CH5. Energy Methods

5.1. Principle of Stationary Potential Energy

- DOF : $x_1, x_2, \cdots, x_n$
- virtual displacement : $\delta x_1, \delta x_2, \cdots, \delta x_n$
- generalized load : $Q_1, Q_2, \cdots, Q_n$.
- virtual work

virtual work done by external forces

virtual work done by internal forces

- generalized external load : $P_1, P_2, \cdots, P_n$

$$\delta W_e =$$

- elastic system with closed path $(\delta x_1 = \delta x_2 = \cdots = \delta x_n = 0)$

$\delta W_e = 0$. Then,

For the closed path, system returns to its initial state.

$\Rightarrow$

$\Rightarrow$ System is conservative

: principle of stationary potential energy

- Conservative System

$\delta W =$

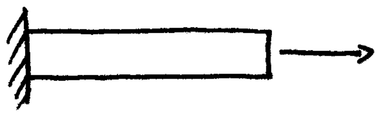
$\Rightarrow$ $\qquad\qquad\qquad\qquad\qquad i = 1, 2, \cdots, n$

- When system is in equilibrium $\delta W = 0 \Rightarrow Q_i = 0$

$i = 1, 2, \cdots, n$

: Castigliano's first theorem

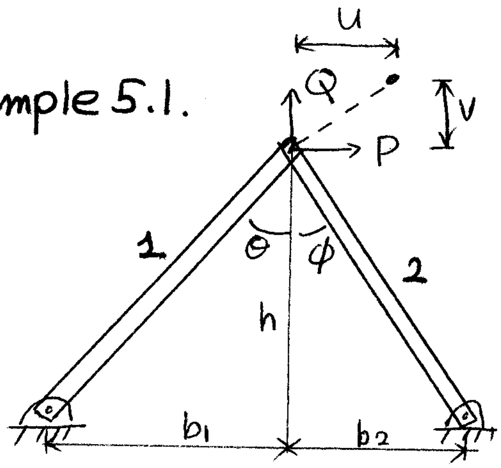
- Valid for nonlinear elastic system (conservative)
- Also valid for inelastic system with monotonic loading.

ex) 

$U = \int P\, de \qquad P = \frac{\partial U}{\partial e}$

$U_0 = \int \sigma\, d\varepsilon \qquad \sigma = \frac{\partial U_0}{\partial \varepsilon}$

Example 5.1.



$e_1 = \sqrt{(b_1+u)^2 + (h+v)^2} - L_1$

$e_2 = \sqrt{(b_2-u)^2 + (h+v)^2} - L_2$

$L_1 = \sqrt{b_1^2 + h^2} \qquad L_2 = \sqrt{b_2^2 + h^2}$

$U =$

$P =$

$Q =$

$$\frac{\partial e_1}{\partial u} = \frac{b_1 + u}{\sqrt{(b_1+u)^2 + (h+v)^2}} \qquad \frac{\partial e_2}{\partial u} = \frac{-(b_2 - u)}{\sqrt{(b_2-u)^2 + (h+v)^2}}$$

$$\frac{\partial e_1}{\partial v} = \frac{h+v}{\sqrt{(b_1+u)^2 + (h+v)^2}} \qquad \frac{\partial e_2}{\partial v} = \frac{+(h+v)}{\sqrt{(b_2-u)^2 + (h+v)^2}}$$

5.2. Castigliano's Theorem on Deflections

- Based on complementary energy C. (same with U for linear elastic system).
- For properly constrained system with concentrated loads $F_1, F_2 \cdots F_n$, the displacement q_i at the point F_i is applied is

$$i = 1, \cdots, n$$

- Limited to small deformation

$$C = \sum_{i=1}^{n} C_i$$

- generalization to rotational angle

Example 5.2. Same with example 5.1.

- nonlinear elastic with $\varepsilon = \varepsilon_0 \sinh\left(\frac{\sigma}{\sigma_0}\right)$.

