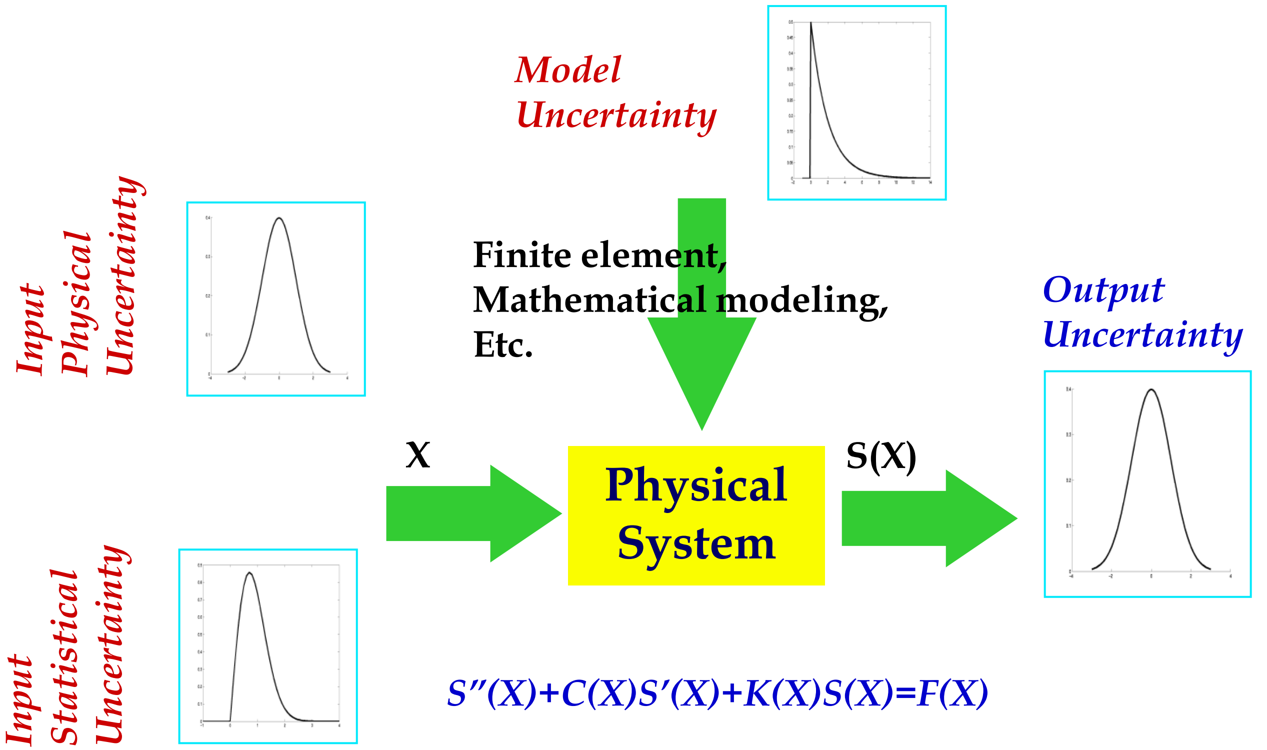
- Or, using reliability

$$P_S = 1 - P_F = 1 - \int_{-\infty}^{\infty} F_R(s) f_S(s) ds$$



Reliability Analysis

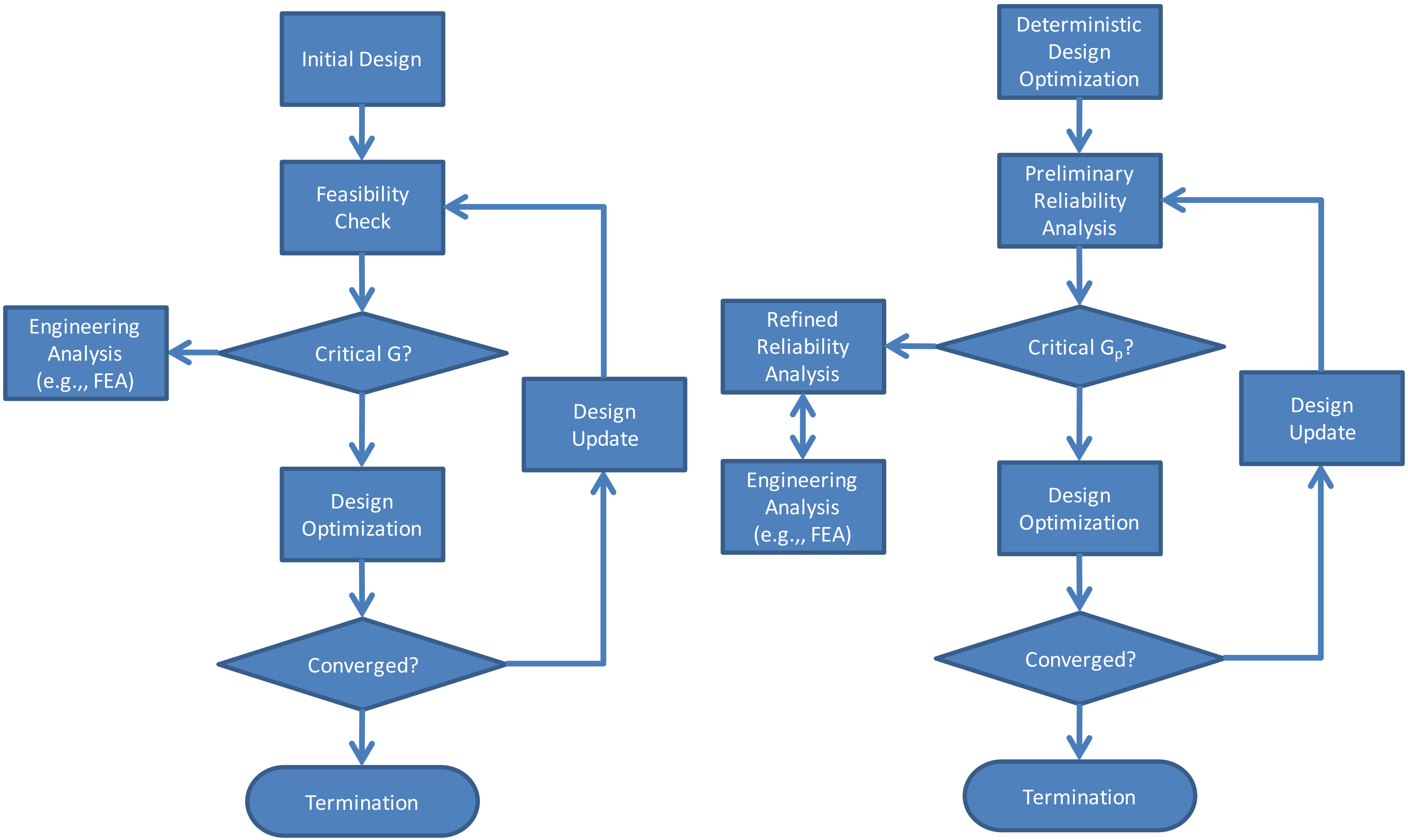
- Uncertainty propagate through physical system



$$G(\mathbf{X}) = R(\mathbf{X}) - S(\mathbf{X})$$

$$P_F = P\{G(\mathbf{X}) \le 0\} = \int_{-\infty}^{0} f_G(g)dg = \int \underset{G(\mathbf{X})\le 0}{\cdots} \int f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$$

