Review for Exam 2

Optimization conditions

- Definitions of Global and local minima
  - We want to find a global but only afford to have local
- Unconstrained optimization problem
  - KT condition (f' = 0)
  - 2nd order necessary condition (f'' PSD)
  - Sufficient condition (f'' PD)
- Condition for global minimum
  - Convex objective on convex constraint set
  - When the obj and constraint set become convex?
- Equality constrained problem
  - Introduce Lagrangian $L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum \lambda_i h_i(\mathbf{x})$
  - KT condition

$$\frac{\partial L}{\partial \mathbf{x}} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

Optimization conditions

- Inequality constrained problem
  - Introduce slack variables: $L(\mathbf{x}, \lambda, s) = f(\mathbf{x}) + \sum \lambda_i (g_i + s_i^2)$
  - KT condition

$$\frac{\partial L}{\partial \mathbf{x}} = 0, \quad \frac{\partial L}{\partial \lambda} = 0, \quad \frac{\partial L}{\partial s} = 0$$

  - Complementary slackness ($\lambda_i g_i = 0$)
- 2nd-order necessary condition
  - $\nabla_x^2 L$ is P.S.D. for all feasible directions
- Sufficient condition
  - $\nabla_x^2 L$ is P.D. for all feasible directions

Numerical Method for Optimization

- Basic algorithm
  - Move from one design to another until can't reduce objective further
  - Need function values (objective & constraints) and their gradient
  - Need to find search direction and step size

$$\Delta \mathbf{x}^{(k)} = \alpha_k \mathbf{d}^{(k)}$$

- Unconstrained problem
  - Descent condition: New objective function must be smaller than previous one

$$\mathbf{c}^{(k)} \cdot \mathbf{d}^{(k)} < 0$$

  - Line search: find α_k that minimize the objective function for given direction

$$\text{minimize} \quad \phi(\alpha_k) = f(\mathbf{x}^{(k)} + \alpha_k \mathbf{d}_k)$$

  - Step size termination criterion:

$$\mathbf{c}^{(k+1)} \cdot \mathbf{d}^{(k)} = 0$$

Numerical Method for Optimization

- Search direction
  - Search direction should reduce the objective function
  - Different algorithms are available for different ways of calculating the search direction
- Steepest descent method
  - The objective function can be reduced the most in the negative gradient direction

$$\mathbf{d}^{(k)} = -\mathbf{c}^{(k)} = -\nabla f^{(k)}$$

  - Although this method seems to reduce f(x) the most, its convergence is slow due to consecutive orthogonal search directions $\mathbf{c}^{(k+1)} \perp \mathbf{c}^{(k)}$
  - This method converges slowly because the previous information is not used in finding the search direction

Numerical Method for Optimization

- Newton method
  - Very fast convergence when the initial design is close to the optimum design (quadratic convergence)
  - Need Hessian information $\Delta \mathbf{x}^{(k)} = -\mathbf{H}^{(k)^{-1}} \cdot \mathbf{c}^{(k)}$
  - If the Hessian in P.D., then new design will reduce f(x)
  - Difficulty in convergence when the Hessian changes its sign
  - Often line search is included (modified Newton method)
- Conjugate gradient method
  - Use previous gradient information $\mathbf{d}^{(k)} = -\mathbf{c}^{(k)} + \beta_k \mathbf{d}^{(k-1)}$
- Quasi-Newton method
  - Calculating Hessian is expensive -> Approximate Hessian or its inverse using gradient information
  - BFGS or DFP update
  - Maintain P.D. property of updated Hessian

Constrained Optimization

- Constrained optimization problem
  - Can convert to the unconstrained optimization problem
  - Can solve directly with constraints
- SUMT (Sequential Unconstrained Minimization Tech)
  - Penalize the objective function with violated constraints by multiplying with penalty parameter
  - Gradually increase the penalty parameter
  - When r becomes too big, Hessian becomes ill-conditioned
- Lagrange multiplier method
  - Minimize Lagrangian with x and λ

Constrained Optimization

- Direct method
  - Minimize the objective function with given feasible set
  - Can either follow interior or boundary of the feasible set
  - Epsilon-active strategy: for numerical purpose, consider a constraint active when it approaches zero
- Sequential linear programming (SLP)
  - Linearize the objective and constraints at the current design and solve for design change
- Quadratic programming subproblem (QP)
  - Quadratic objective with linear constraints for solving design change: convex problem and global optimum

- SLP and QP are used to calculate design change $\Delta\mathbf{x}$, followed by line search for step size

