CH3. Linear Stress – Strain – Temperature Relations

3.1. 1st-Law of Thermodynamics

- Assumption : isotropic (same for all direction) and linear.
- 1st-Law of thermodynamics (energy balance)

Work done by external force
Heat input
Internal energy increment
Kinetic energy increment

- Specialization for adiabatic ($\delta H = 0$) and static ($\delta K = 0$) condition.

- A member in equilibrium
  - displacement : u, v, w
  - displacement variation : $\delta u, \delta v, \delta w$ (arbitrary)
  - strain variation
  - work done by surface stress $\underline{\sigma}_p = [\sigma_{px}, \sigma_{py}, \sigma_{pz}]$

$$\delta W_s =$$

From pp. 9 $\Longrightarrow$

$$= \int_S T_x \, \delta u \, dS + \int_S T_y \, \delta v \, dS + \int T_z \, \delta w \, dS$$

$$= \int_S (\sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n) \, \delta u \, dS$$

$$+\int_S (\sigma_{xy} l + \sigma_{yy} m + \sigma_{yz} n)\,\delta v\, dS$$

$$+\int_S (\sigma_{xz} l + \sigma_{yz} m + \sigma_{zz} n)\,\delta w\, dS$$

※ Divergence theorem (surface integ. ⇒ volume integ.)

$$\int_S (f\underbrace{l}_{} + g\underbrace{m}_{} + h\underbrace{n}_{})\,dS =$$

direction cosine

$$\therefore\ \delta W_S = \int_V \frac{\partial}{\partial x}(\sigma_{xx}\delta u + \sigma_{xy}\delta v + \sigma_{xz}\delta w)\,dV$$

$$+\int_V \frac{\partial}{\partial y}(\sigma_{xy}\delta u + \sigma_{yy}\delta v + \sigma_{yz}\delta w)\,dV$$

$$+\int_V \frac{\partial}{\partial z}(\sigma_{xz}\delta u + \sigma_{yz}\delta v + \sigma_{zz}\delta w)\,dV$$

- Work done by body force $\underline{B} = [B_x \ \ B_y \ \ B_z]$

$$\delta W_B = \int_V (B_x\delta u + B_y\delta v + B_z\delta w)\,dV$$

$$\therefore\ \delta W = \delta W_S + \delta W_B$$

$$= \int_V \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} + B_x\right)\delta u\, dV$$

$$+\int_V \left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} + B_y\right)\delta v\, dV$$

$$+\int_V \left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + B_z\right)\delta w\, dV$$

$$+\int_V \sigma_{xx}\frac{\partial\delta u}{\partial x} + \sigma_{xy}\left(\frac{\partial\delta v}{\partial x} + \frac{\partial\delta u}{\partial y}\right) + \sigma_{xz}\left(\frac{\partial\delta w}{\partial x} + \frac{\partial\delta u}{\partial z}\right)$$

$$+\sigma_{yy}\frac{\partial\delta v}{\partial y} + \sigma_{yz}\left(\frac{\partial\delta v}{\partial z} + \frac{\partial\delta w}{\partial y}\right) + \sigma_{zz}\frac{\partial\delta w}{\partial z}\Big)\,dV$$

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$\Rightarrow$ $\boxed{\quad}$

※ using index notation

$\delta W =$

____________________ sum on i & j.

- Internal energy : integ. of internal energy density U_0.

energy per volume.

$$U = \int_V U_0 \, dV$$

$$\delta U = \int_V \delta U_0 \, dV$$

- From $\delta W = \delta U$

$\Rightarrow$ $\boxed{\delta U_0}$

1. Elasticity and U_0.

- elastic material : potential energy $= U$.

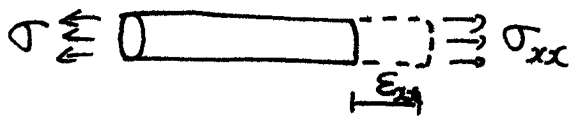
$$U_0 = U_0(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{xz}, x, y, z, T).$$

- Variation of internal energy density (chain rule)