Deterministic Optimization vs RBDO



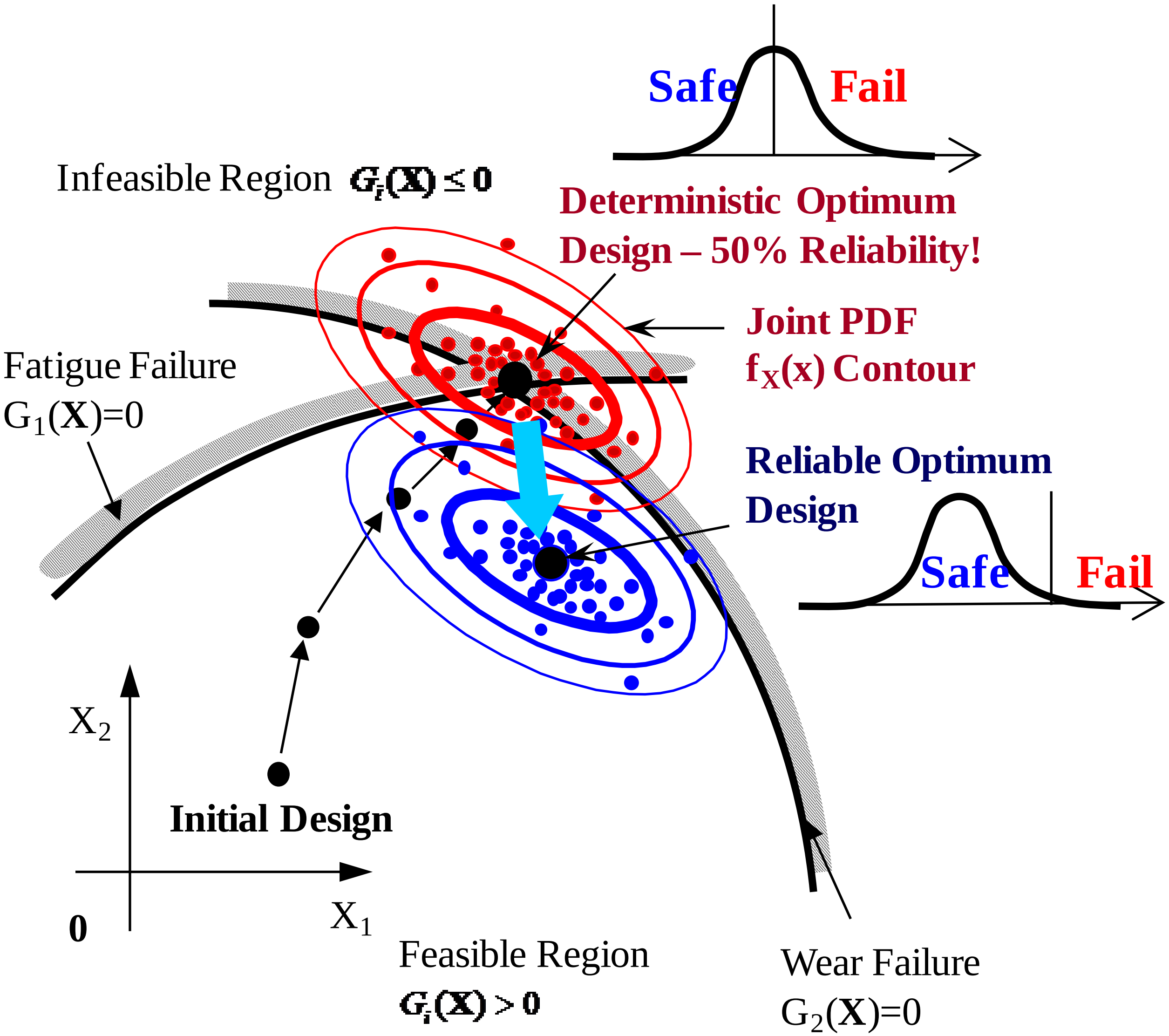
Reliability-Based Design Optimization (RBDO)

- Problem formulation

$$\begin{aligned} &\textbf{Minimize} && \text{Cost}(\mathbf{d}) \\ &\textbf{subject to} && P\{G_i\{\mathbf{X};\mathbf{d}(\mathbf{X})\} \le 0\} \le P_{F_i}^t,\ i = 1,\cdots,nc \\ & && \mathbf{d}_\mathrm{L} \le \mathbf{d} \le \mathbf{d}_\mathrm{U},\quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{aligned}$$

- Limit state: $G_i\{\mathbf{X};\mathbf{d}(\mathbf{X})\} = R_i\{\mathbf{X};\mathbf{d}(\mathbf{X})\} - S_i\{\mathbf{X};\mathbf{d}(\mathbf{X})\}$
- Design variable: **d** Random variable: **X**
- Target probability of failure: $P_F^t = \Phi(-\beta^t)$
- Target reliability index: β^t
- Constraint is given in terms of probability of failure
  - Need to evaluate P_F at every design iteration