- Equilibrium

ΣF_x ;

ΣF_y ;

$$\sin\theta = \frac{b_1}{L_1} \quad \cos\theta = \frac{h}{L_1} \quad \sin\phi = \frac{b_2}{L_2} \quad \cos\phi = \frac{h}{L_2}$$

$$\begin{cases} P\cos\phi - N_1 \sin\theta\cos\phi + N_2 \sin\phi\cos\phi = 0 \\ Q\sin\phi - N_1\cos\theta\sin\phi - N_2\cos\phi\sin\phi = 0 \end{cases}$$

$$N_1 = \frac{P\cos\phi + Q\sin\phi}{\sin\theta\cos\phi + \cos\theta\sin\phi} = \frac{P\frac{h}{L_2} + Q\frac{b_2}{L_2}}{\frac{b_1}{L_1}\frac{h}{L_2} + \frac{h}{L_1}\frac{b_2}{L_2}}$$

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$C =$ $\qquad e_1 = \varepsilon_1 L_1 \ , \quad e_2 = \varepsilon_2 L_2$

$$= \int_0^{N_1} L_1 \varepsilon_0 \sinh\left(\frac{N_1}{A_1\sigma_0}\right) dN_1 + \int_0^{N_2} L_2 \varepsilon_0 \sinh\left(\frac{N_2}{A_2\sigma_0}\right) dN_2$$

$$u = \frac{\partial C}{\partial P} = L_1\varepsilon_0 \sinh\left(\frac{N_1}{A_1\sigma_0}\right)\frac{\partial N_1}{\partial P} + L_2\varepsilon_0\sinh\left(\frac{N_2}{A_2\sigma_0}\right)\frac{\partial N_2}{\partial P}$$

$$v = \frac{\partial C}{\partial Q} = L_1\varepsilon_0 \sinh\left(\frac{N_1}{A_1\sigma_0}\right)\frac{\partial N_1}{\partial Q} + L_2\varepsilon_0\sinh\left(\frac{N_2}{A_2\sigma_0}\right)\frac{\partial N_2}{\partial Q}$$

$$\frac{\partial N_1}{\partial P} = \frac{L_1 h}{h(b_1+b_2)} \ , \quad \frac{\partial N_1}{\partial Q} = \frac{L_1 b_2}{h(b_1+b_2)}$$

$$\frac{\partial N_2}{\partial P} = \frac{-L_2 h}{h(b_1+b_2)} \ , \quad \frac{\partial N_2}{\partial Q} = \frac{L_2 b_2}{h(b_1+b_2)}$$

5.3. Linear Load-Deflection Relations

$$U = \int U_0 \, dV \qquad \text{function of generalized loads}$$

$$U_0 = \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx})$$

$$+ \frac{1}{2G}(\sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2).$$

Or add individual contribution

$$U_0 = U_{axial} + U_{bending} + U_{torsion} + \cdots$$

1. U_N : Strain energy for Axial Loading

$$de = \varepsilon \, dz = \frac{\sigma}{E} dz = \frac{N}{EA} dz$$

$\therefore$ $\boxed{}$ superposition is possible.

- Axially Loaded Spring
  - Load Q, elongation δ
  - Strain energy $U = \int dU = \int_0^\delta Q \, dx$
  - Complementary energy $C = \int dC = \int_0^P x \, dQ$
  - Linear system

$$U = C = \int_0^\delta Q \, dx \qquad \text{for spring } Q = F = kx$$

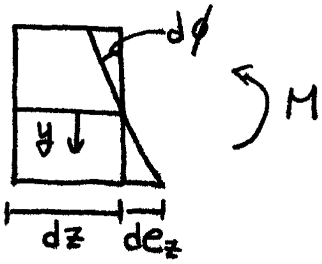
- $U = C =$

2. U_M & U_S : Bending & Shear Strain Energy

- Strain energy by bending

$$U_M =$$

$$d\phi = \frac{de_z}{y} = \frac{\varepsilon\, dz}{y} = \frac{\sigma\, dz}{E y} = \frac{M y}{I}\frac{dz}{E y}$$