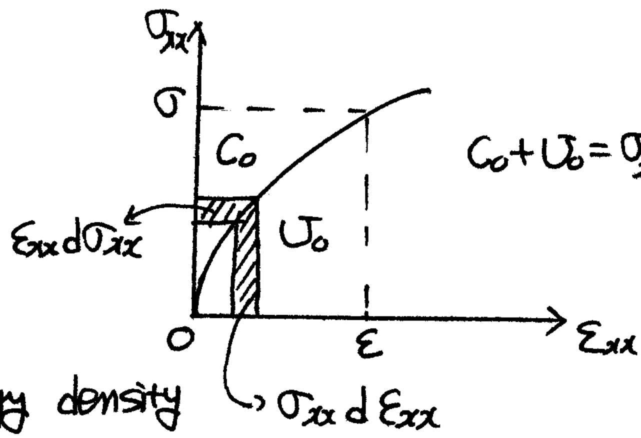
∴ For elastic material

2. Complementary Internal Energy Density, C_o.

- Uni-axial tension problem



$$\therefore \quad U_o = \int_0^{\varepsilon} \sigma_{xx} d\varepsilon_{xx}$$



$C_o + U_o = \sigma_{xx} \varepsilon_{xx}$

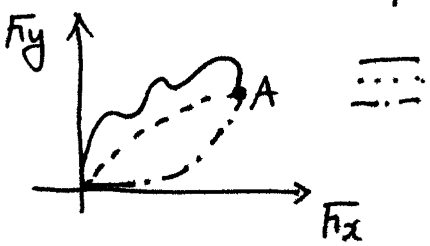
↑

Complementary internal energy density
or Complementary strain energy density

$\varepsilon_{xx} =$

- 3D problem

※ U_o & C_o are independent of loading path.



} all of them have the same U_o & C_o at point A.

3.2. Hooke's Law : Anisotropic Elasticity

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{16} \\ C_{21} & C_{22} & \cdots & C_{26} \\ \vdots & \vdots & \ddots & \vdots \\ C_{61} & C_{62} & \cdots & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

$\to$ not ε_{xy}, ε_{yz}, ε_{xz} $\because$ remove $\frac{1}{2}$.

$\Rightarrow$

↳ elastic coefficients.

- Relation with U_0.

$$\frac{\partial U_0}{\partial \varepsilon_{xx}} = \sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + \cdots + C_{16}\gamma_{xz}$$

$\Rightarrow C_{11} =$ $\qquad\qquad$ $C_{12} = C_{21} =$

$\cdots C_{56} = C_{65} =$

$\therefore C_{ij} = C_{ji}$: symmetric (21 components)

- general anisotropic material has 21 independent coefs.

3.3. Hooke's Law : Isotropic Elasticity

- Isotropy : same material property for all directions.
  Homogeneous : 〃 at everywhere.
- Isotropic material $\Rightarrow$ use principal strain (Invariants)

* Strain-Energy Density.

U_0

Since material properties are same for 3 principal directions

$$C_{11} = C_{22} = C_{33} := C_1$$
$$C_{12} = C_{13} = C_{23} := C_2$$

$U_0 =$

$$= \frac{1}{2} C_2 (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 2\varepsilon_1\varepsilon_2 + 2\varepsilon_1\varepsilon_3 + 2\varepsilon_2\varepsilon_3)$$
$$+ \frac{1}{2}(C_1 - C_2)(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$$
$$= \frac{1}{2} C_2 (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + \frac{1}{2}(C_1 - C_2)(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$$

Lame's constants

$$\begin{cases} \lambda = C_2 \\ G = \frac{1}{2}(C_1 - C_2) \end{cases}$$

$$\therefore \boxed{U_0 = \qquad\qquad}$$

From invariants of strains (pp. 21)

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \bar{I}_1$$
$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = \bar{I}_1^2 - 2\bar{I}_2$$

$\Rightarrow U_0 =$

$\Rightarrow \underline{U_0 =}$

Now, express $\bar{I}_1$ & $\bar{I}_2$ in terms of Cartesian components

$$U_0 = \frac{1}{2}\lambda\left(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\right)^2 + G\left(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2+2\varepsilon_{xy}^2+2\varepsilon_{xz}^2+2\varepsilon_{yz}^2\right)$$

Obtain stress-strain relation for linear elastic isotropic material

$$\boxed{\begin{array}{lll} \sigma_{xx} = \dfrac{\partial U_0}{\partial \varepsilon_{xx}} = \lambda \bar{I}_1 + 2G\varepsilon_{xx} & \sigma_{yy} = \lambda\bar{I}_1 + 2G\varepsilon_{yy} & \sigma_{zz} = \lambda\bar{I}_1 + 2G\varepsilon_{zz} \\ \sigma_{xy} = \dfrac{1}{2}\dfrac{\partial U_0}{\partial \varepsilon_{xy}} = 2G\varepsilon_{xy} & \sigma_{xz} = 2G\varepsilon_{xz} & \sigma_{yz} = 2G\varepsilon_{yz} \end{array}}$$

※ For isotropic material, only <u>two</u> elastic constants exist.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} \lambda+2G & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2G & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

$\Rightarrow$

Invert stress-strain relation

$E =$

$\nu =$

$\lambda =$

G :

- Bulk modulus

mean stress $\sigma_m =$

volumetric strain $e = (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$

$\sigma_m = K e \qquad K = \underline{\qquad\qquad}$

what happen when $\nu = 0.5$?