RBDO

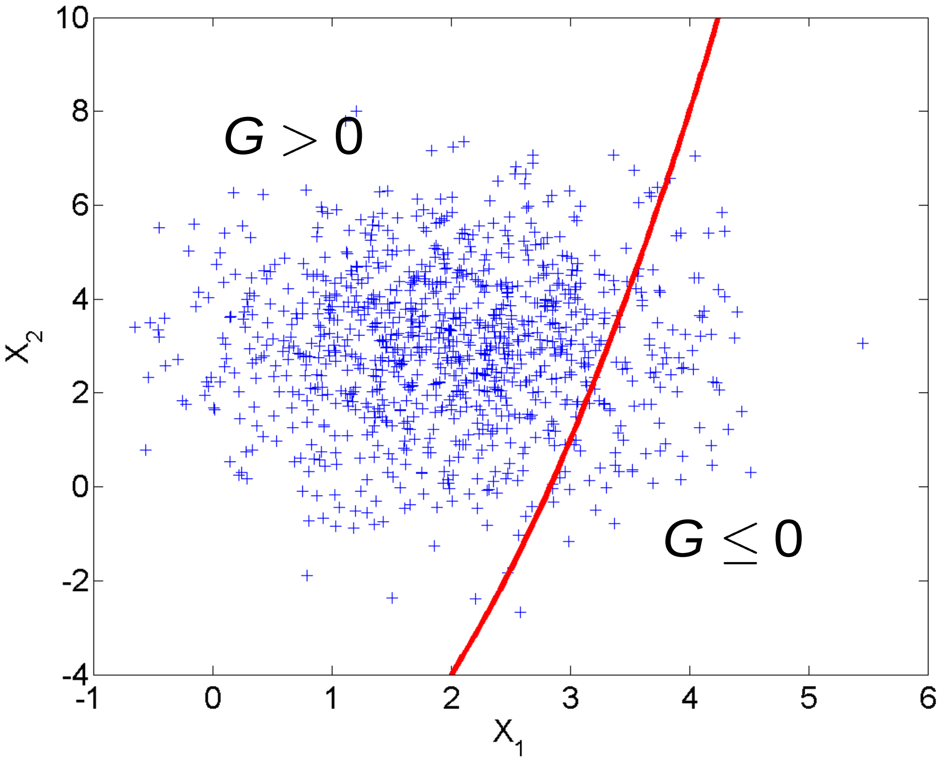


Safe
Fail
Infeasible Region $G_i(\mathbf{X}) \le 0$
Deterministic Optimum
Design – 50% Reliability!
Joint PDF
$f_X(x)$ Contour
Fatigue Failure
$G_1(\mathbf{X})=0$
Reliable Optimum
Design
Safe
Fail
X_2
Initial Design
0
X_1
Feasible Region
$G_i(\mathbf{X}) > 0$
Wear Failure
$G_2(\mathbf{X})=0$

