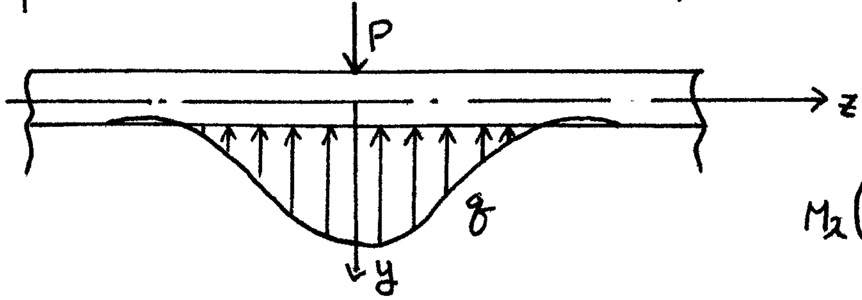
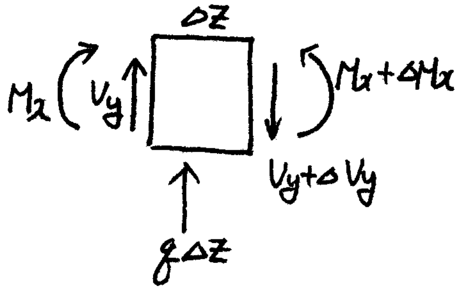
CH 10. BEAM on ELASTIC FOUNDATION

10.1. General Theory

~ Elastic foundation produces distributed load q proportional to the beam deflection.





- For linear elastic foundation

$$q = k y$$
$$= b k_o y$$

b : beam width

k_o : modulus of foundation $[F/L^3]$

$\Rightarrow$ $\boxed{}$ Differential Eq.

- General Solution for concentrated load.

$$y(z) = e^{\beta z}(C_1 \sin\beta z + C_2 \cos\beta z) + e^{-\beta z}(C_3 \sin\beta z + C_4 \cos\beta z)$$

$C_1 = C_2 = 0$ $\because$ displacement $\to 0$ as $z \to 0$

$\boxed{}$

where $\beta = \sqrt[4]{\dfrac{k}{4EI_x}}$

10.2. Boundary Conditions

(a) at $z=0$, slope $\theta=0$ $\because$ symmetry

(b) Force equilibrium in y-direction

$$\frac{dy}{dz} = -\beta e^{-\beta z}(C_3 \sin\beta z + C_4 \cos\beta z) + e^{-\beta z}(\beta C_3 \cos\beta z - \beta C_4 \sin\beta z)\Big|_{z=0} = 0.$$

$$\Rightarrow -\beta C_4 + \beta C_3 = 0.$$

$$\Rightarrow C_3 = C_4 = C.$$

$\Rightarrow y(z) =$

$$\int_0^\infty ky\,dz = \int_0^\infty Cke^{-\beta z}(\sin\beta z + \cos\beta z)dz = \frac{P}{2}$$

$$\text{※} \int e^{az}\sin bz\,dz = \frac{1}{a^2+b^2}e^{az}[a\sin bz - b\cos bz]$$

$$\int e^{az}\cos bz\,dz = \frac{1}{a^2+b^2}e^{az}[a\cos bz + b\sin bz]$$

$$Ck\int_0^\infty e^{-\beta z}(\sin\beta z + \cos\beta z)dz$$

$$= Ck\left[\frac{1}{2\beta^2}e^{-\beta z}(-\beta\sin\beta z - \beta\cos\beta z - \beta\cos\beta z + \beta\sin\beta z)\right]_0^\infty$$

$$= Ck\frac{1}{2\beta^2}2\beta = \frac{P}{2}$$

ZZO.

$$\theta(z) = \quad = -\frac{P\beta^2}{k} e^{-\beta z} \sin\beta z$$

$$M_x(z) = \quad = EI_x \frac{P\beta^3}{k} e^{-\beta z}(\cos\beta z - \sin\beta z)$$

$$= \frac{P}{4\beta} e^{-\beta z}(\cos\beta z - \sin\beta z)$$

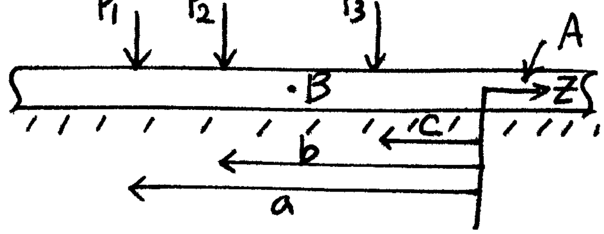
$$V_y(z) = -\frac{P}{2} e^{-\beta z} \cos\beta z$$

$y(-z) = y(z)$, $\theta(-z) = -\theta(z)$, $M_x(-z) = M_x(z)$

$V_y(-z) = -V_y(z)$.