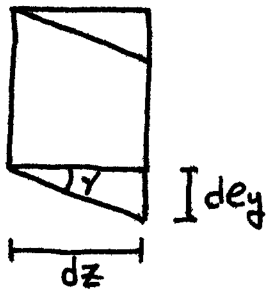
- Approximate shear energy U_S

$$U_S = \int k\, dU_S' = \int k \cdot \frac{1}{2} V_y\, de_y$$

linear approx

correction factor

$$de_y = \gamma\, dz = \frac{\tau}{G} dz = \frac{V_y}{GA} dz$$



$\Rightarrow$ strain energy for shear loading of a beam

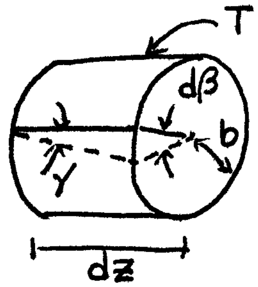
- U_S is usually small compared to U_M.
- k is the ratio between accurate shear stress $\frac{V_y Q}{I b}$ and averaged shear stress $\frac{V_y}{A}$. at the neutral axis. Q: 1st moment of area above y.

approximate

$k =$ $= 1.50$ for rectangular cross-section.

3. Strain Energy for Torsion : U_T

- Circular cross-section

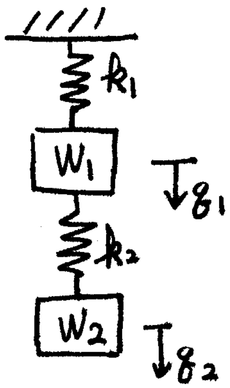


$U_T =$

$$b \cdot d\beta = \gamma dz = \frac{\tau}{G} dz = \frac{Tb}{GJ} dz$$

$\Rightarrow$ $\boxed{}$ strain energy for torsion.

Example 5.3.



$U =$

We need $U = U(W_1, W_2)$ to calculate δ_1 & δ_2.

Use $F_1 = k_1 \delta_1$, $F_2 = k_2(\delta_2 - \delta_1)$

internal force of springs

$$W_1 + W_2 = F_1 = k_1 \delta_1$$

$$W_2 = F_2 = k_2(\delta_2 - \delta_1)$$

$\Rightarrow$ $U =$

Use Castigliano's theorem

$$\delta_1 = \frac{\partial U}{\partial W_1} =$$

$$\delta_2 = \frac{\partial U}{\partial W_2} =$$

Example 5.4. Nonlinear springs.

- Force-elongation relation : $F = k\delta^2$
- U & C are different for nonlinear system.
- Use C.

$$C =$$

$$= \int_0^{F_1} \left(\frac{F}{k_1}\right)^{1/2} dF + \int_0^{F_2} \left(\frac{F}{k_2}\right)^{1/2} dF$$

$$=$$

- Use $F_1 = W_1 + W_2$, $F_2 = W_2$

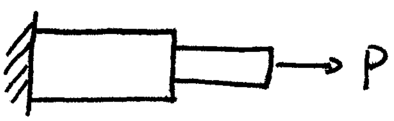
$$C = \frac{2}{3}\left(\frac{(W_1+W_2)^{3/2}}{k_1^{1/2}} + \frac{W_2^{3/2}}{k_2^{1/2}}\right)$$

$$\therefore \quad \delta_1 = \frac{\partial C}{\partial W_1} =$$

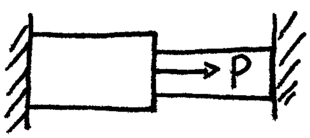
$$\delta_2 = \frac{\partial C}{\partial W_2} =$$

5.4. Statically Determinate System

~ member force & stress can be determined from static equilibrium, not considering deformation.