- Plane strain Problem

$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0.$

$$\Rightarrow \sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right]$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy}\right]$$

$$\sigma_{zz} = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy})$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}, \qquad \sigma_{xz} = \sigma_{yz} = 0.$$

- Plane Stress Problem

$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy})$$

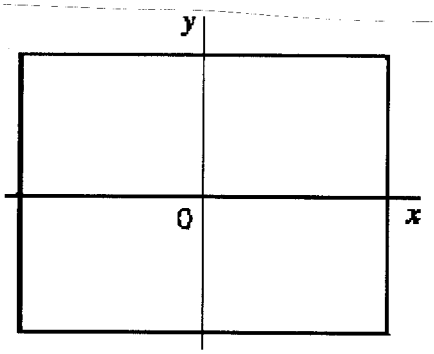
$$\sigma_{yy} = \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{yy})$$

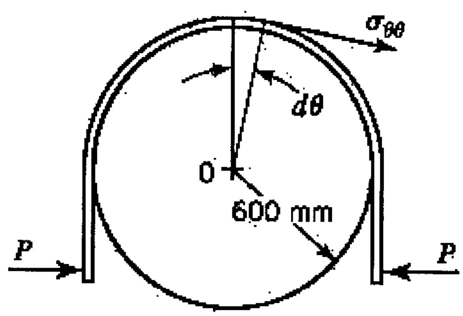
$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

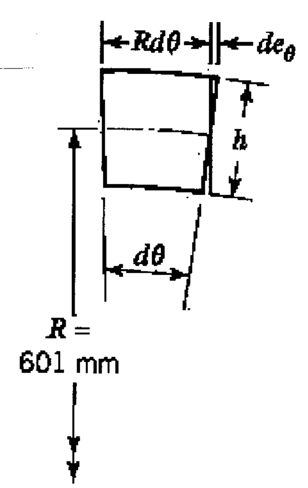
H.W. Derive this relation. from general 3D relation.

※ Principal directions for stress & strain are identical for isotropic materials.

Ex 3.1.







(a) No shear strains. Assume plane strain in y-dir.

From pp. 30

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu \cdot \sigma_{yy} - \nu \cancel{\sigma_{rr}}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{\theta\theta} - \nu\cancel{\sigma_{rr}}\right) = 0 \quad \text{plane strain}$$

$$\varepsilon_{rr} = \frac{1}{E}\left(\cancel{\sigma_{rr}} - \nu\sigma_{\theta\theta} - \nu\sigma_{yy}\right)$$

$$\therefore \quad \sigma_{yy} = \nu\sigma_{\theta\theta}$$

$$\Rightarrow \varepsilon_{\theta\theta} =$$

$$\tan(d\theta) = \frac{Rd\theta}{R} = \frac{de_\theta}{h/2} = \frac{\varepsilon_{\theta\theta}^{max} \cdot Rd\theta}{h/2}$$

$$\therefore \quad \varepsilon_{\theta\theta}^{max} = \frac{h}{2R}$$

$$\sigma_{\theta\theta}^{max} = \frac{E}{1-\nu^2} \cdot \varepsilon_{\theta\theta}^{max} = \frac{Eh}{2(1-\nu^2)R}$$

(b) $K = \frac{1}{R} = \frac{M}{EI}$ for unit width.

$$K = \frac{1}{R} = \frac{\sigma_{\theta\theta}^{max} \cdot 2(1-\nu^2)}{Eh} \quad \Leftarrow \sigma_{\theta\theta}^{max} = \frac{M \cdot \frac{h}{2}}{I}$$

$$K = \frac{1}{R} =$$

______________________ //

3.4. Thermo-Elasticity

- Temperature increase, ΔT, expands same amount in all directions for isotropic, homogeneous material.

: no shape change.

- combined effect of force & temp. change.

ε_{xx} : total strain. ε''_{xx} : due to the applied force.

- stress

$$
\begin{cases}
\sigma_{xx} = \lambda e'' + 2G\varepsilon''_{xx} = \lambda(e - 3\alpha\Delta T) + 2G(\varepsilon_{xx} - \alpha\Delta T) \\
\quad = \lambda e + 2G\varepsilon_{xx} - \underbrace{(3\lambda + 2G)\alpha}_{= \boxed{C = \frac{E\alpha}{1-2\nu}}}\Delta T \qquad e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \\
\sigma_{yy} = \lambda e + 2G\varepsilon_{yy} - C\Delta T \\
\sigma_{zz} = \lambda e + 2G\varepsilon_{zz} - C\Delta T \\
\sigma_{xy} = 2G\varepsilon_{xy}, \quad \sigma_{xz} = 2G\varepsilon_{xz}, \quad \sigma_{yz} = 2G\varepsilon_{yz}.
\end{cases}
$$

or

$$
\begin{cases}
\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] + \alpha\Delta T \\
\varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] + \alpha\Delta T \\
\varepsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] + \alpha\Delta T \\
\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}, \quad \varepsilon_{xz} = \frac{1+\nu}{E}\sigma_{xz}, \quad \varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{yz}.
\end{cases}
$$

- Strain Energy Density

$$U_0 = \left(\tfrac{1}{2}\lambda + G\right)\bar{I}_1^2 - 2G\bar{I}_2 - c\bar{I}_1 \Delta T + \tfrac{3}{2}c\alpha(\Delta T)^2$$

$$\boxed{\begin{aligned} U_0 = {} & \tfrac{1}{2}\lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + G(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2 + 2\varepsilon_{xy}^2 + 2\varepsilon_{xz}^2 + 2\varepsilon_{yz}^2) \\ & - c(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})\Delta T + \tfrac{3}{2}c\alpha(\Delta T)^2. \end{aligned}}$$

