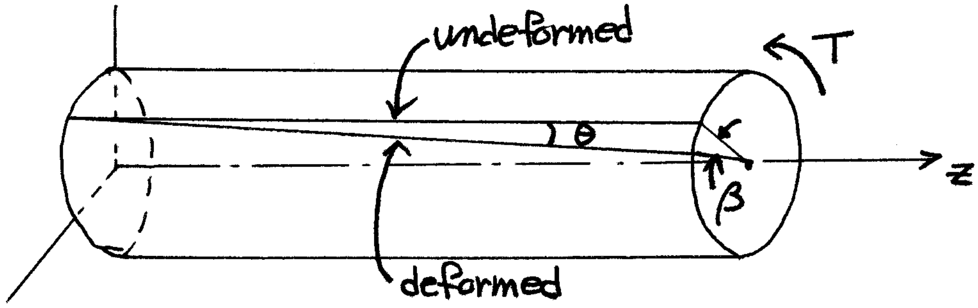
CH 6. Torsion

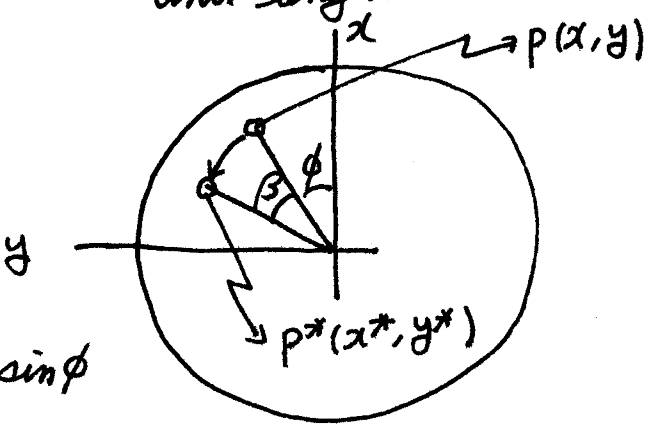
6.1. Torsion of Circular Shaft



- Assumptions
  - Plane cross-section remains plane
  - All radii remains straight and same magnitude. (rigid body rotation)

- Rotation of a cross-section

θ : twist angle per unit length



- Deformation. ($w=0$)

$$\begin{cases} u= \\ v= \end{cases}$$

$$\begin{cases} \cos(\beta+\phi) = \cos\beta\cos\phi - \sin\beta\sin\phi \\ \sin(\beta+\phi) = \sin\beta\cos\phi + \cos\beta\sin\phi \end{cases}$$

$$u = r\cos\beta\cos\phi - r\sin\beta\sin\phi - r\cos\phi$$

$$= \underbrace{r\cos\phi}_{x}(\cos\beta - 1) - \underbrace{r\sin\phi}_{y}\sin\beta$$

- For small deformation ($\sin\beta \approx \beta$, $\cos\beta = 1$)

$$\begin{cases} u = -y\beta \\ v = x\beta \\ \omega = 0 \end{cases} \qquad \Leftarrow \beta = \theta z$$

Displacement caused by Torsion T.

- Strains

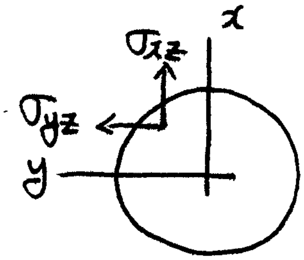
$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0$, $\quad \varepsilon_{yy} = 0$, $\quad \varepsilon_{zz} = 0$.

$\gamma_{xy} = \theta z - \theta z = 0$, ______ ______.

- Stresses

$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$

______ ______.



- Net torque T caused by stresses.

$\Sigma M_z = T =$

$$= \int_A (G\theta x^2 + G\theta y^2)\, dA$$

$$= G\theta \int_A (x^2 + y^2)\, dA$$

$x^2 + y^2 = r^2$.

$\int r^2 dA = J = \frac{\pi b^4}{2}$

GJ : torsional rigidity

θ : twist angle per unit length.

J : torsional constant
= polar moment of inertia J_0

- Stress vector on the cross-section

$$\underline{\tau} = -\theta G y \hat{i} + \theta G x \hat{j}$$

$$|\underline{\tau}| = \theta G \sqrt{x^2+y^2} = \theta G r = \tau, \quad |\underline{\tau}|_{max} = \theta G b.$$

$\Rightarrow$

※ For hollowed section, $J=$

1. Design of Shaft

- Shaft with frequency f and power P.

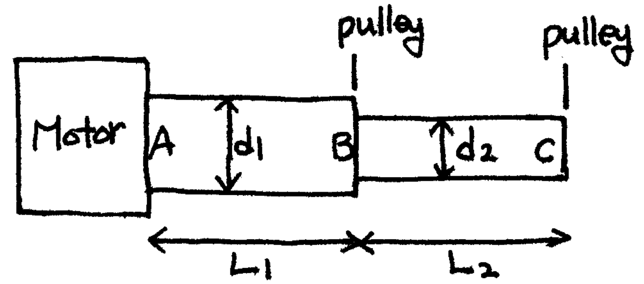
$$P = T\omega = 2\pi f T$$

$\Rightarrow$

$P = [N \cdot \frac{m}{s}]$

$\omega =$ rad/sec.

Example 6.2. Drive Shaft Design



$T_{pulley} = 113 N = T$

$T_{AB} = 2T$

$T_{BC} = T$ $\qquad\qquad \tau_{max} = \frac{1}{2}\frac{Y}{2} = \frac{Y}{4}$

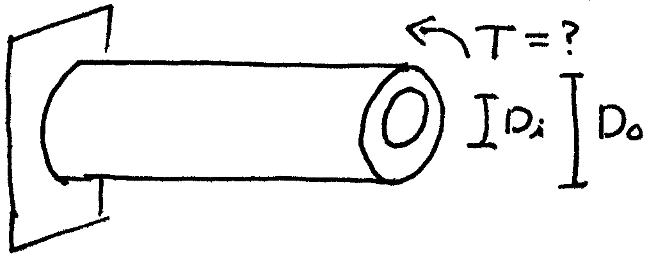
$$d_1 = 2\left(\frac{2 \cdot 2T}{\pi Y/4}\right)^{1/3} = 2\left(\frac{16T}{\pi Y}\right)^{1/3}$$

$$d_2 = 2\left(\frac{2T}{\pi Y/4}\right)^{1/3} = 2\left(\frac{8T}{\pi Y}\right)^{1/3}$$

$$\theta_{AB} = \frac{T_{AB}}{GJ_{AB}} \qquad \theta_{BC} = \frac{T_{BC}}{GJ_{BC}}$$

angle of twist $\beta_C = \theta_{AB} L_1 + \theta_{BC} L_2$.

Example 6.3. Torque Design of Hollow Shaft



SF = 2.0
$\beta \leq 0.2$ rad.