statically determinate



statically indeterminate

- When $U_j = U_{N_j} + U_{M_j} + U_{S_j} + U_{T_j}$

- Deflection δ_i at the location of applied force F_i:

$$\delta_i = \frac{\partial U}{\partial F_i} = \sum_{j=1}^{m} \left(\int \frac{N_j}{E_j A_j} \frac{\partial N_j}{\partial F_i} dz + \int \frac{k_j V_j}{G_j A_j} \frac{\partial V_j}{\partial F_i} dz + \int \frac{M_j}{E_j I_j} \frac{\partial M_j}{\partial F_i} dz + \int \frac{T_j}{G_j J_j} \frac{\partial T_j}{\partial F_i} dz \right)$$

- Slope change θ_i at the location of moment M_i

$$\theta_i = \frac{\partial U}{\partial M_i}$$

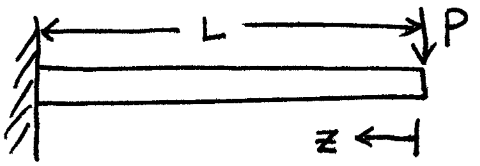
∘ Procedure

(1) Write internal actions (force, shear, moment, torque) in terms of applied loads

(2) Calculate $\frac{\partial N_j}{\partial F_i}$, $\frac{\partial V_j}{\partial F_i}$, $\frac{\partial M_j}{\partial F_i}$, $\frac{\partial T_j}{\partial F_i}$, $\frac{\partial N_j}{\partial M_i}$, ...

(3) Calculate $\delta_i = \frac{\partial U}{\partial F_i}$ from (1) & (2).

Example 5.5 Cantilevered Beam



· Ignore shear.

$U =$

$\delta_P =$ $\Leftarrow \left(\begin{array}{l} M = P \cdot z \\ \frac{\partial M}{\partial P} = z . \end{array} \right.$

$=$

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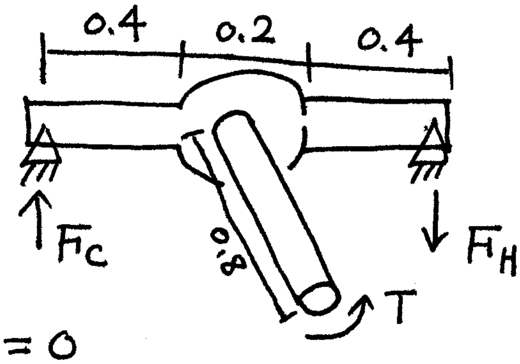
Example 5.7



ΣF_y ; $F_C - F_H = 0$

ΣM_z ; $0.5 F_C + 0.5 F_H - T = 0$ $\qquad \therefore\ F_C = F_H = T.$

$U = U_M + U_T =$

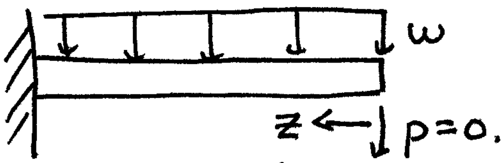
$\theta_B = \dfrac{\partial U}{\partial T} =$

$$= 2\int_0^{0.4} \frac{Tz^2}{2EI}\,dz + \int_0^{0.8} \frac{T}{2GJ}\,dz$$

$$= \frac{1}{3}\frac{T \cdot 0.4^3}{EI} + \frac{T}{2GJ} \cdot 0.8$$

Example 5.8. Cantilevered beam with distributed load

- Introduce a dummy load P at the tip. $P=0$.



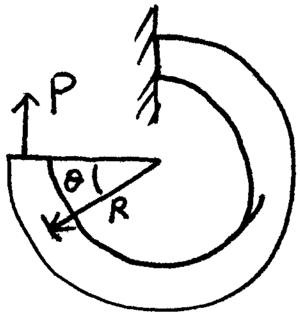
$U =$ $\qquad \delta_P =$

$M(x) = Pz + \frac{1}{2}\omega z^2 \qquad \dfrac{\partial M}{\partial P} = z.$

$\delta_P =$ $\qquad = \displaystyle\int_0^L \frac{\omega z^3}{2EI}\,dz$

$$= \frac{\omega L^4}{8EI}$$ //

Example 5. 9 Curved Beam



- Approx. as a straight beam
- At the cross-section with angle θ

$N =$

$V =$

$M =$

$\frac{\partial N}{\partial P} =$ $\qquad$ $\frac{\partial V}{\partial P} =$ $\qquad$ $\frac{\partial M}{\partial P} =$