Monte Carlo Simulation

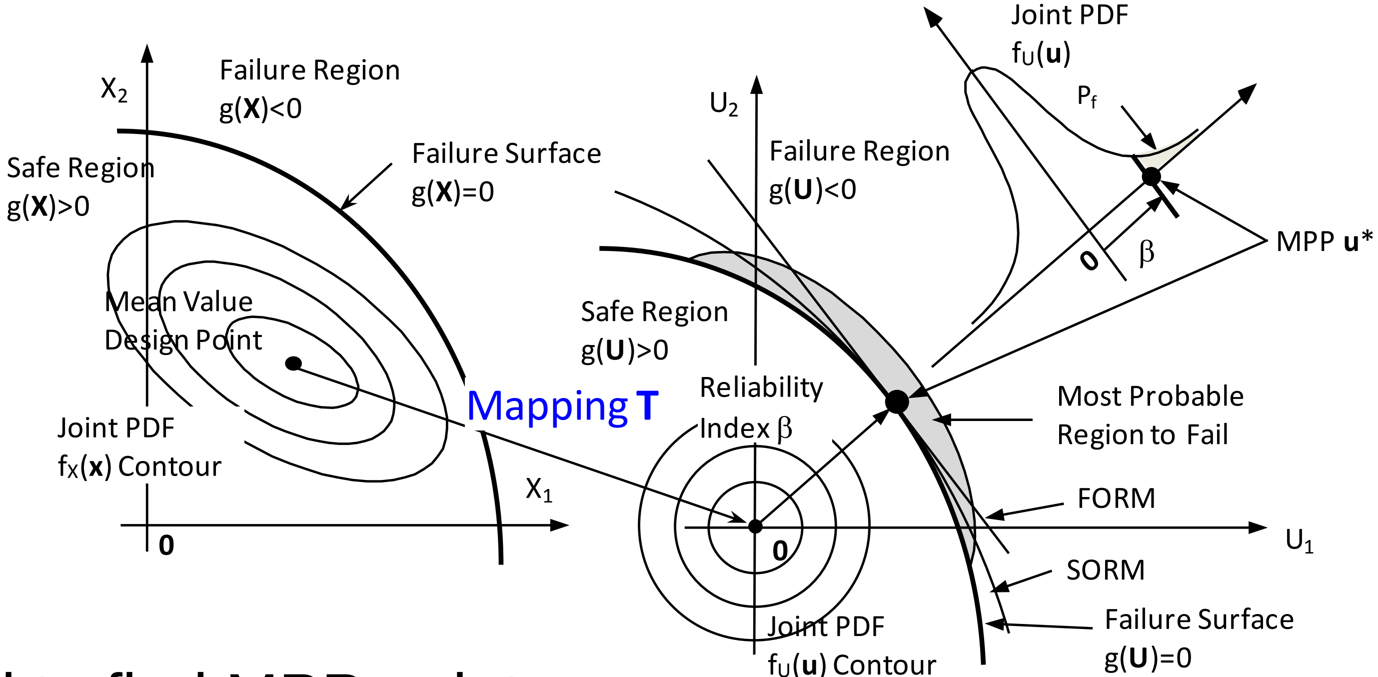
- Probability of failure

$$P\{G_i(\mathbf{X};\mathbf{d}) \le 0\} = F_{G_i}(0) = \int_{G_i(\mathbf{X})\le 0} \cdots \int f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \approx \frac{\text{Number of failed trials}}{\text{Number of total trials}}$$