T ≤

and

T ≤

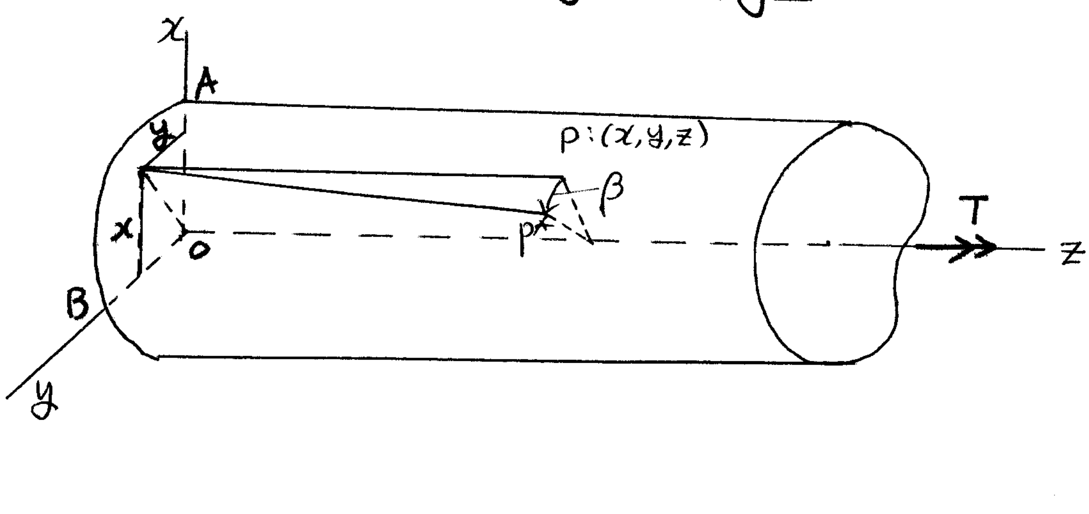
choose smaller one.

6.2. St Vernant's Semiinverse Method

- Non-circular cross-section warps after deformation.
- From St. Vernant's principle, boundary effect can be ignored far from the ends.

1. Geometry of Deformation

- Assumption : constant cross-section member has an axis of twist, about which the cross-section rotates as a rigid body.



- $p(x,y,z) \xrightarrow[\text{deform}]{} p^*$ $\quad (\beta = \theta \cdot z)$

  - same with circular cross-section

    warping

    $\psi(x,y)$: warping fn.

  - Strains

    $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0$

    ※ $\gamma_{xz} = -\theta y$, $\gamma_{yz} = \theta x$ for circular.

  - Remove ψ by $\dfrac{\partial \gamma_{xz}}{\partial y} - \dfrac{\partial \gamma_{yz}}{\partial x}$

    $\Rightarrow$ compatibility condition

2. Kinetics

$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$.

From Equilibrium Eq. in pp. 20 with no body force

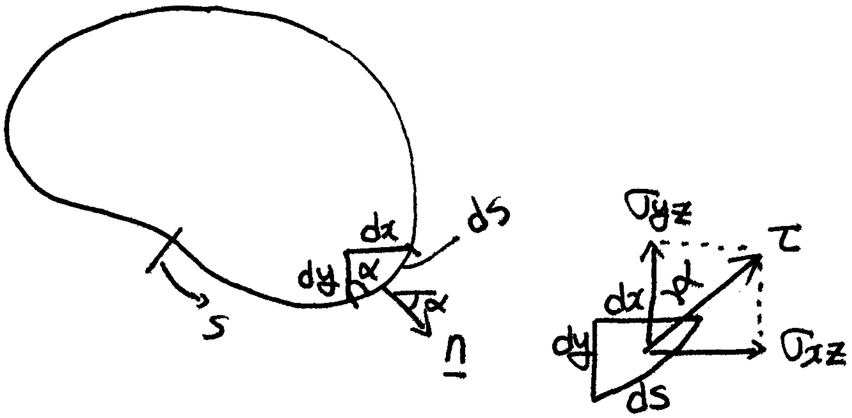
$$\left.\begin{aligned} \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{yz}}{\partial z} &= 0 \end{aligned}\right\} \Rightarrow \sigma_{xz},\ \sigma_{yz} \text{ independent of } z.$$

$\Rightarrow$ in order to satisfy this relation, there exists a stress function $\phi(x,y)$ (Prandtl stress function), such that

Torsion problem is to determine $\phi(x,y)$.

3. Boundary Conditions

- shear stress on the outer surface is tangent to surface.



$$\sigma_{xz} = \tau \sin\alpha \qquad \sin\alpha = \frac{dx}{ds} \qquad \cos\alpha = \frac{dy}{ds}$$

$$\sigma_{yz} = \tau \cos\alpha \qquad \underline{n} = (\cos\alpha, -\sin\alpha).$$

- Since $\underline{\tau} \perp \underline{n}$

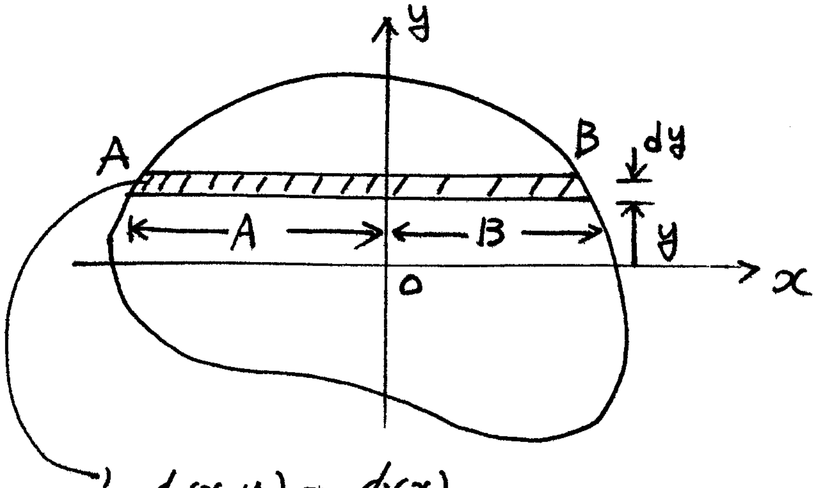
$$\Rightarrow \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial x}\frac{dx}{ds} = \frac{d\phi}{ds} = 0.$$

$\Rightarrow$ $\phi = \text{const}$ on the boundary S.

choose $\boxed{}$ because the derivative of ϕ is interested.