3.5. Orthotropic Materials

- wood, laminated plate, cold rolled steel, reinforced concrete, ..
- 3 orthogonal planes of symmetry

$$[\underset{\sim}{C}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \qquad \text{9 constants}$$

$$\begin{cases} \varepsilon_{xx} = \dfrac{1}{E_x}\sigma_{xx} - \dfrac{\nu_{yx}}{E_y}\sigma_{yy} - \dfrac{\nu_{zx}}{E_z}\sigma_{zz} \\ \varepsilon_{yy} = -\dfrac{\nu_{xy}}{E_x}\sigma_{xx} + \dfrac{1}{E_y}\sigma_{yy} - \dfrac{\nu_{zy}}{E_z}\sigma_{zz} \\ \varepsilon_{zz} = -\dfrac{\nu_{xz}}{E_x}\sigma_{xx} - \dfrac{\nu_{yz}}{E_y}\sigma_{yy} + \dfrac{1}{E_z}\sigma_{zz} \\ \gamma_{xy} = \dfrac{1}{G_{xy}}\sigma_{xy}, \quad \gamma_{xz} = \dfrac{1}{G_{xz}}\sigma_{xz}, \quad \gamma_{yz} = \dfrac{1}{G_{yz}}\sigma_{yz} \end{cases}$$

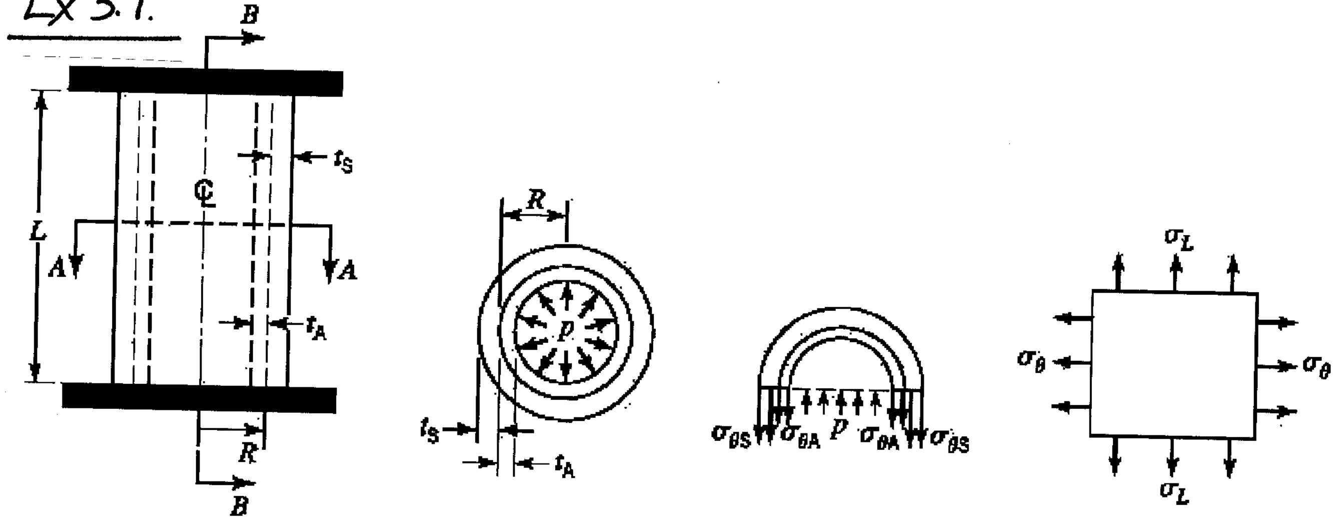
Due to symmetry

E_x, E_y, E_z : orthotropic moduli of elasticity.

G_{xy}, G_{xz}, G_{yz} : " shear moduli

ν_{xy} : Poisson's ratio (strain in y-dir caused by stress in x-dir.)

Ex 3.7.



From FBD ; $\Sigma F = 2pRL - 2\sigma_{\theta s} tL - 2\sigma_{\theta A} tL = 0$

(1) $\Rightarrow \sigma_{\theta A} + \sigma_{\theta s} = \frac{R}{t} p$ also $\sigma_r \approx 0$. plane stress.

From pp. 33

$$\begin{cases} E\varepsilon_L = \sigma_L - \nu\sigma_\theta + E\alpha\Delta T = 0 & \text{; fixed top \& bottom.} \\ E\varepsilon_\theta = \sigma_\theta - \nu\sigma_L + E\alpha\Delta T \end{cases}$$

Interface condition $\varepsilon_{\theta A} = \varepsilon_{\theta s}$ at R.

Since $t_A = t_s \ll R$, $\varepsilon_{\theta A} = \varepsilon_{\theta s} =$ constant in t.