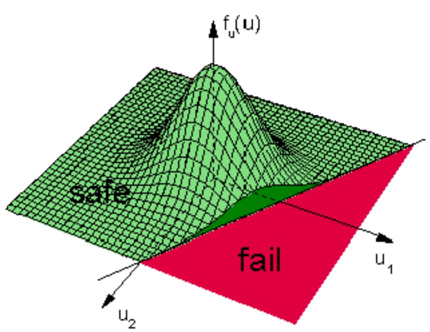


Most Probable Point (MPP)-based Method

- Transform the limit state to the U-space



- Need to find MPP point
- Calculate reliability index

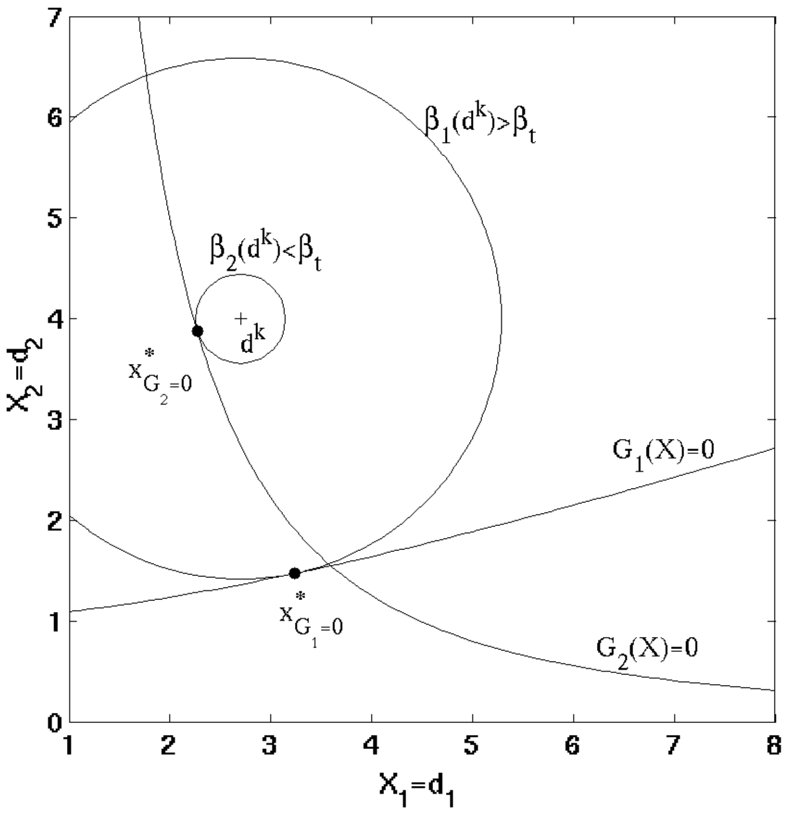


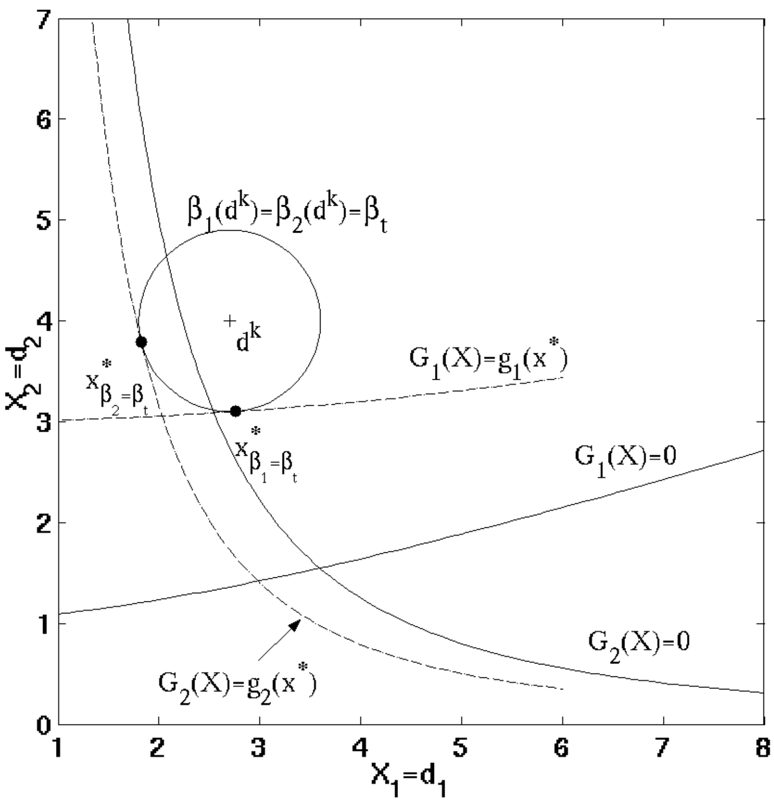
MPP Search Method

- Reliability appears as a constraint in optimization
- Reliability index approach vs Performance measure approach

$$\beta_s = -\Phi^{-1}\{F_G(0)\} \geq \beta_t$$

$$G_p = F_G^{-1}\{\Phi(-\beta_t)\} \geq 0$$





Reliability Index Approach (RIA)

- Find $\beta_{S,\mathrm{FORM}}$ using FORM in U-space