- Equilibrium

$$\begin{cases} \Sigma F_x = \int \sigma_{xz}\, dx\,dy = \int \frac{\partial \phi}{\partial y}\, dx\,dy = 0 \\ \Sigma F_y = \int \sigma_{yz}\, dx\,dy = -\int \frac{\partial \phi}{\partial x}\, dx\,dy = 0 \\ \Sigma M_z = \end{cases}$$



$\phi(x,y) = \phi(x).$

- $\Sigma F_y = -\int \frac{\partial \phi}{\partial x}\, dx\,dy = -dy \int \frac{d\phi}{dx}\, dx = -dy \int_{\phi(A)}^{\phi(B)} d\phi$

$$= -dy\,[\,\phi(B) - \phi(A)\,] = 0.$$

- Similarly $\Sigma F_x = 0$
- Moment equilibrium (for the strip)

$$-\int x \frac{\partial \phi}{\partial x}\, dx\,dy = -dy \int x \frac{d\phi}{dx}\, dx = -dy \int_{\phi(A)}^{\phi(B)} x\, d\phi$$

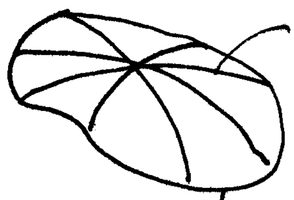
$$= -dy \left\{ x\phi \Big|_A^B - \int_{x_A}^{x_B} \phi\, dx \right\} = dy \int_{x_A}^{x_B} \phi\, dx$$

integration by part

- Summing for all strips $\Rightarrow$ $-\iint x \frac{\partial \phi}{\partial x}\, dx\,dy = \iint \phi\, dx\,dy.$
- Apply for the vertical strips $\Rightarrow$ $\iint y \frac{\partial \phi}{\partial y}\, dx\,dy = \iint \phi\, dx\,dy$

- $\Sigma M_z = 0$;

- Physical interpretation :



$\iint \phi \, dx dy$: volume under $\phi(x,y)$.

$\therefore T = 2 \times$ volume under $\phi(x,y)$.

6.3. Linear Elastic Solutions

$$\sigma_{xz} = \frac{\partial \phi}{\partial y} = G \gamma_{xz}$$

$$\sigma_{yz} = -\frac{\partial \phi}{\partial x} = G \gamma_{yz}$$

From $\dfrac{\partial \gamma_{xz}}{\partial y} - \dfrac{\partial \gamma_{yz}}{\partial x} = -2\theta$

$\Rightarrow$

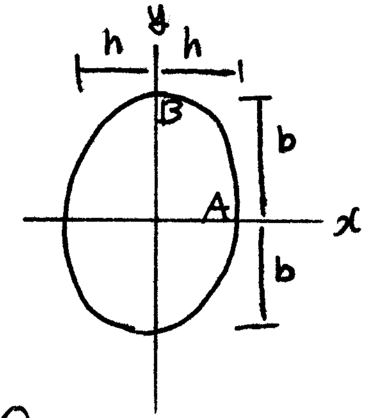
B.C. $\phi = 0$ on Γ $\quad \theta =$ unit angle of twist.

Solve for $\phi \Rightarrow \sigma_{xz}, \sigma_{yz}$

$\Rightarrow T$

- Indirect approach,
  - Let a function $F(x,y)$ represent the boundary of cross section by $F(x,y) = 0$.
  - Let stress function $\phi(x,y) = B \cdot F(x,y)$.
  - If $\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2} =$ const. Then, $\phi(x,y)$ is the solution of a torsion problem.

1. Elliptical Cross-section



$\phi =$

$\underbrace{\qquad\qquad\qquad}_{F(x,y)=0 \text{ on } \Gamma}$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2B\left(\frac{1}{h^2}+\frac{1}{b^2}\right) = -2G\theta$$

$\therefore B =$

$\sigma_{xz} = \frac{\partial\phi}{\partial y} = 2\frac{By}{b^2} =$

$\sigma_{yz} = -\frac{\partial\phi}{\partial x} =$

- Max. shear stress occurs at A

$\tau_{max} = \sigma_{yz(x=h)} =$

$$T = 2\iint\phi\, dx\, dy = \frac{2B}{h^2}\iint x^2 dA + \frac{2B}{b^2}\iint y^2 dA - 2B\iint dA$$

$$= \frac{2B}{h^2}I_y + \frac{2B}{b^2}I_x - 2BA$$

$I_x = \frac{\pi b^3 h}{4}$, $I_y = \frac{\pi b h^3}{4}$, $J_0 = \frac{\pi bh(h^2+b^2)}{4}$

polar moment of inertia

$=$

$T =$ $\qquad B = -\frac{T}{\pi bh}$

different.

$\tau_{max} = -\frac{\partial\phi}{\partial x}\Big|_{x=h} = -2\frac{B}{h} = -\frac{2}{h}\left(-\frac{T}{\pi bh}\right)$

$\tau_{max} = \frac{2T}{\pi bh^2}$, $\quad \theta =$

$$T = GJ\theta = G\underbrace{\frac{\pi b^3 h^3}{b^2+h^2}}_{J \,:\, \text{torsional constant.}}\cdot\theta$$

GJ : torsional rigidity

- Warping ω

$$\begin{cases} \sigma_{xz} = \dfrac{\partial \phi}{\partial y} = B\dfrac{2y}{b^2} = -\dfrac{2T}{\pi b^3 h} y = \\ \sigma_{yz} = -\dfrac{\partial \phi}{\partial x} = -B\dfrac{2x}{h^2} = \dfrac{2T}{\pi b h^3} x = \end{cases}$$

$\uparrow$ pp. 75

$$\Rightarrow \begin{cases} \theta \dfrac{\partial \psi}{\partial x} = -\dfrac{2T}{G\pi b^3 h} y + \theta y \\ \theta \dfrac{\partial \psi}{\partial y} = \dfrac{2T}{G\pi b h^3} x - \theta x \end{cases}$$

From $\quad \omega = \theta \psi$

$$\frac{\partial \omega}{\partial x} = \theta \frac{\partial \psi}{\partial x} = -\frac{2T}{G\pi b^3 h} y + \theta y$$

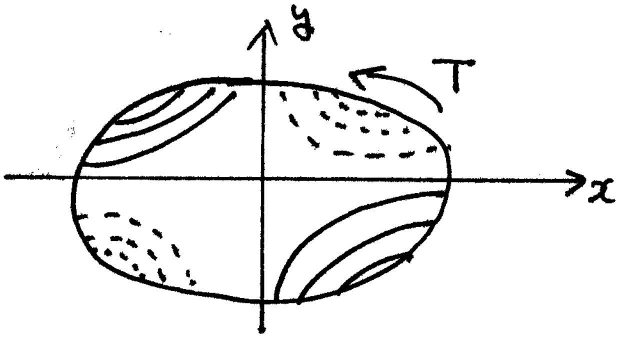
$$\frac{\partial \omega}{\partial y} = \theta \frac{\partial \psi}{\partial y} = \frac{2T}{G\pi b h^3} x - \theta x.$$

$$\omega = \frac{2T}{G\pi b h^3} xy - \theta xy + c$$

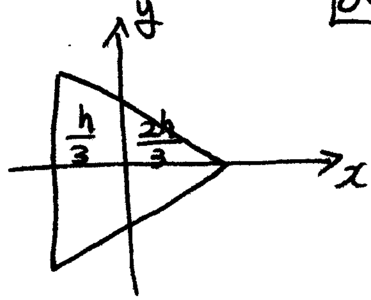
$$\omega(0,0) = c = 0 \qquad \therefore \ c = 0.$$

Also, $\quad \theta = \dfrac{T(b^2+h^2)}{G\pi b^3 h^3}$

$$\therefore \ \boxed{\omega = \qquad\qquad\qquad\qquad}$$



2. Equilateral Triangle

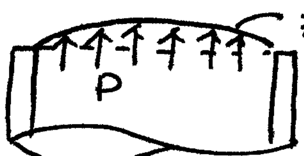


$\phi =$

$\tau_{max} = \dfrac{15\sqrt{3}\,T}{2h^3}$, $\quad \theta = \dfrac{15\sqrt{3}\,T}{Gh^4}$

6.4. Soap-Film Analogy (Membrane Analogy)

- Membrane with pressure $\overset{}{\underset{\text{analogy}}{\Longleftrightarrow}}$ torsional Eq.
- Visualization of shear stress distribution.



