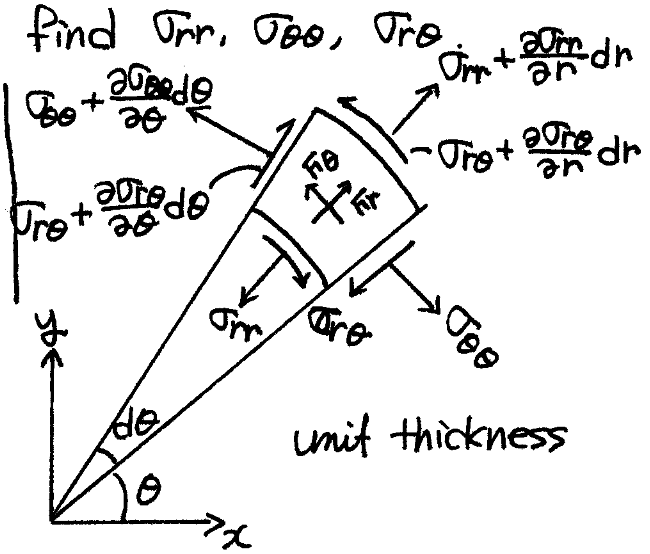
CH 11. THICK-WALL CYLINDER

11.1. General Equations

- Stress transformation

Given σ_{xx}, σ_{yy}, σ_{xy}, & θ , find σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$



From $[\sigma]_{x'y'} = [Q]^T[\sigma]_{xy}[Q]$

$$\begin{cases} \sigma_{rr} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \\ \sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \\ \sigma_{r\theta} = -(\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta-\sin^2\theta) \end{cases}$$

Alternatives

- Force equilibrium

$$\Sigma F_r = (\cancel{\sigma_{rr}}+\frac{\partial\sigma_{rr}}{\partial r}dr)(r+dr)d\theta - \cancel{\sigma_{rr}}\, r\, d\theta$$
$$+ (\cancel{\sigma_{r\theta}}+\frac{\partial\sigma_{r\theta}}{\partial\theta}d\theta)dr\cos\frac{d\theta}{2} - \cancel{\sigma_{r\theta}}\, dr\cos\frac{d\theta}{2}$$
$$-(\sigma_{\theta\theta}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}d\theta)dr\sin\frac{d\theta}{2} - \sigma_{\theta\theta}\, dr\sin\frac{d\theta}{2}$$
$$+ F_r(r+\frac{dr}{2})d\theta dr = 0.$$

Simplify by ignoring higher-order terms.

$$\Rightarrow \sigma_{rr}\,dr\,d\theta + \frac{\partial \sigma_{rr}}{\partial r} r\,dr\,d\theta + \frac{\partial \sigma_{r\theta}}{\partial \theta} dr\,d\theta$$

$$- \sigma_{\theta\theta}\,dr\,d\theta + F_r\, r\,dr\,d\theta = 0$$

$\Rightarrow$

$$\Sigma F_\theta = 0 = \left(\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} d\theta\right) dr \cos\frac{d\theta}{2} - \sigma_{\theta\theta}\, dr \cos\frac{d\theta}{2}$$

$$+ \left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta\right) dr \sin\frac{d\theta}{2} + \sigma_{r\theta}\, dr \sin\frac{d\theta}{2}$$

$$+ \left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial r} dr\right)(r+dr)\,d\theta - \sigma_{r\theta}\, r\,d\theta + F_\theta \left(r + \frac{dr}{2}\right) dr\,d\theta = 0.$$

$$\Rightarrow \frac{\partial \sigma_{\theta\theta}}{\partial \theta} dr\,d\theta + \sigma_{r\theta}\,dr\,d\theta + \sigma_{r\theta}\,dr\,d\theta + \frac{\partial \sigma_{r\theta}}{\partial r} r\,dr\,d\theta + F_\theta\, r\,dr\,d\theta = 0$$