$$U = \int_0^{\frac{3}{2}\pi} \frac{M^2}{2EI} \cdot R d\theta + \int_0^{\frac{3}{2}\pi} \frac{kV^2}{2GA} R d\theta + \int_0^{\frac{3}{2}\pi} \frac{N^2}{2EA} R d\theta$$

$$\delta_P = \frac{\partial U}{\partial P} = \int_0^{\frac{3}{2}\pi} \frac{M}{EI} R(1-\cos\theta) R d\theta + \int_0^{\frac{3}{2}\pi} \frac{kV}{GA} \sin\theta R d\theta$$

$$+ \int_0^{\frac{3}{2}\pi} \frac{N}{2EA} \cos\theta R d\theta$$

* Dummy Load Method

When no force is applied at the point of interest,

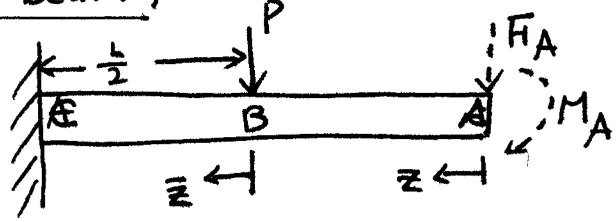
(1) Apply a fictitious force F_i (or M_i)

(2) Derive U and differentiate it with F_i

(3) Set $F_i = 0$.

Example 5.11 Cantilevered beam.

Tip displ. & rotation



$M_{AB} =$

$M_{BC} =$

$\dfrac{\partial M_{AB}}{\partial F_A} = \qquad \dfrac{\partial M_{AB}}{\partial M_A} =$

$\dfrac{\partial M_{BC}}{\partial F_A} = \qquad \dfrac{\partial M_{BC}}{\partial M_A} =$

$$U = \int_0^{L/2} \frac{M_{AB}^2}{2EI}\, dz + \int_0^{L/2} \frac{M_{BC}^2}{2EI}\, dz$$

$$\delta_A = \frac{\partial U}{\partial F_A} = \int_0^{L/2} \frac{M_{AB}}{EI} \cdot z\, dz + \int_0^{L/2} \frac{M_{BC}}{EI}\left(\bar{z} + \frac{L}{2}\right) dz \Bigg|_{\substack{M_A = 0 \\ F_A = 0}}$$

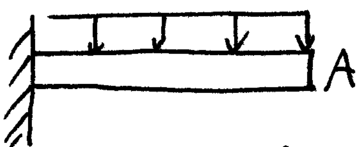
$$= 0 + \int_0^{L/2} \frac{P\bar{z}}{EI}\left(\bar{z} + \frac{L}{2}\right) dz$$

$$= \frac{P}{EI}\left(\frac{1}{3}\bar{z}^3 + \frac{L}{4}z^2\right)\Bigg|_0^{L/2} = \frac{P}{EI}\left(\frac{L^3}{24} + \frac{L^3}{16}\right) = \frac{5PL^3}{48EI}$$

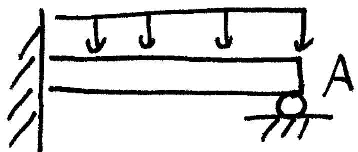
$$\theta_A = \frac{\partial U}{\partial M_A}\Bigg|_{\substack{M_A = 0 \\ F_A = 0}} = \int_0^{L/2} \frac{M_{AB}}{EI} \cdot 1\, dz + \int_0^{L/2} \frac{P\bar{z}}{EI} \cdot 1\, dz = \frac{PL^2}{8EI}$$

5.5. Statically Indeterminate Structures

- Equilibrium equation is not sufficient.