$z(x,y) = \phi(x,y)$.

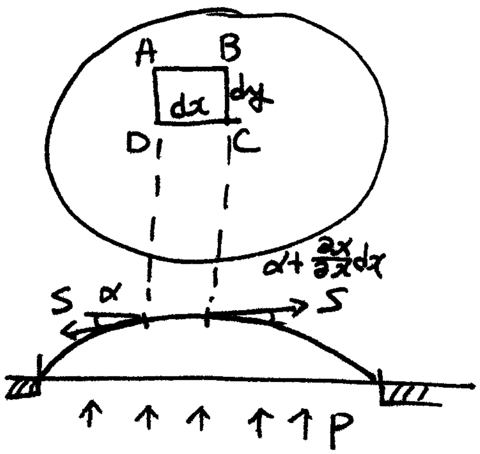
- membrane displacement $z(x,y)$ = stress function $\phi(x,y)$.

∘ Torsional Eq.

Elastic membrane $\Big($

p: pressure

S: tension.



Edge AD

$-S\,dy \sin\alpha \approx -S\,dy\dfrac{\partial z}{\partial x}$,

Edge BC

$S\,dy\dfrac{\partial}{\partial x}\left(z + \dfrac{\partial z}{\partial x}dx\right)$

Edge DC

$-S\,dx\dfrac{\partial z}{\partial y}$

Edge AB

$S\,dx\dfrac{\partial}{\partial y}\left(z + \dfrac{\partial z}{\partial y}dy\right)$

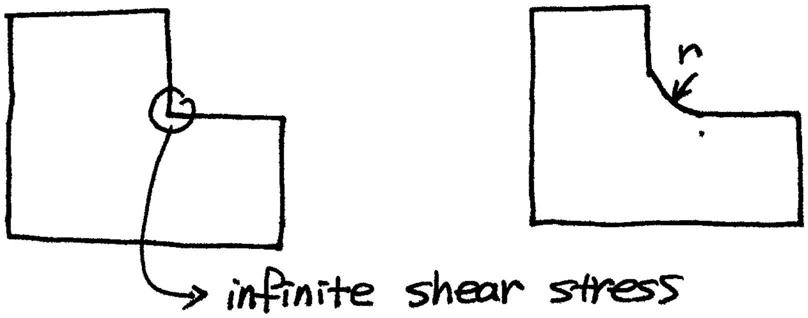
ΣF_z ; $\quad S\dfrac{\partial^2 z}{\partial x^2}dx\,dy + S\dfrac{\partial^2 z}{\partial y^2}dx\,dy + p\,dx\,dy = 0$

$\Rightarrow \boxed{}$ membrane Eq.

- Analogous quantities :

- Stresses(σ_{xz}, σ_{yz}) are the slope of the membrane.
- Torque is propotional to the volume enclosed by membrane

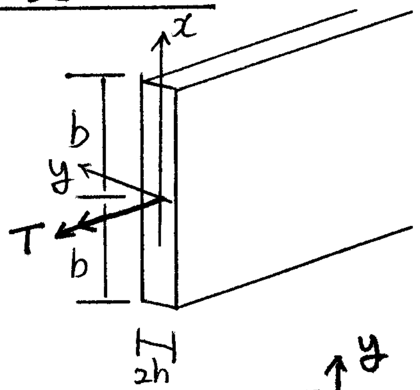
∘ Stress concentration



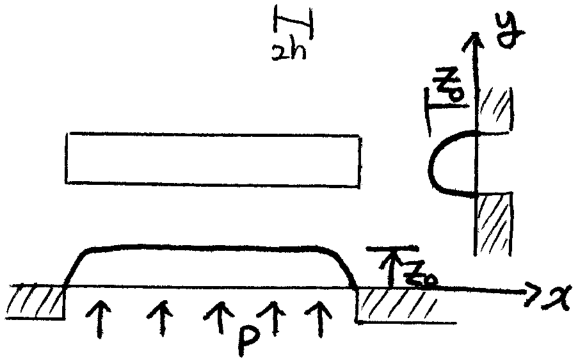
6.5. Narrow Rectangular Cross Section

· solid rectangle, $b \gg h$

· Assume deflection is independent of x, parabolic in y



$z(y) =$



$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -2\frac{z_0}{h^2}$$

$$\frac{p}{S} = 2\frac{z_0}{h^2} = 2cG\theta$$

$$c = \frac{z_0}{G\theta h^2} \Longrightarrow$$

$$\sigma_{xz} = \frac{\partial \phi}{\partial y} = -2G\theta y \,, \qquad \sigma_{yz} = -\frac{\partial \phi}{\partial x} = 0$$

$\tau_{max} =$

$$T = \quad = 2\cdot(2b)\int_{-h}^{h} G\theta(h^2-y^2)\,dy$$

$$= 2\cdot(2b)\cdot G\theta\cdot\left(2h^3-\frac{2}{3}h^3\right)$$

$$T = \frac{1}{3}G\theta(2b)(2h)^3 = GJ\theta$$

$$\therefore\ J = \frac{1}{3}(2b)(2h)^3.$$

$$J_o = \frac{1}{12}\left[(2b)(2h)^3 + (2h)(2b)^3\right], \qquad J_o > J$$

$$\tau_{max} = 2G\theta h = 2\frac{Th}{J}$$

* Generalization : Cross-section with long narrow rectangles

$J =$ $\qquad\qquad$ c; section constant

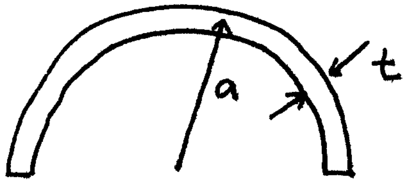
for $b_i > 10h_i$, $\quad c \approx 1.$

$b_i < 10h_i$, $\quad c \approx 0.91$ (recommended)

$n=1$, $b > 10h$, $\quad c = 1.$

$$\tau_{max} = \frac{2Th_{max}}{J}, \qquad \theta = \frac{T}{GJ}.$$

Example 6.6.



circumference $\quad 2b = \pi a$

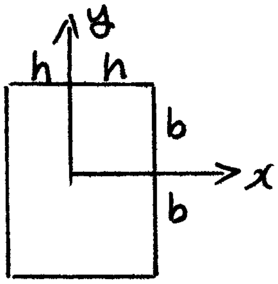
thickness $\quad 2h = t$

$$\tau_{max} = \frac{2Th}{J} \qquad J = \frac{1}{3}\pi a t^3$$

$$= \frac{3T}{\pi a t^2}$$

$$\theta = \frac{T}{GJ} = \frac{3T}{\pi a t^3 G}$$

6.6. Torsion of Rectangular Cross-Section



$$\boxed{\begin{aligned} \nabla^2\phi &= -2G\theta \\ \phi &= 0 \quad \text{on boundary} \end{aligned}}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