1. Method of Superposition

Let
$$\begin{cases} A_{\beta z} = e^{-\beta z}(\sin\beta z + \cos\beta z) \\ B_{\beta z} = e^{-\beta z} \sin\beta z \\ C_{\beta z} = e^{-\beta z}(\cos\beta z - \sin\beta z) \\ D_{\beta z} = e^{-\beta z} \cdot \cos\beta z \end{cases}$$



$y(z) =$

superposition

- Point A:

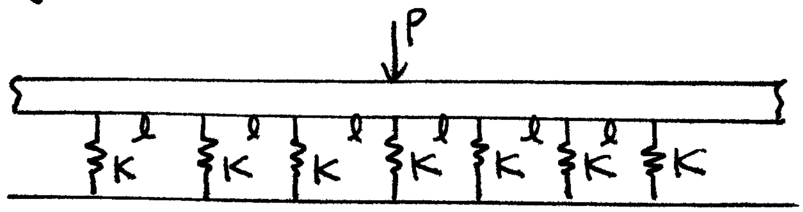
$$\begin{cases} y_A = \frac{\beta}{2k}\left[P_1 A_{\beta(z_A+a)} + P_2 A_{\beta(z_A+b)} + P_3 A_{\beta(z_A+c)}\right] \\ \theta_A = -\frac{\beta^2}{k}\left[P_1 B_{\beta(z_A+a)} + P_2 B_{\beta(z_A+b)} + P_3 B_{\beta(z_A+c)}\right] \end{cases}$$

- Point B : $z_B + c < 0$. use $|z_B + c|$

$$\begin{cases} y_B = \dfrac{\beta}{2k}\left[P_1 A_{\beta(z_B+a)} + P_2 A_{\beta(z_B+b)} + P_3 A_{\beta|z_B+c|}\right] \\ \theta_B = -\dfrac{\beta^2}{k}\left[P_1 B_{\beta(z_B+a)} + P_2 B_{\beta(z_B+b)} - P_3 B_{\beta|z_B+c|}\right] \end{cases}$$

$\hookrightarrow \because \theta$ is antisymmetric.

2. Equally Spaced Discrete Elastic Support



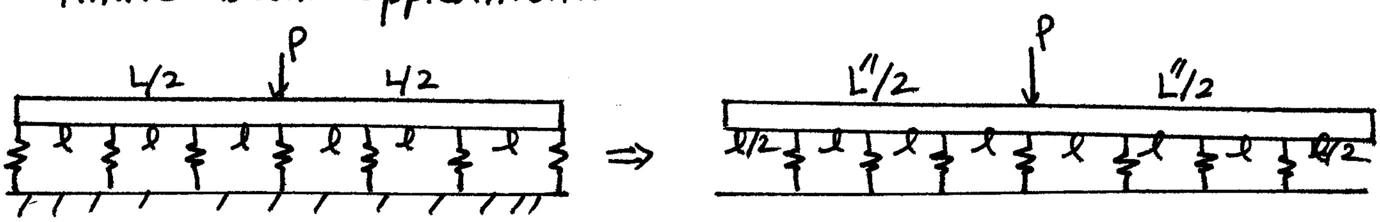
- Reaction force from spring

$$R = Ky$$

- Approximate the discrete reaction force by a distributed load by modulus of foundation

- The approximation error is small when

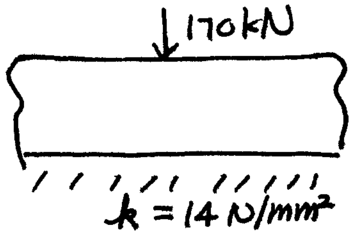
- Finite beam approximation

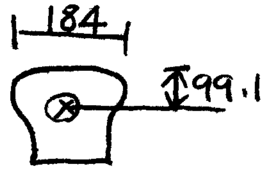


$$L'' = m\,l$$

$\uparrow$ # of springs

Example 10.1. Train Wheel on Rail





$E = 200\,GPa$

$I_{xx} = 36.9\times10^{6}\ mm^4$

(a) $P = 170\ kN.$ $\qquad \beta = \sqrt[4]{\dfrac{k}{4EI_x}} = 8.3\times10^{-4}\ mm^{-1}$