$\Rightarrow$

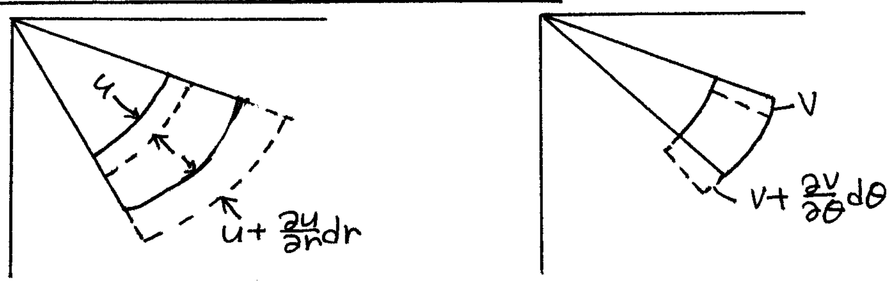
Summary of Governing Differential Eq.

(1)

- Strains

u: radial displ.

v: tangential displ. (2)



- Compatibility Eq.

$$\boxed{} \qquad (3)$$

- Stress strain relation
  - Plane stress

$$\begin{cases} \varepsilon_{rr} = \\ \varepsilon_{\theta\theta} = \\ \gamma_{r\theta} = \end{cases}$$

  - Plane strain

$$\begin{cases} \varepsilon_{rr} = \\ \varepsilon_{\theta\theta} = \\ \gamma_{r\theta} = \end{cases}$$

  - Put stress-strain relation into compatibility eq.

$$\Rightarrow \frac{\partial^2}{\partial r^2}(\sigma_{\theta\theta} - \nu\sigma_{rr}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\sigma_{rr} - \nu\sigma_{\theta\theta}) + \frac{2}{r}\frac{\partial}{\partial r}(\sigma_{\theta\theta} - \nu\sigma_{rr})$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(\sigma_{rr} - \nu\sigma_{\theta\theta}) = 2(1-\nu)\left[\frac{\partial^2\sigma_{r\theta}}{r\,\partial r\,\partial\theta} + \frac{1}{r^2}\frac{\partial\sigma_{r\theta}}{\partial\theta}\right]$$

$$\Rightarrow \boxed{} \qquad (4)$$

$$\Rightarrow$$

______ in cylindrical coord. system.

- **Airy's Stress Function (no body force)**

(5)

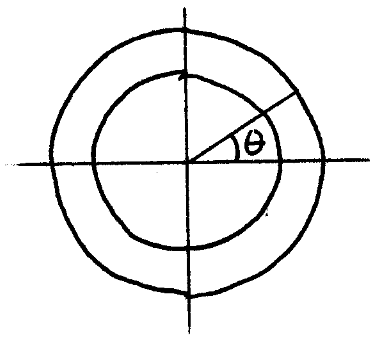
Automatically satisfy equilibrium.

$$\sigma_{rr} + \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \nabla^2 \phi$$

$\therefore$ Eq. (4) Compatibility Equation becomes

(6)

11.2. Axi-symmetric Problems



$\sigma_{r\theta} = 0$

$\sigma_{\theta\theta}$ = independent of θ

σ_{rr} = independent of θ

- Equilibrium

- Compatibility

$$\Rightarrow \left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} \right)\phi = 0$$

$$\Rightarrow \quad \frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0$$

- Since all terms are same order with respect to r,

try $r = e^t$ or $t = \ln r$.

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = \frac{1}{r}\frac{d\phi}{dt}$$

$$\frac{d^2\phi}{dr^2} = \frac{d}{dr}\left(\frac{d\phi}{dr}\right) = \frac{1}{r^2}\left(\frac{d^2\phi}{dt^2} - \frac{d\phi}{dt}\right)$$

$$\frac{d^3\phi}{dr^3} = \frac{1}{r^3}\left(\frac{d^3\phi}{dt^3} - 3\frac{d^2\phi}{dt^2} + 2\frac{d\phi}{dt}\right)$$

$$\frac{d^4\phi}{dr^4} = \frac{1}{r^4}\left(\frac{d^4\phi}{dt^4} - 6\frac{d^3\phi}{dt^3} + 11\frac{d^2\phi}{dt^2} - 6\frac{d\phi}{dt}\right)$$

Compatibility Eq. $\Rightarrow$

$$\frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0$$

※ General soln : let $\phi = e^{\alpha t} \Rightarrow \alpha^4 - 4\alpha^3 + 4\alpha^2 = 0.$

$\Rightarrow \alpha^2(\alpha-2)^2 = 0. \Rightarrow \alpha = 0, 2$ but double root.