- From the narrow rectangle, $\phi = G\theta(h^2 - x^2)$ is a particular solution. Assume $\phi(x,y)$ to be

$$\nabla^2\phi = -2G\theta + \nabla^2 V = -2G\theta$$

$$\therefore \quad \nabla^2 V = 0 \qquad \text{over the cross-section.}$$

$$V \begin{cases} = 0 & \text{at } x = \pm h \\ = G\theta(x^2 - h^2) & \text{at } y = \pm b \end{cases}$$

$V(x,y)$ is an <u>even fn.</u>

- Use separation of variable

$$V(x,y) = f(x)g(y)$$

$$\nabla^2 V(x,y) = f''g + fg'' = 0.$$

$$f'' = \frac{\partial^2 f}{\partial x^2}. \qquad g'' = \frac{\partial^2 g}{\partial y^2}.$$

$$\Rightarrow \frac{f''}{f} = -\frac{g''}{g} = \text{const} = -\lambda^2$$

$$\Rightarrow \begin{cases} \dfrac{d^2f}{dx^2} + \lambda^2 f = 0 \\ \dfrac{d^2g}{dy^2} - \lambda^2 g = 0 \end{cases}$$

$$\Rightarrow \begin{cases} f = \\ g = \end{cases}$$

$B = D = 0$

$\because V(x,y)$ is an even fn.

$\Rightarrow \cdot\ V(x,y) = A\cos\lambda x \cosh\lambda y$ (satisfies $\nabla^2 V = 0$)

$V(\pm h, y) = A\cos\lambda(\pm h)\cos\lambda y = 0$

$n = 1, 2, \cdots$

$$\cdot\ V(x,y) = \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi x}{2h}\cosh\frac{(2n-1)\pi y}{2h}$$

(satisfies $V=0$ at $x=\pm h$)

$$V(x,\pm b) = \sum A_n \cosh\frac{(2n-1)\pi b}{2h}\cos\frac{(2n-1)\pi x}{2h}$$

$$= \sum C_n \cos\frac{(2n-1)\pi x}{2h} = G\theta(x^2 - h^2) = F(x).$$

$\cdot$ Use Fourier series by mul. $\cos((2n-1)\pi x/2h)$ and integrate $[-h, h]$

$$\int_{-h}^{h}\sum_{n=1}^{\infty} C_n \underbrace{\cos\frac{(2n-1)\pi x}{2h}\cdot\cos\frac{(2k-1)\pi x}{2h}}_{=0 \text{ if } n \neq k.}\,dx = \int_{-h}^{h} F(x)\cos\frac{(2k-1)\pi x}{2h}\,dx$$

$$C_k = \frac{1}{h}\int_{-h}^{h} F(x)\cos\frac{(2k-1)\pi x}{2h}\,dx$$

$$= \frac{2G\theta}{h}\int_0^h (x^2 - h^2)\cos\frac{(2k-1)\pi x}{2h}\,dx$$

$$= -\frac{(-1)^{k-1}\,32\,G\theta h^2}{(2k-1)^3\pi^3}$$

$$\Rightarrow A_k = -\frac{(-1)^{k-1}\,32\,G\theta h^2}{(2k-1)^3\pi^3\cosh\frac{(2k-1)\pi b}{2h}}$$

$$\therefore \phi(x,y) = G\theta(h^2-x^2) + \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi x}{2h} \cosh\frac{(2n-1)\pi y}{2h}$$

$\frac{b}{h} \to \infty$: narow section $\Rightarrow \phi(x,y) \cong G\theta(h^2-x^2)$.

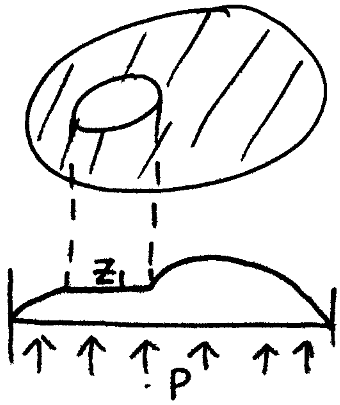
- $J = k_1 (2b)(2h^3)$

$$k_1 = \frac{1}{3}\left[1 - \frac{192}{\pi^5}\left(\frac{h}{b}\right)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\tanh\frac{(2n-1)\pi b}{2h}\right]$$

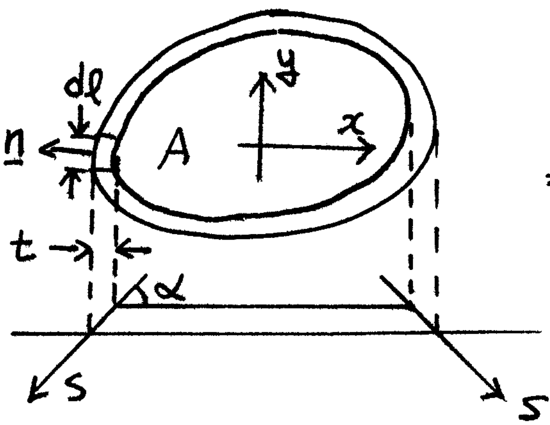
$$\tau_{max} = \frac{T}{k_2(2b)(2h)^2} = 2G\theta h\frac{k_1}{k_2}$$

6.7. Hollow Thin-Wall Member

~ stress function is constant in the hollow region



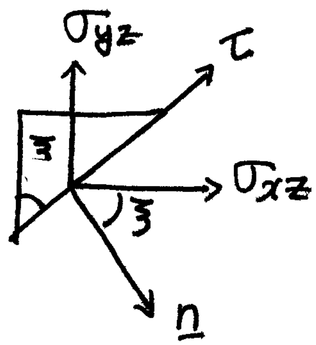
- Non-circular thin wall shaft.



Assume z increase linearly through thickness t.

$\Rightarrow$ shear stress is constant through t.

$$\frac{\partial \phi}{\partial n} = \underbrace{\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial n}}_{\cos \xi} + \underbrace{\frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n}}_{-\sin \xi}$$



$$\frac{\partial \phi}{\partial n} = -\sigma_{yz}\cos \xi - \sigma_{xz}\sin \xi$$

$$= -\tau \cos^2 \xi - \tau \sin^2 \xi$$

$$= -\tau$$

- If only magnitude is considered

$$\tau = \frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n}\left(\frac{2G\theta S}{p} z\right) = \frac{2G\theta S}{p}\frac{\partial z}{\partial n}.$$

$$\tau = \frac{2G\theta S}{p}\tan\alpha = \frac{1}{c}\tan\alpha \approx \frac{1}{c}\sin\alpha.$$

- Shear flow

[F/L] : same as ϕ, constant.

τ varies with thickness t. $\tau_1 = \frac{q}{t_1}$, $\tau_2 = \frac{q}{t_2}$ ...

- Torque : Volume under the membrane

A: area enclosed by mean perimeter.