$$\begin{aligned}&\text{minimize} && \|\mathbf{u}\| \\ &\text{subject to} && G(\mathbf{u}) = 0\end{aligned} \qquad \beta_{S,\mathrm{FORM}} = \left\|\mathbf{u}^{*}_{G(\mathbf{u})=0}\right\|$$

- HL-RF Method $\mathbf{u}^{(k+1)} = \left(\mathbf{u}^{(k)} \bullet \mathbf{n}^{(k)} - \dfrac{G(\mathbf{u}^{(k)})}{\left\|\nabla_U G(\mathbf{u}^{(k)})\right\|}\right)\mathbf{n}^{(k)}$

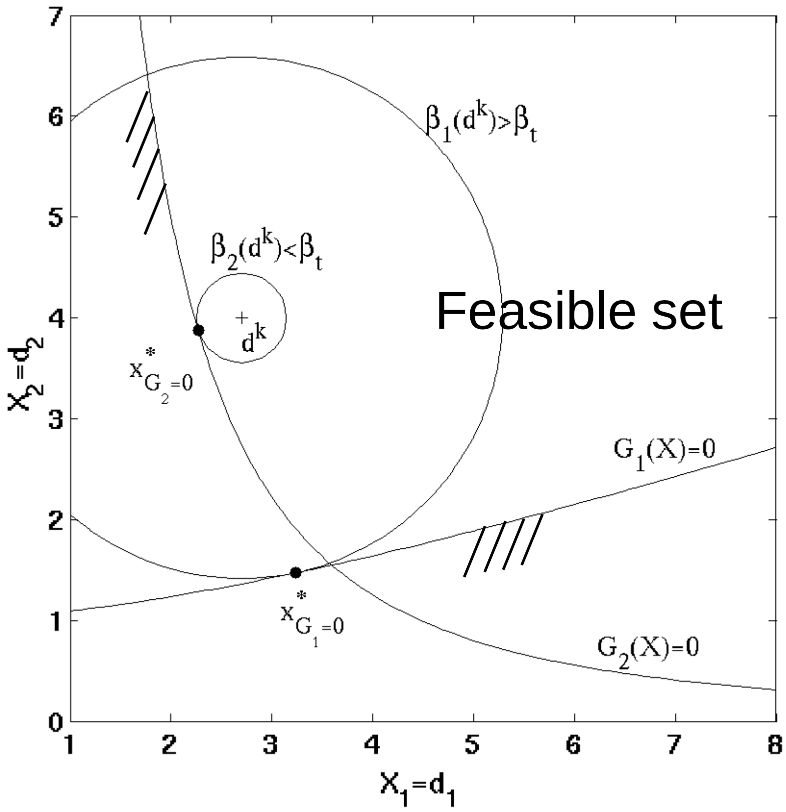
$$= \left[\nabla_U G(\mathbf{u}^{(k)}) \bullet \mathbf{u}^{(k)} - G(\mathbf{u}^{(k)})\right] \frac{\nabla_U G(\mathbf{u}^{(k)})}{\left\|\nabla_U G(\mathbf{u}^{(k)})\right\|^2}$$