(2) $\varepsilon_{LA} = \frac{1}{E_A}(\sigma_{LA} - \nu_A \sigma_{\theta A}) + \alpha_A \Delta T = 0$

(3) $\varepsilon_{LS} = \frac{1}{E_S}(\sigma_{LS} - \nu_S \sigma_{\theta S}) + \alpha_S \Delta T = 0$

(4) $\varepsilon_{\theta A} = \varepsilon_{\theta S} \Rightarrow \frac{1}{E_A}(\sigma_{\theta A} - \nu_A \sigma_{LA}) + \alpha_A \Delta T = \frac{1}{E_S}(\sigma_{\theta S} - \nu_S \sigma_{LS}) + \alpha_S \Delta T$

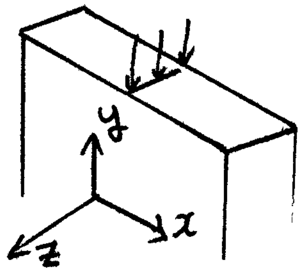
4 Eqs & 4 unknowns (σ_{LS}, σ_{LA}, $\sigma_{\theta S}$, $\sigma_{\theta A}$)

3.6. Plane Stress & Plane Strain Problems

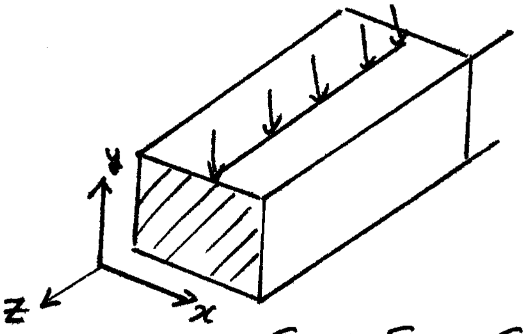
1. 2-D Problems

∘ Plane stress

-※· Plane strain



$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$



$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$

· Equations of Equilibrium

$$\Bigg(\qquad (1)$$

· Compatibility eq.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \qquad (2)$$

· Stress - Strain Relation

(3)

· Substitute (3) into (2)

$$\frac{\partial^2}{\partial y^2}(\sigma_{xx} - \nu\sigma_{yy}) + \frac{\partial^2}{\partial x^2}(\sigma_{yy} - \nu\sigma_{xx}) = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \qquad (4)$$

From (1)

$$\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial b_x}{\partial x} = 0$$

$$+\ \frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial y^2} + \frac{\partial b_y}{\partial y} = 0$$

$$2\frac{\partial^2\sigma_{xy}}{\partial x\partial y} =$$

Substitute into (4)

$$\frac{\partial^2}{\partial y^2}(\sigma_{xx}-\nu\sigma_{yy}) + \frac{\partial^2}{\partial x^2}(\sigma_{yy}-\nu\sigma_{xx}) = -(1+\nu)\left(\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}+\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{xx}}{\partial y^2}+\frac{\partial^2\sigma_{yy}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2} = -(1+\nu)\left(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\right)$$

$$\Rightarrow \underbrace{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)}_{\equiv\nabla^2}(\sigma_{xx}+\sigma_{yy}) = -(1+\nu)\left(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\right)$$

$\therefore$ $\boxed{}$

Solution for plane stress problem.

- Plane strain

Use $\Big($

$\Rightarrow$ $\boxed{}$

Solution for plane strain problem

- b_x, b_y constant

$$\boxed{} \qquad (5)$$

Harmonic Eq. Laplace Eq.

※ Same solution for plane strain & plane stress

Solution is independent of material

same stress, but strain, displ. will be different.

2. Airy Stress Functions

- Special Case. $b_x = 0$. $b_y = \rho g$ gravitation

Consider an Airy stress function $\phi(x,y)$ that satisfies

Then, equilibrium Eq. (1) is automatically satisfied.

Substitute into Harmonic Eq. (5).

$$\nabla^2 \left(\frac{\partial^2 \phi}{\partial y^2} + \rho g y + \frac{\partial^2 \phi}{\partial x^2} - \rho g y \right) = 0$$

bi-harmonic Eq.

∴ When a body force is caused by gravity,

(6)

Solution $\phi \Rightarrow \sigma_{xx}, \sigma_{yy}, \sigma_{xy}$.

3. Solution by Polynomials

(good for rectangular shape with continuous loading)

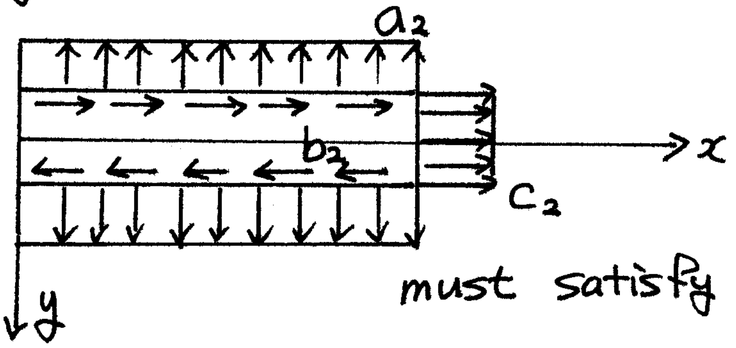
- 2nd-order

$\nabla^4 \phi_2 = 0$: satisfy automatically

$$\sigma_{xx} = \frac{\partial^2 \phi_2}{\partial y^2} = c_2$$

$$\sigma_{yy} = \frac{\partial^2 \phi_2}{\partial x^2} = a_2 \qquad \therefore \phi_2 : \text{uniform stress field.}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -b_2$$