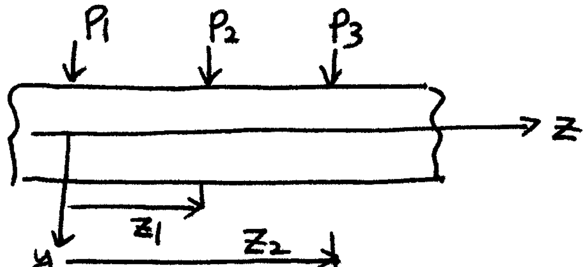
Max deflection at $z=0$ $\quad A_{\beta z} = C_{\beta z} = 1\ ,\ B_{\beta z} = D_{\beta z} = 0.$

$y_{max} = \qquad = 5.039\ mm$

$M_{max} = \qquad = 51.21\ kN\cdot m$

$\sigma_{max} = \qquad = 137.5\ MPa.$

(b)



$z_1 = 1.7\times10^3\ mm$

$z_2 = 3.4\times10^3\ mm$

$\beta z_1 = 1.411\ ,\qquad \beta z_2 = 2.822$

$A_{\beta z_1} = 0.2797\ ,\quad C_{\beta z_1} = -0.2018$

$A_{\beta z_2} = -0.0377\ ,\quad C_{\beta z_2} = -0.0752$

$$\begin{cases} y(z=0) = \dfrac{P\beta}{2k}\left(A_{\beta z_0} + A_{\beta z_1} + A_{\beta z_2}\right) = 6.258\ mm \\ M(z=0) = \dfrac{P}{4\beta}\left(C_{\beta z_0} + C_{\beta z_1} + C_{\beta z_2}\right) = 37.02\ kN\cdot m \end{cases}$$

same for $z = z_2$ $\because$ symmetry.

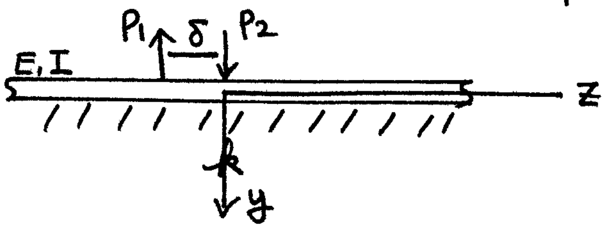
$$\begin{cases} y(z=z_1) = \dfrac{P\beta}{2k}\left(A_{\beta z_0} + 2A_{\beta z_1}\right) = 7.858 \\ M(z=z_1) = \dfrac{P}{4\beta}\left(C_{\beta z_0} + 2C_{\beta z_1}\right) = 30.54\ kN\cdot m \end{cases}$$

$\therefore\ y_{max} = y_{center} = 7.858\ mm$

$M_{max} = M_{end} = 37.02\ kN \cdot m$

$$\sigma_{max} = \frac{M_{max} \cdot c}{I_x} = 99.4\ MPa$$

Example 10.3. Beam with a Couple



$y(z) =$ $\qquad z \geq 0$

$M_x(z) =$ $\qquad z \geq 0$

Let $|P_1| = |P_2| = P$.

- $$y(z) = \frac{-(P\delta)\beta}{2k}\left[\frac{A_{\beta(z+\delta)} - A_{\beta z}}{\delta}\right]$$

Let $\delta \to 0$. $\quad P\delta \to M_0$

$$y(z) = -\lim_{\delta \to 0}\left\{\frac{(P\delta)\beta}{2k}\left[\frac{A_{\beta(z+\delta)} - A_{\beta z}}{\delta}\right]\right\} = -\frac{M_0\beta}{2k}\frac{dA_{\beta z}}{dz}$$

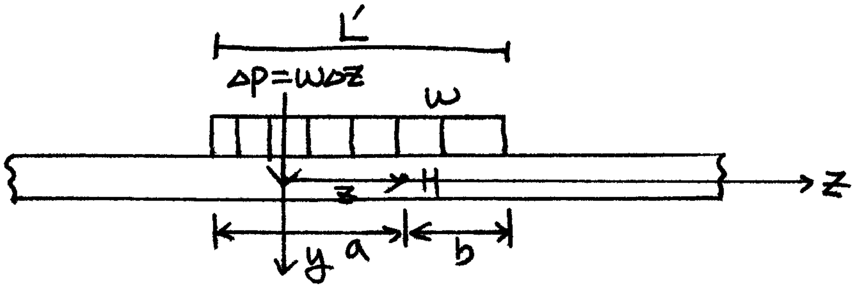
$\boxed{} \quad \Leftarrow \frac{dA_{\beta z}}{dz} = -2\beta B_{\beta z}$