ℓ: perimeter.

$$\Sigma F_z = pA - \oint_\ell \underbrace{S\sin\alpha}_{\tau \cdot c}\, d\ell = 0$$

$$\frac{1}{A}\oint_\ell \tau\, d\ell = \frac{p}{Sc} = 2G\theta$$

$\Rightarrow$

- For segments $l_1, l_2 \cdots$ of constant thicknesses $t_1, t_2, \cdots$

$$\theta = \frac{q}{2GA}\left(\frac{l_1}{t_1} + \frac{l_2}{t_1} + \cdots\right)$$

$$\uparrow q = \frac{T}{2A}$$

$$\theta = \frac{T}{4GA^2}\left(\frac{l_1}{t_1} + \frac{l_2}{t_1} + \cdots\right)$$

$$\Rightarrow T = GJ\theta = \frac{4GA^2\theta}{\left(\frac{l_1}{t_1} + \frac{l_2}{t_2} + \cdots\right)}$$

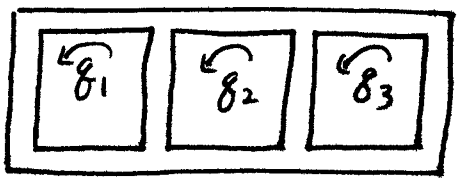
$\therefore$ for general case

- Constant thickness hollow tube

- Hollow circular cross-section

$$A = \pi R^2, \quad l = 2\pi R, \quad \Rightarrow \underline{J = 2\pi R^3 t}$$

∘ <u>Hollow Member with many compartments</u>



N compartments $\Rightarrow$ N+1 unknowns

: $\theta, q_i, i = 1 \cdots N.$

~ N+1 Equations

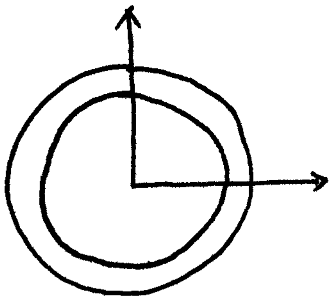
$$\boxed{\begin{aligned} T &= 2\sum_{i=1}^{N} A_i q_i \\ \theta &= \frac{1}{2GA_i}\int_{l_i} \frac{q_i - q'}{t}\, dl \end{aligned}} \qquad i = 1, \cdots, N$$

A_i : area of ith compartment

q' : shear flow of neighbor compartmend.

· Max. shear stress occurs at the greates slope.
i.e., $(q_i - q')/t$ has max. value.

Example 6.9. Circular Hollow Section



$D_o = 22$ mm, $D_i =$ mm, $t = 2$ mm

(a) $\tau = 70$ MPa at mean diameter.

$$A = \frac{\pi D^2}{4} = 100\pi \text{ mm}^2$$

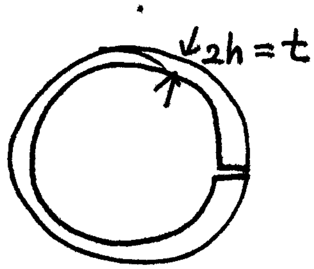
$$T = 2A\tau t = 2 \cdot 100\pi \cdot 70 \cdot 2 = 87.96 \text{ N·m}$$

$$\theta = \frac{\tau l}{2GA} = \frac{\tau \cdot \pi D}{2 \cdot G \cdot \pi D^2/4} = \frac{2\tau}{GD} = 9.03\times10^{-5} \text{ rad/mm}$$

From elasticity

$$\tau = \frac{Tr}{J} \Rightarrow T = \frac{\tau J}{r} = 88.84 \text{ N·m}$$

$$\theta = \frac{T}{GJ} = \frac{\tau}{Gr} = 9.03\times10^{-5} \text{ rad/mm}$$



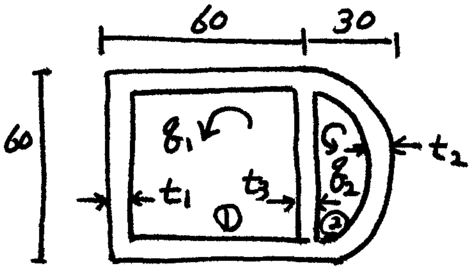
(b) Long narrow rectangle.

$$\tau_{max} = 2G\theta h \Rightarrow \theta = \frac{\tau_{max}}{2Gh} = 4.516\times10^{-4}$$

$$T = GJ\theta = \frac{J\tau_{max}}{2h} = 5.864 \text{ N·m}$$

T is reduced significantly, while θ increases significantly

Example 6.10 Two Compartments Member



$$T = 2(A_1 q_1 + A_2 q_2)$$

$$\theta = \frac{1}{2GA_1}\left[\frac{q_1 l_1}{t_1} + \frac{(q_1 - q_2) l_3}{t_3}\right]$$

$$\theta = \frac{1}{2GA_2}\left[\frac{q_2 l_2}{t_2} + \frac{(q_2 - q_1) l_3}{t_3}\right]$$

$$\frac{q_1}{q_2} = 1.22$$

~ $\frac{q_1}{q_2} = 1.22$, $\frac{t_1}{t_2} = 1.5$ ⇒ max. may occur at t_2.

$q_2 = \tau_{max} \cdot t_2 = 120$ N/mm

$q_1 = 1.22 q_2 = 146.4$ N/mm

$\tau_1 = \frac{q_1}{t_1} = 32.5$ MPa, $\tau_2 = \frac{q_2}{t_2} = 40$ MPa

$\tau_3 = \frac{q_1 - q_2}{t_3} = 17.6$ MPa since $\tau_3 < \tau_2$, initial assump. is correct.

$\therefore T = 2(A_1 q_1 + A_2 q_2) = 1.393$ kN·m

$$\theta = \frac{1}{2GA_1}\left(\frac{q_1 l_1}{t_1} + \frac{(q_1 - q_2) l_3}{t_3}\right) = 0.0369 \text{ rad/m}.$$

6.8. Torsion with Restrained Ends

~ Clamped end produces bending moment due to the prevented warping

6.9. Numerical Methods

- Solve Prandtl's formula

$$\nabla^2\phi = -2G\theta \quad \text{on region } R$$

$$\phi = 0 \quad \text{on boundary } C$$

- Finite Difference Method (rectangular grid)

$$(\nabla^2\phi)_{i,j} = \frac{1}{h^2}\left(\phi_{i-1,j} + \phi_{i+1,j} - 4\phi_{i,j} + \phi_{i,j-1} + \phi_{i,j+1}\right)$$

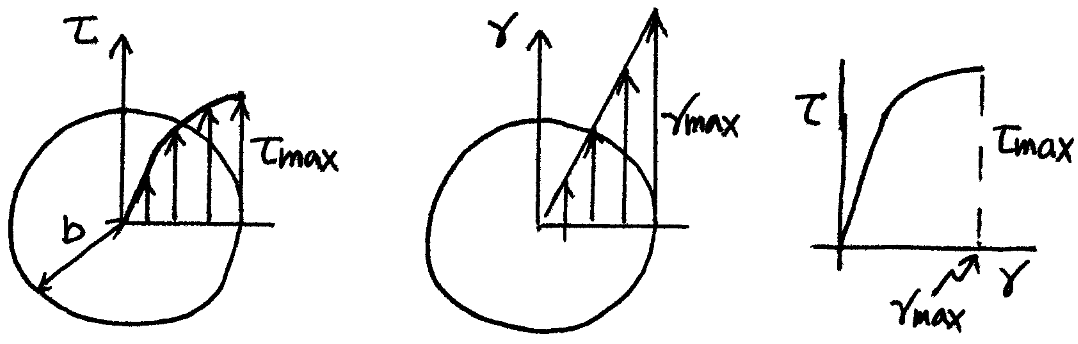
- Finite Element Method (General shape)

EML5526.