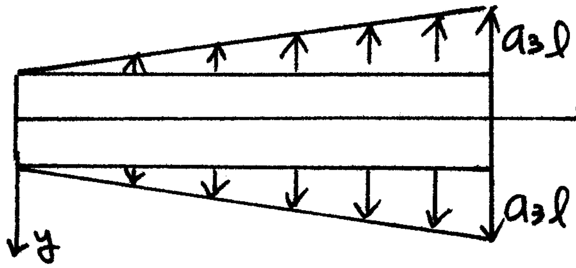
must satisfy B.C.s.

- 3rd-order

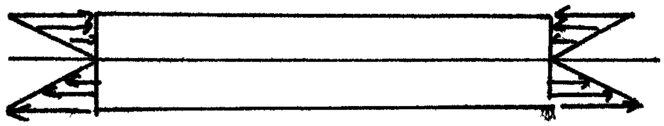
$\nabla^4 \phi_2 = 0$: automatically satisfy

$$\begin{cases} \sigma_{xx} = \\ \sigma_{yy} = \\ \sigma_{xy} = \end{cases}$$

ex) $\phi_3 = \frac{a_3}{3!}x^3$ $\sigma_{xx} = 0$ $\sigma_{yy} = a_3 x$ $\sigma_{xy} = 0$



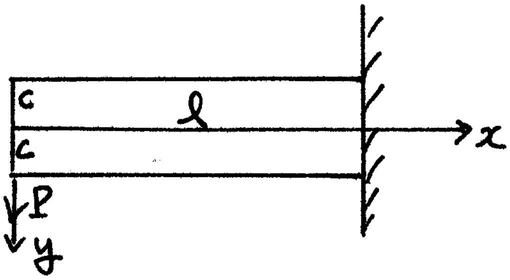
$\phi_3 =$ $\sigma_{xx} = d_3 y$ $\sigma_{yy} = 0$ $\sigma_{xy} = 0$



pure bending

4. Bending of a Cantilever Beam

$$\sigma_{xx} = \frac{M_z}{I} y$$



$$M_z = P \cdot x \equiv kx$$

$$\sigma_{xx} = \frac{k}{I} xy \equiv d_4 xy$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = d_4 xy$$ integ.

$\Rightarrow$

$\Rightarrow$

$\therefore$

$$\Rightarrow f_1(x) = b_1 + b_2 x + b_3 x^2 + b_4 x^3$$

$$f_2(x) = b_8 + b_7 x + b_6 x^2 + b_5 x^3$$

$$\Rightarrow \phi = \frac{d_4}{3\cdot 2} x y^3 + y(\cancel{b_1} + b_2 x + b_3 x^2 + b_4 x^3)$$
$$+ (\cancel{b_8} + \cancel{b_7} x + b_6 x^2 + b_5 x^3)$$

$\because$ no effect on stress. (linear & const)

$\Rightarrow$ (

- **Apply B.C.**

  - $\sigma_{yy} = 0$ at $y = \pm c$

$$\sigma_{yy}\big|_{y=c} = 6(b_4 c + b_5)x + 2(b_3 c + b_6) = 0$$
$$\sigma_{yy}\big|_{y=-c} = 6(-b_4 c + b_5)x + 2(-b_3 c + b_6) = 0$$
$$\therefore b_3 = b_4 = b_5 = b_6 = 0$$

$\therefore \ \phi =$

$$\begin{cases} \sigma_{xx} = d_4 x y \\ \sigma_{yy} = 0 \\ \sigma_{xy} = -\frac{d_4}{2} y^2 - b_2 \end{cases}$$

  - $\sigma_{xy} = 0$ at $y = \pm c$

$$\sigma_{xy}\big|_{y=c} = \sigma_{xy}\big|_{y=-c} = -\frac{d_4}{2} c^2 - b_2 = 0 \qquad \therefore b_2 = -\frac{d_4}{2} c^2$$