- Good for reliability analysis
- Expensive with MCS and MPP-based method when reliability is high
- MPP-based method can be unstable when reliability if high or the limit state is highly nonlinear

RIA-RBDO

- RIA-RBDO:

$$\begin{aligned} &\text{Minimize} && \text{Cost}(\mathbf{X};\mathbf{d}) \\ &\text{subject to} && g_{RIA}(\mathbf{X};\mathbf{d}) = \beta_{t_i} - \beta_{s_i}(\mathbf{X};\mathbf{d}) \le 0,\ \ i = 1,\cdots,nc \\ &&& \mathbf{d}_L \le \mathbf{d} \le \mathbf{d}_U,\quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{aligned}$$



Performance Measure Approach (PMA)

- Inverse reliability analysis

$$\begin{aligned} &\textbf{maximize} && G(\mathbf{u}) \\ &\textbf{subject to} && \|\mathbf{u}\| = \beta_t \end{aligned}$$

Optimum: $G_p(\mathbf{u}^*)$

- Advanced mean value method

$$\mathbf{u}_{\text{AMV}}^{(1)} = \mathbf{u}_{\text{MV}}^{*}, \; \mathbf{u}_{\text{AMV}}^{(k+1)} = \beta_t \mathbf{n}(\mathbf{u}_{\text{AMV}}^{(k)})$$

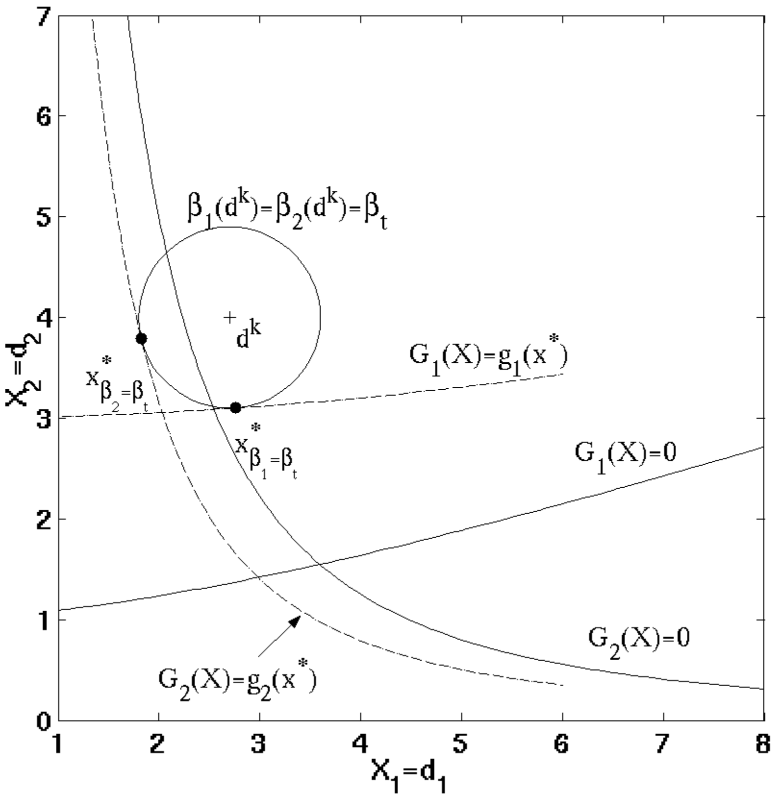
$$\mathbf{n}(\mathbf{u}_{\text{AMV}}^{(k)}) = \frac{\nabla_U G(\mathbf{u}_{\text{AMV}}^{(k)})}{\left\| \nabla_U G(\mathbf{u}_{\text{AMV}}^{(k)}) \right\|}$$