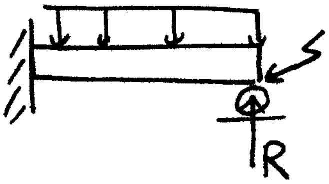
Determinate



Indeterminate
Redundant constraint.

- Need more information.
  use zero displ. at A to solve the problem.

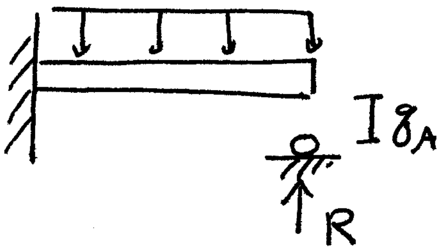
- For redundant forces (internal or external)



since no displ. $\frac{\partial U}{\partial R} = 0.$

$\therefore$ easy to show for redundant constraint.

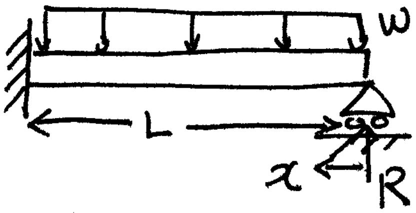
- Structure with initial gap or overlap.



Solve with R and use cond.

$$\delta_1 = -\frac{\partial U}{\partial R}$$

Example 5.15 Propped Cantilevered Beam



$M =$

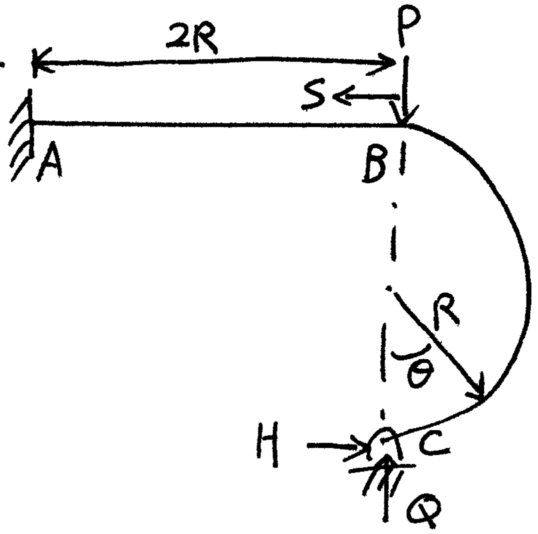
$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\delta_R = 0 = \frac{\partial U}{\partial R} = \int_0^L \frac{(Rx - \frac{1}{2}wx^2)}{EI}\cdot x\,dx$$

$$= \frac{1}{3}\frac{RL^3}{EI} - \frac{wL^4}{8EI} = 0$$

$$\therefore R = \frac{3wL}{8}$$

Example 5.16



consider bending only

$M_{AB} =$

$M_{BC} =$

$$\frac{\partial M_{AB}}{\partial Q} = s \,, \quad \frac{\partial M_{AB}}{\partial H} = 2R$$

$$\frac{\partial M_{BC}}{\partial Q} = R\sin\theta, \quad \frac{\partial M_{BC}}{\partial H} = -R(1-\cos\theta)$$

$$\frac{\partial U}{\partial Q} = \int_0^{2R} \frac{(Q-P)s + 2RH}{EI} \cdot s\, ds + \int_0^{\pi} \frac{QR\sin\theta - HR(1-\cos\theta)}{EI} R\sin\theta \, R\, d\theta = 0$$

$\Rightarrow$

$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$ ①

$$\frac{\partial U}{\partial H} = \int_0^{2R} \frac{(Q-P)s + 2RH}{EI} \cdot 2R\, ds + \int_0^{\pi} \frac{QR\sin\theta - HR(1-\cos\theta)}{EI} [-R(1-\cos\theta)] \, R\, d\theta = 0$$

$\Rightarrow$

$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$ ②

Solve for Q & H.

$$\delta_P = \frac{\partial U}{\partial P} = \int_0^{2R} \frac{(Q-P)s + 2RH}{EI} \cdot (-s)\, ds$$

$=$

$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$ //

HW 5: Solve Problems: 5.25, 5.26, 5.33, 5.51, 5.66

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5.25. Find the vertical deflection of point C in the truss shown in Figure P5.25. All members have the same cross section and are made of the same material.