- **Apply equilibrium**

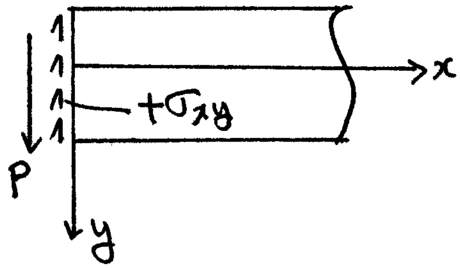
$$P = -\int_A \sigma_{xy}\, dA =$$

Moment of inertia $I = \frac{b(2c)^3}{12} = \frac{bc^3}{\frac{3}{2}}$

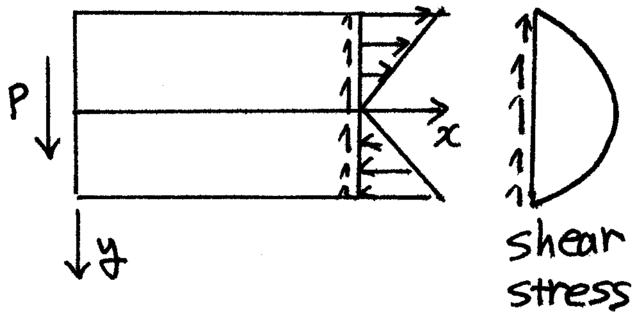
$\therefore \; P =$

$d_4 =$

$b_2 =$



$\therefore \; \phi(x,y) =$



- Displacement

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) = \frac{\sigma_{xx}}{E} = -\frac{P}{EI}xy = \frac{\partial u}{\partial x} \qquad -(a)$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) = -\frac{\nu\sigma_{xx}}{E} = \nu\frac{P}{EI}xy = \frac{\partial v}{\partial y} \qquad -(b)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy} = -\frac{(1+\nu)P}{EI}(c^2-y^2) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad -(c)$$

- Integrate

$u =$

$v =$) put into (c)

$$-\frac{P}{2EI}x^2 + \frac{dg_1}{dy} + \frac{\nu P}{2EI}y^2 + \frac{dg_2}{dx} = -\frac{(1+\nu)P}{EI}(c^2-y^2)$$

$$\Rightarrow \underbrace{-\frac{P}{2EI}x^2 + \frac{dg_2}{dx} + \frac{(1+\nu)P}{EI}c^2}_{\text{fn of } x} = \underbrace{\frac{(1+\nu)P}{EI}y^2 - \frac{\nu P}{2EI}y^2 - \frac{dg_1}{dy}}_{\text{fn of } y} = \text{const} = a_1$$

$$\Rightarrow \begin{cases} \dfrac{dg_2}{dx} = \dfrac{P}{2EI}x^2 - \dfrac{(1+\nu)P}{EI}c^2 + a_1 \\ \dfrac{dg_1}{dy} = \dfrac{1+\nu}{EI}Py^2 - \dfrac{\nu P}{2EI}y^2 - a_1 \end{cases}$$

$$\begin{cases} g_2(x) = \\ g_1(y) = \end{cases}$$

$$\therefore \begin{cases} u(x,y) = \\ v(x,y) = \end{cases}$$

- Apply B.C.s (to prevent rigid body motion)

$u\big|_{\substack{x=l \\ y=0}} = 0$; $a_2 = 0$

$v\big|_{\substack{x=l \\ y=0}} = 0$; $\dfrac{P}{6EI}l^3 - \dfrac{1+\nu}{EI}Pc^2 l + a_1 l + a_3 = 0$

Need more B.C.s.

- Case 1) No slope change on the wall

$$\frac{\partial v}{\partial x}\Big|_{\substack{x=l \\ y=0}} = 0 = \frac{P}{2EI}l^2 - \frac{1+\nu}{EI}Pc^2 + a_1$$

$$\therefore a_1 = -\frac{Pl^2}{2EI} + \frac{1+\nu}{EI}Pc^2$$

$$a_3 = \frac{Pl^3}{3EI}$$

$$\therefore \boxed{\begin{aligned} u(x,y) &= -\frac{P}{2EI}x^2 y - \frac{\nu P}{6EI}y^3 + \frac{P}{6GI}y^3 + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2GI}\right)y \\ v(x,y) &= \frac{\nu P}{2EI}xy^2 + \frac{P}{6EI}x^3 - \frac{Pl^2}{2EI}x + \frac{Pl^3}{3EI} \end{aligned}}$$

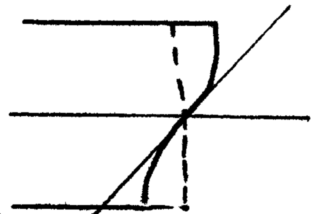
$$v\big|_{y=0} = \frac{P}{3EI}x^3 - \frac{Pl^2}{2EI}x + \frac{Pl^3}{3EI}$$

$$u\big|_{y=0} = 0$$

$$\frac{1}{\rho} = \frac{d^2v}{dx^2} = \frac{Px}{EI} = \frac{M}{EI}$$

$$u\big|_{x=l} = -\frac{\nu P}{6EI}y^3 + \frac{P}{6GI}y^3 - \frac{Pc^2}{2GI}y$$

$$\frac{\partial u}{\partial y}\Big|_{x=l} = -\frac{\nu P}{2EI}y^2 + \frac{P}{2GI}y^2 - \frac{Pc^2}{2GI}$$