Constrained Optimization

- Feasible direction method
  - Combine both feasible direction (satisfying constraints) and usable direction (reducing objective)
- Constrained quasi-Newton method (Sequential quadtratic programming, SQP)
  - Solve the QP subproblem with approximate Hessian

$$\begin{aligned} \min imize \quad & f = \mathbf{c}^{\mathrm{T}}\mathbf{d} + \tfrac{1}{2}\mathbf{d}^{\mathrm{T}}(\nabla_{xx}L)\mathbf{d} \\ s.t. \quad & \mathbf{N}^{\mathrm{T}}\mathbf{d} = \mathbf{e} \\ & \mathbf{A}^{\mathrm{T}}\mathbf{d} \le \mathbf{b} \end{aligned}$$

  - Linear search for step size

Reliability Analysis and Design

- Study basic terminology of statistics (PDF, CDF, Normal…)
- Conditional probability $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
- Transformation of RV (X -> Y)
  - For given statistical property of X, calculate property of Y
  - Linear transformation (Y = aX + b)

$$\mu_y = \mathbf{a}^T \mu_X + b \qquad \sigma_Y^2 = \mathbf{a}^T \Sigma_X \mathbf{a}$$

- Nonlinear transformation (Y = g(X))
  - Linear approximation at mean: good when g is almost linear and uncertainty in X is small
  - Equivalent linearization: minimize expected value of square error

Transformation of Distribution

- Monotonic function Y = g(X)
- CDF

$$F_Y(y) = \begin{cases} F_X(g^{-1}(y)) & g \uparrow \\ 1 - F_X(g^{-1}(y)) & g \downarrow \end{cases}$$

- PDF

$$f_Y(y) = \begin{cases} \dfrac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y)) & g \uparrow \\ -\dfrac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y)) & g \downarrow \end{cases}$$

- General nonlinear function
  - Need to find a region $g(x) \le y$ and integrate f_X on that region

$$F_Y(y) = \int_{\{g(x) \le y\}} f_X(x)\,\mathrm{d}x$$

Reliability Analysis

- Limit state $g(X) = 0$; Failed state $g(X) < 0$
- Probability of failure (P_F) and reliability index (β_{HL})

$$P_F = P[g(X) \leq 0] \equiv \Phi(-\beta_{HL})$$

- For general nonlinear limit state

$$P_F = F_Y(0) = \int_{\{g(x)\leq 0\}} f_X(x)\,\mathrm{d}x$$

- Standard normal random variable with linear limit state

$$g(\mathbf{U}) = a_1 U_1 + a_2 U_2 + b$$

$$P_F = \Phi\left(-\frac{\mu_G}{\sigma_G}\right), \qquad \frac{\mu_G}{\sigma_G} = \frac{b}{\sqrt{a_1^2 + a_2^3}}$$

Approximate Reliability Analysis

- First Order Reliability Analysis (FORM)
  - Transform all input RVs (X) into SNRVs (U)
  - Find the closest point of $g(U) = 0$ from the origin
  - Approximate $g(U) = 0$ by tangent line at the closest point $g_L(U) = 0$
  - Reliability index is the distance from origin to the closest point
- Monte Carlo Simulation (MCS)
  - General random samples of input RVs (N)
  - Calculate the limit states samples using input RVs
  - Count the number of limit states less than zero (N_F)

$$P_{F,MCS} = \frac{N_F}{N}$$

Reliability0based Design Optimization (RBDO)

- Reliability appears as a constraint in optimization
- Reliability index approach $\beta_s = -\Phi^{-1}\{F_G(0)\} \geq \beta_t$

$$\begin{array}{ll} \text{minimize} & \|\mathbf{u}\| \\ \text{subject to} & G(\mathbf{u}) = 0 \end{array} \qquad \beta_{S,FORM} = \left\| \mathbf{u}^*_{G(\mathbf{u})=0} \right\|$$

  - Good for reliability analysis, but expensive and unstable when reliability is high or the limit state is highly nonlinear
- Performance measure approach $G_p = F_G^{-1}\{\Phi(-\beta_t)\} \geq 0$

$$\begin{array}{ll} \text{maximize} & G(\mathbf{u}) \\ \text{subject to} & \|\mathbf{u}\| = \beta_t \end{array} \qquad \text{Optimum: } G_p(\mathbf{u}^*)$$

  - Not suitable for assessing reliability (β_t is fixed), but efficient and stable for design optimization
- Sensitivity of reliability can be calculated eaily