- $$M_x(z) = -\lim_{\delta \to 0}\frac{P\delta}{4\beta}\left[\frac{C_{\beta(z+\delta)} - C_{\beta z}}{\delta}\right] = -\frac{M_0}{4\beta}\frac{dC_{\beta z}}{dz}$$

$\boxed{} \quad \Leftarrow \frac{dC_{\beta z}}{dz} = -2\beta D_{\beta z}$

10.3. Beam with Distributed Load

1. Uniformly Distributed Load



- Infinitesimal Δz, treat $\Delta P = w\Delta z$ as a concentrate force
- Point H locates at z from ΔP. Deflection of H caused by $P = \Delta P = w\Delta z$

$$\Delta y_H =$$

- Total deflection by superposition.

$$y_H = \sum \Delta y_H$$

$$= \int \frac{w\beta}{2k} e^{-\beta z}(\cos\beta z + \sin\beta z)\,dz$$

$$+ \int_0^b \frac{w\beta}{2k} e^{-\beta z}(\cos\beta z + \sin\beta z)\,dz$$

$$= \frac{w\beta}{2k}\left[\frac{1}{2\beta^2} e^{-\beta z}(\cancel{-\beta\sin\beta z} - \beta\cos\beta z - \beta\cos\beta z + \cancel{\beta\sin\beta z})\right]_0^a$$

$$+ \frac{w\beta}{2k}\left[\frac{1}{2\beta^2} e^{-\beta z}(-2\beta\cos\beta z)\right]_0^b$$

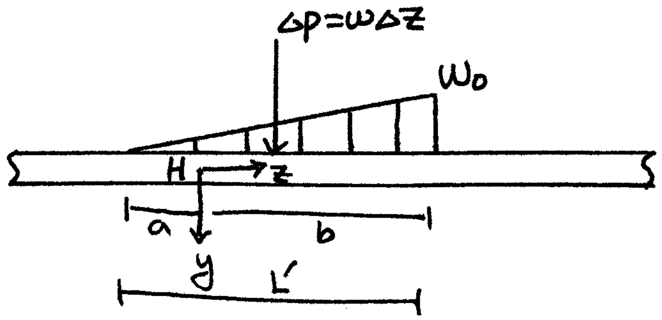
$$= \frac{w}{2k}\left[-\frac{1}{2} e^{-\beta a}\cdot 2\cos\beta a + \frac{1}{2}\cdot 1\cdot 2 - \frac{1}{2} e^{-\beta b}\cdot 2\cdot\cos\beta b + 1\right]$$

$$y_H = \_\_$$

$$\therefore \boxed{\begin{aligned} y_H &= \\ \theta_H &= \\ M_H &= \\ V_H &= \end{aligned}}$$

- Max. deflection occurs at the center of L'.
- $\beta L' \leq \pi$: M_{max} at the center of L'.
- $\beta L' > \pi$: M_{max} at $\frac{\pi}{4\beta}$ from either end of L'.

5. Triangular Load



$w = \frac{w_o}{L'}(a-z)$ over length a

or

$w = \frac{w_o}{L'}(a+z)$ over length b

$$\Delta y_H = \frac{w \Delta z \beta}{2k} A_{\beta z}$$

$$y_H = \frac{w_o \beta}{2kL'}\left[\int_0^a (a-z) A_{\beta z}\,dz + \int_0^b (a+z) A_{\beta z}\,dz\right]$$

$$=$$

$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$ //

Solve problems 10.1 10.9 10.20

10.1. By the method used in Example 10.3, derive formulas for the slope θ and the shear V_y of the beam.

10.9. An infinitely long beam rests on an elastic foundation and is loaded by two equal forces P spaced at a distance L. The beam has bending stiffness EI and the foundation has a spring constant k.

10.20. For the beam on a linearly elastic foundation shown in Figure 10.1, replace the concentrated load P by a concentrated (counterclockwise) moment M_0 at point 0. The beam has bending stiffness EI and the foundation has a spring constant k (force/area). Derive analytical expressions for the deflected shape $y(z)$, rotation $\theta(z)$, internal moment $M(z)$, and shear $V(z)$. Sketch each of the four expressions as is Figure 10.1 (see Example 10.3).