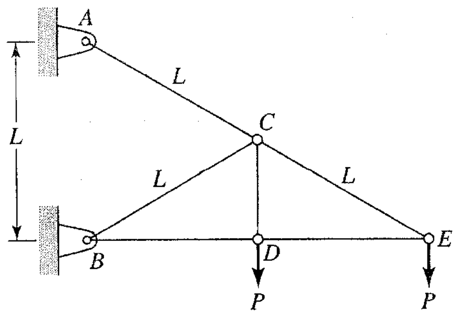

FIGURE P5.25

5.26. The beam in Figure P5.26 has its central half enlarged so that the moment of inertia I is twice the value for each end section. Determine the deflection at the center of the beam.

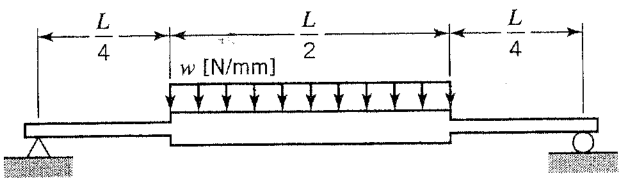

FIGURE P5.26

5.33. The structure in Figure P5.33 is made up of a cantilever beam AB (E_1, I_1, A_1) and two identical rods BC and CD (E_2, A_2). Let A_1 be large compared with A_2 and L_1 be large compared with the beam depth.

a. Determine the component of the deflection of point C in the direction of load P.

b. If $E_1 = E_2 = E$, the beam and rods have solid circular cross sections with radii r_1 and r_2, respectively, and $L_1 = L_2 = 25r_1$, determine the ratio of r_1 to r_2 such that the beam and rods contribute equally to q_P.

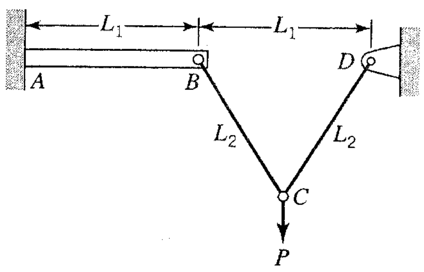

FIGURE P5.33

5.51. A structure (Figure P5.51) is made by welding a circular cross section steel shaft (E = 200 GPa and G = 77.5 GPa), of length 1.2 m and diameter 60 mm, to a rectangular cross section steel beam of length 1.5 m and cross-section dimensions 70 mm by 30 mm. A torque T_0 = 2.50 kN • m is applied to the free end of the shaft as shown. Determine the rotation of the free end of the shaft.

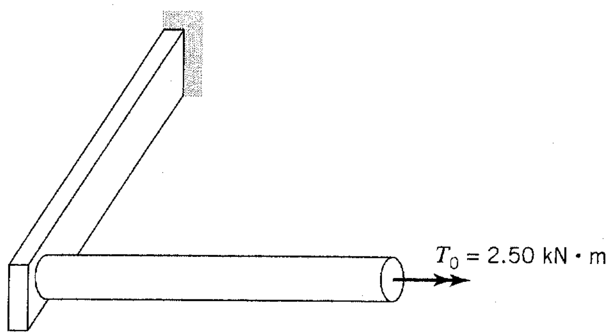

FIGURE P5.51

5.66. The beam in Figure P5.66 is fixed at the right end and rests on a coil spring with spring constant k at the left end. Assuming that the beam length is large compared to its depth, determine the force R in the spring.

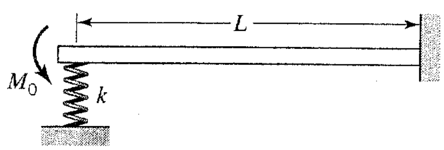

FIGURE P5.66