6.10. Inelastic Torsion

- previous formulas are valid only for elastic torsion.

shear stress – nolinear ; shear strain – linear.



- Moment Equilibrium

$$T = \int r\tau\,dA = \int_0^b r\tau(2\pi r)\,dr$$

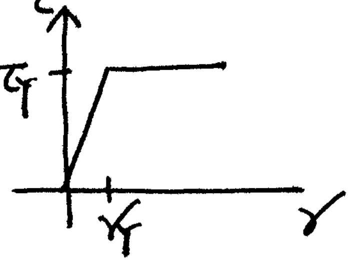
1. Modulus of Rupture

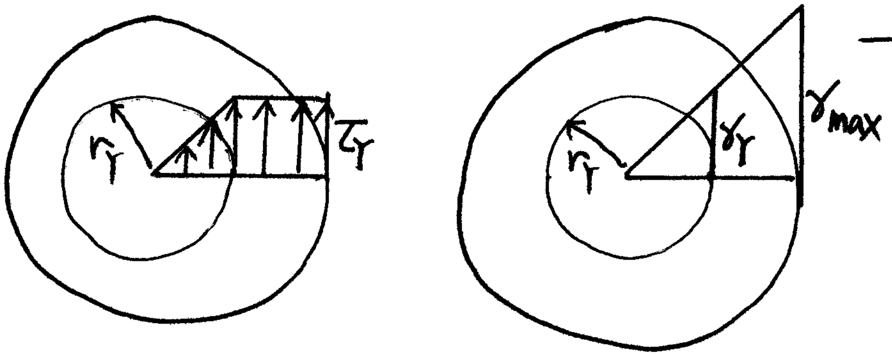
~ Ultimate shear strength τ_u.

Modulus of rupture τ_r (fictitious, linear)

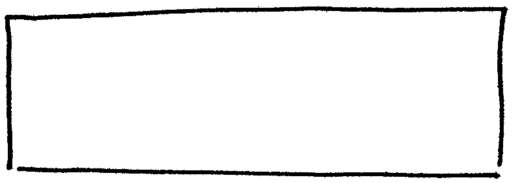
2. Elastic-Plastic & Fully Plastic Torsion

∘ Elastic-perfectly plastic material





$$\tau_Y = G\gamma_Y$$



$k \geq 1$.

∘ Torque $T_{EP} = \underset{\text{elastic}}{T_E} + \underset{\text{plastic}}{T_P}$

$$T_E = \frac{\tau_Y \cdot J_E}{r_Y} = \frac{\tau_Y}{r_Y}\frac{\pi r_Y^3}{2} = \frac{\pi}{2}\tau_Y r_Y^3$$

$$T_P = \int_{r_Y}^{b} \tau_Y \cdot r\,dA = \int_{r_Y}^{b} \tau_Y\, r(2\pi r)\,dr = \frac{2}{3}\pi\tau_Y(b^3 - r_Y^3)$$

$$T_{EP} = \frac{\pi}{2}\tau_Y r_Y^3 + \frac{2}{3}\pi\tau_Y(b^3 - r_Y^3) = \frac{\pi\tau_Y b^3}{6}\left(3\left(\frac{r_Y}{b}\right)^3 + 4 - 4\left(\frac{r_Y}{b}\right)^3\right)$$

$$= \frac{2}{3}\pi\tau_Y b^3\left(1 - \frac{1}{4k^3}\right)$$

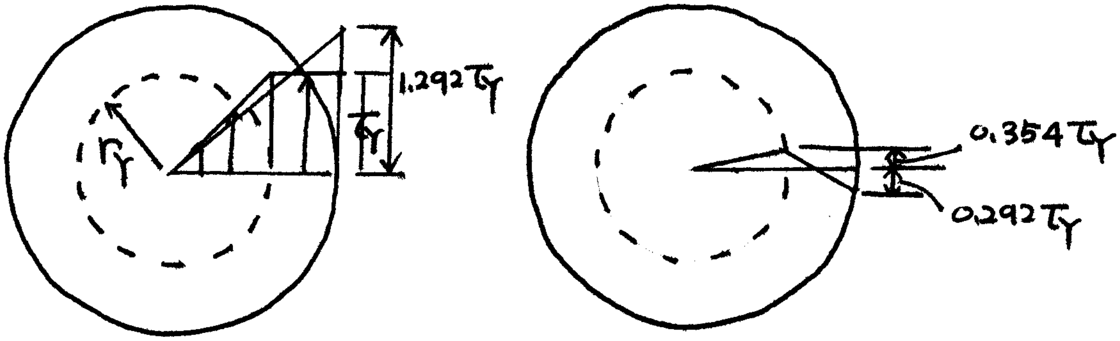
- Max. elastic Torque T_Y at $k=1$ (initial yielding)

$$T_Y = \frac{1}{2}\pi \tau_Y b^3$$

Fully plastic torque $T_{FP} =$

3. Residual Shear Stress

~ Loading → elastic-plastic → unloading → residual stress



- $k=2$.

$$T_{EP} = \frac{4}{3}T_Y\left(1 - \frac{1}{4\cdot 2^3}\right) = 1.292\, T_Y$$

- At unloading, $r=b \Rightarrow \tau|_b = \tau_Y - 1.292\tau_Y = -0.292\tau_Y$

$$r = r_Y \Rightarrow \tau|_{r_Y} = \tau_Y - \frac{1}{2}(1.292)\tau_Y = +0.354\tau_Y$$

∘ Hollow member

$$T_{FP} = 2\pi\tau_Y \int_{r_i}^{r_o} r^2 dr = \frac{2}{3}\pi\tau_Y (r_o^3 - r_i^3)$$

Example 6.11. Angle of Twist Given

$D = 40\,mm$, $L = 1.5\,m$, $\tau_Y = 130\,MPa$, $G = 77.5\,GPa$, $\psi = 0.2\,rad$

(a) Yield?

$$\gamma_{max} = \frac{b}{L}\psi \quad \leftarrow \text{given.} \qquad \gamma_Y = \frac{\tau_Y}{G}$$

$$k = \frac{\gamma_{max}}{\gamma_Y} = \frac{G}{\tau_Y}\frac{b}{L}\psi = 1.59 \quad > 1 \qquad \therefore \text{yield.}$$

(b) $r_Y = ?$ $\quad k = \frac{b}{r_Y} \qquad r_Y = \frac{b}{k} = 12.6\ mm$

(c) $T = ?$ $\quad T = T_{EP} = \frac{2}{3}\pi\tau_Y b^3\left(1 - \frac{1}{4k^3}\right) = 2.043\ kN\cdot m$

(d) permanent angle of twist ψ_s ? residual shear stress?