straight cross-section doesn't remain straight

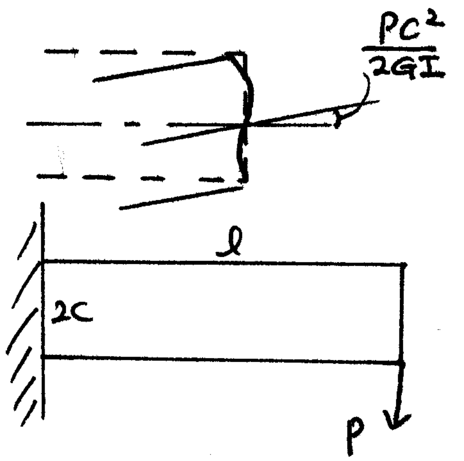
- Case 2)

$$\frac{\partial u}{\partial y}\Big|_{\substack{x=l \\ y=0}} = 0 \quad \Longrightarrow \quad a_1 = -\frac{Pl^2}{2EI}$$

$$u(x,y) = -\frac{P}{2EI}x^2y + \frac{P}{3EI}\left(1+\frac{\nu}{2}\right)y^3 + \frac{Pl^2}{2EI}y$$

$$v(x,y) = \frac{\nu P}{2EI}xy^2 + \frac{P}{6EI}x^3 + \frac{1+\nu}{EI}Pc^2(l-x) - \frac{Pl^2}{2EI}x + \frac{Pl^3}{3EI}$$

$$v\big|_{y=0} = \underbrace{\frac{P}{6EI}x^3 - \frac{Pl^2}{2EI}x + \frac{Pl^3}{3EI}}_{\text{bending}} + \underbrace{\frac{Pc^2}{2GI}(l-x)}_{\text{shear}}$$



$$v\big|_{\substack{x=0 \\ y=0}} = \frac{Pl^3}{3EI} + \frac{Pc^2l}{2GI}$$

$$\text{Ratio} \quad \frac{\frac{Pc^2l}{2GI}}{\frac{Pl^3}{3EI}} = \begin{cases} 1 & \text{when } \frac{l}{2c} = 1 \\ \frac{1}{10} & \text{when } \frac{l}{2c} = 3.12 \\ \approx 0 & \text{when } \frac{l}{2c} = 50 \end{cases}$$

HW 3 : Derive the relations in pp. 31 36

Solve problems 3.5 3.15 3.21

3.5. For an isotropic elastic medium subjected to a hydrostatic state of stress, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ and $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$, where p denotes pressure [FL^{-2}]. Show that for this state of stress $p = -Ke$, where $K = E/[3(1-2\nu)]$ is the bulk modulus and $e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ is the classical small-displacement cubical strain (also called the volumetric strain).

3.15. An airplane wing spar (Figure P3.15) is made of an aluminum alloy (E = 72 GPa and ν = 0.33), and it has a square cross section perpendicular to the plane of the figure. Stress components σ_{xx} and σ_{yy} are uniformly distributed as shown.

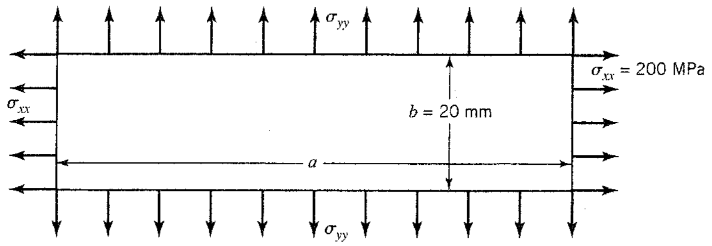


FIGURE P3.15

3.21. A member whose material properties remain unchanged (invariant) under rotations of 90° about axes (x, y, z) is called a *cubic material* relative to axes (x, y, z) and has three independent elastic coefficients (C_1, C_2, C_3). Its stress–strain relations relative to axes (x, y, z) are (a special case of Eq. 3.50)

$$\sigma_{xx} = C_1\epsilon_{xx} + C_2\epsilon_{yy} + C_2\epsilon_{zz}$$
$$\sigma_{yy} = C_2\epsilon_{xx} + C_1\epsilon_{yy} + C_2\epsilon_{zz}$$
$$\sigma_{zz} = C_2\epsilon_{xx} + C_2\epsilon_{yy} + C_1\epsilon_{zz}$$
$$\sigma_{xy} = C_3\gamma_{xy}$$
$$\sigma_{xz} = C_3\gamma_{xz}$$
$$\sigma_{yz} = C_3\gamma_{yz}$$

Although in practice aluminum is often assumed to be an isotropic material (E = 72 GPa and ν = 0.33), it is actually a cubic material with C_1 = 103 GPa, C_2 = 55 GPa, and C_3 = 27.6 GPa. At a point in an airplane wing, the strain components are ϵ_{xx} = 0.0003, ϵ_{yy} = 0.0002, ϵ_{zz} = 0.0001, ϵ_{xy} = 0.00005, and $\epsilon_{xz} = \epsilon_{yz} = 0$.

a. Determine the orientation of the principal axes of strain.

b. Determine the stress components.

c. Determine the orientation of the principal axes of stress.

d. Calculate the stress components and determine the orientation of the principal axes of strain and stress under the assumption that the aluminum is isotropic.