- Not suitable for assessing reliability (β_t is fixed)
- Efficient and stable for design optimization

PMA-RBDO

- PMA-RBDO

$$\begin{aligned} &\textbf{Minimize} && \mathrm{Cost}(\mathbf{X};\mathbf{d}) \\ &\textbf{subject to} && g_{PMA}(\mathbf{X};\mathbf{d}) = -G_{p_i}(\mathbf{X};\mathbf{d}) \le 0,\ i = 1,\cdots,nc \\ & && \mathbf{d}_L \le \mathbf{d} \le \mathbf{d}_U,\quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{aligned}$$



Reliability-based Sensitivity Analysis

- During optimization, gradient (sensitivity) needs to be calculated at each iteration
- For RIA, gradient of reliability index w.r.t. design variable (DV)
- For PMA, gradient of performance function w.r.t. DV
- RIA:

$$\frac{\partial \beta_{s,FORM}}{\partial d_i} = \left.\frac{\partial (\mathbf{U}^T\mathbf{U})^{1/2}}{\partial d_i}\right|_{\mathbf{U}=\mathbf{u}^*_{G(\mathbf{U})=0}}$$

$$= \left.\frac{\partial (\mathbf{U}^T\mathbf{U})^{1/2}}{\partial \mathbf{U}} \cdot \frac{\partial \mathbf{U}}{\partial d_i}\right|_{\mathbf{U}=\mathbf{u}^*_{G(\mathbf{U})=0}}$$

$$= \left.\frac{1}{2}(\mathbf{U}^T\mathbf{U})^{-1/2} \cdot (2\mathbf{U}^T) \cdot \frac{\partial \mathbf{U}}{\partial d_i}\right|_{\mathbf{U}=\mathbf{u}^*_{G(\mathbf{U})=0}}$$

$$= \left.\frac{\mathbf{U}^T}{\beta_{s,FORM}} \cdot \frac{\partial \mathbf{U}}{\partial d_i}\right|_{\mathbf{U}=\mathbf{u}^*_{G(\mathbf{U})=0}}$$

From **U** = T(**X**; **d**)

$$\frac{\partial \beta_{s,FORM}}{\partial d_i} = \left.\frac{\mathbf{T}(\mathbf{X};\mathbf{d})^T}{\beta_{s,FORM}} \frac{\partial \mathbf{T}(\mathbf{X};\mathbf{d})}{\partial d_i}\right|_{\mathbf{X}=\mathbf{x}^*_{G(\mathbf{X})=0}}$$

Reliability-based Sensitivity Analysis

- Example – Normal distribution
  - Design variables are **d** = [μ, σ] of Normal distribution

$$U = T(X,\mathbf{d}) = \frac{X-\mu}{\sigma}$$

  - Thus,

$$\frac{\partial T}{\partial \mu} = \frac{1}{\sigma} \qquad \frac{\partial T}{\partial \sigma} = -\frac{X-\mu}{\sigma^2} = -\frac{U}{\sigma}$$

- Example – Log-Normal distribution

$$U = \frac{1}{\sigma}\left[\log(X-a)-\mu\right]$$

$$\frac{\partial T}{\partial \mu} = -\frac{1}{\sigma} \qquad \frac{\partial T}{\partial \sigma} = -\frac{1}{\sigma^2}\left[\log(X-a)-\mu\right] = -\frac{U}{\sigma}$$

Reliability-based sensitivity analysis

- PMA:

$$\frac{\partial G_{p,\text{FORM}}}{\partial d_i} = \left.\frac{\partial G(\mathbf{U})}{\partial d_i}\right|_{\mathbf{U}=\mathbf{u}^*_{\beta=\beta_t}}$$

$$\frac{\partial G_{p,\text{FORM}}}{\partial d_i} = \left.\frac{\partial G(\mathbf{T}(\mathbf{X};\mathbf{d}))}{\partial d_i}\right|_{\mathbf{X}=\mathbf{x}^*_{\beta=\beta_t}}$$

  – Regular sensitivity analysis in optimization can be used
  – The sensitivity needs to be evaluated at MPP point.