Apply T_{EP} linearly $\Rightarrow$ $\tau = \frac{T\cdot b}{\frac{\pi}{2}b^4} = 162.58\ MPa$

$$\tau_1(r) = \begin{cases} \frac{\tau_Y}{r_Y} r & 0 \le r \le r_Y \quad \text{(linear)} \\ \tau_Y & r_Y \le r \le b \quad \text{(constant)} \end{cases}$$

$$\tau_2(r) = \frac{4}{3}\left(1 - \frac{1}{4k^3}\right)\frac{r}{b} \quad \text{(linear)}$$

$$\tau_{residual}(r) = \tau_1(r) - \tau_2(r).$$

Permanent twist ψ_s

$$\psi_s = \psi_{loading} - \psi_{unloading}$$
$$= \psi - \frac{T_{EP}L}{GJ}$$

Example 6.12 Fully Plastically loaded shaft

(a) $\theta_Y = ?$ $\quad T_Y = \frac{\tau_Y J}{b} = \frac{1}{2}\pi\tau_Y b^3 = GJ\theta_Y$

$$\theta_Y = \frac{\tau_Y}{Gb}$$

(b) when $r_Y = b_Y$, $\theta = ?$

At b_Y, $\tau = \tau_Y$. $\Rightarrow$ $b_Y = \dfrac{\tau_Y}{G\theta}$ $\Rightarrow$ $\theta = \dfrac{\tau_Y}{G b_Y}$

(c) $T_{EP} = T_E + T_P$?

$$T_E = \frac{1}{2}\pi \tau_Y b_Y^3$$

$$T_P = \int_{b_Y}^{b} r \cdot \tau_Y (2\pi r)\,dr = \frac{2}{3}\pi \tau_Y (b^3 - b_Y^3)$$

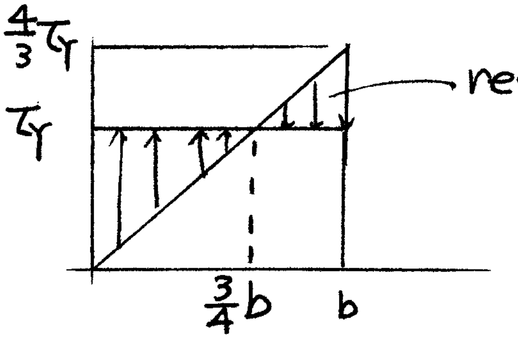
$$T_{EP} = T_E + T_P = \frac{2}{3}\pi \tau_Y \left(b^3 - b_Y^3 + \frac{3}{4} b_Y^3\right)$$

$$= \frac{2}{3}\pi \tau_Y \left(b^3 - \frac{1}{4} b_Y^3\right)$$

(d) $T_{FP} = ?$

$b_Y \to 0$ $\qquad T_{FP} = T_{EP}\big|_{b_Y = 0} = \frac{2}{3}\pi \tau_Y b^3 = \frac{4}{3} T_Y$

(e) Residual stress after T_{FP} ?



6.11. Fully Plastic Torsion

~ Entire region of the cross-section is plastic.

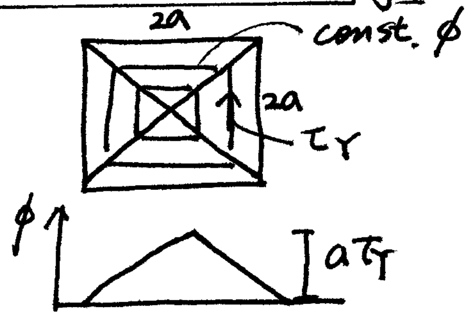
$$\sigma_{xz}^2 + \sigma_{yz}^2 = \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial x}\right)^2 = \tau_Y^2$$

$\Rightarrow$ $\phi = \tau_Y$ * distance from nearest boundary.

$$T_{FP} = 2\iint \phi \,dx\,dy$$

$$= 2\left[\frac{1}{3}(2a)^2 \tau_Y a\right]$$

$$= \frac{8}{3}\tau_Y a^3$$



HW6: Solve Problems

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6.6. The torsion member shown in Figure P6.6 is made of structural steel with a shear yield strength of $\tau_Y = 160$ MPa and is subjected to two torsional moments.

a. Determine the maximum shear stress in the member.

b. Determine the factor of safety for the given loads and a failure mode of general yielding.

6.7. Determine the angle of twist of the free end of a torsion member of Problem 6.6; $G = 77.5$ GPa.

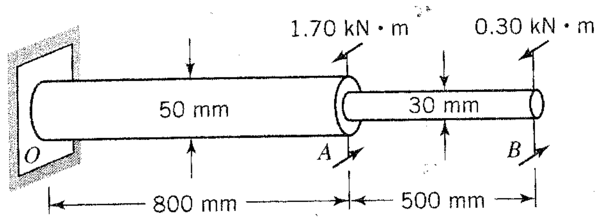

FIGURE P6.6

6.11. The load P produces a downward deflection Δ of point C (Figure P6.11). For small deflections, P is related to Δ by the relation

$$P = k\Delta$$

where k is a constant. Assume that the member BC is rigid. Derive a formula for k in terms of L_1, L_2, d and material properties E and G.

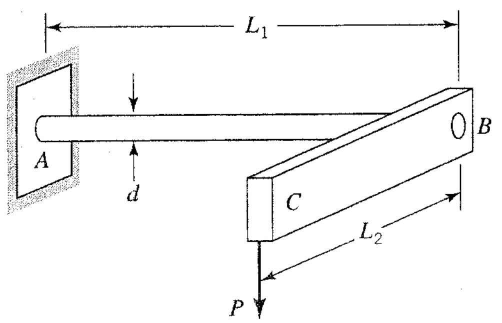

FIGURE P6.11

6.18. A stepped steel shaft ABC has lengths $AB = L_1 = 1.0$ m and $BC = L_2 = 1.27$ m, with diameters $d_1 = 25.4$ mm and $d_2 = 19.05$ mm, respectively. The steel has a yield stress $Y = 450$ MPa and shear modulus $G = 77$ GPa. A twisting moment is applied at the stepped section B. Ends A and C are fixed.

a. Determine the value of T that first causes yielding.

b. For this value of T, determine the angle of rotation ψ_B at section B.

6.57. The aluminum ($G = 27.1$ GPa) hollow thin-wall torsion member in Figure P6.57 has the dimensions shown. Its length is 3.00 m. If the member is subjected to a torque $T = 11.0$ kN • m, determine the maximum shear stress and angle of twist.

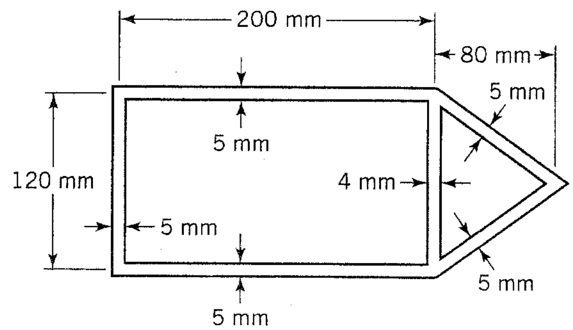

FIGURE P6.57