$\therefore$

$\therefore$

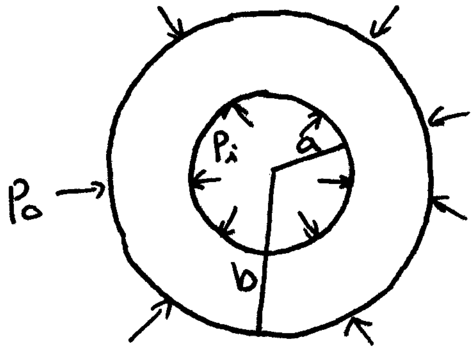
Use boundary conditions to determine

C_1, C_2 & C_3

11.3. Thick Cylinder under Uniform Pressure

Lame's solution (1852)

- Case 1. Thick cylinder



B.C.s

$\sigma_{rr} = -P_i$ at $r = a$

$\sigma_{rr} = -P_o$ at $r = b$.

- Assume $C_1 = 0$.

$\sigma_{rr} =$

$\sigma_{\theta\theta} =$

$$\longrightarrow \left(\begin{array}{l} -P_i = \\ -P_o = \end{array} \right.$$

$$\Rightarrow C_3 = \frac{P_i a^2 - P_o b^2}{b^2 - a^2} + \frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2} \frac{1}{r^2}$$

$$C_3 = (P_o - P_i) \frac{a^2 b^2}{b^2 - a^2} \quad ; \quad C_2 = \frac{1}{2} \frac{P_i a^2 - P_o b^2}{b^2 - a^2}$$

$\Rightarrow$

- If $P_o = 0$

$\sigma_{rr} =$ < 0 compression

$\sigma_{\theta\theta} =$ > 0 tension

$\sigma_{\theta\theta}\big|_{max} =$

※ Since $\sigma_{\theta\theta}|_{max} > P_i$, a compressive pre-stress is often applied by winding wires so that $\sigma_{\theta\theta}|_{initial}$ = compressive.

- Case 2. Thin Cylinder ($a \approx b \approx R$)

$b - a = t \qquad b^2 - a^2 = (b+a)(b-a) = 2Rt$

$\Rightarrow \sigma_{\theta\theta}|_{max} =$

$\sigma_{rr} \approx 0$

- Strain

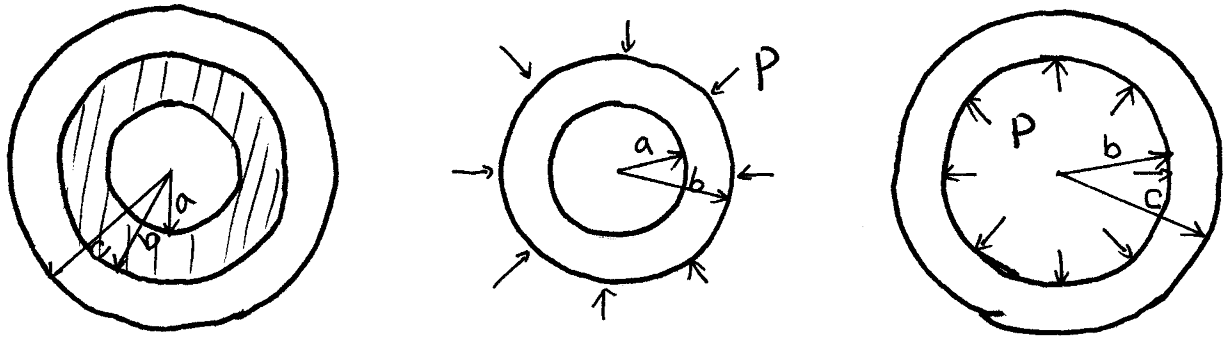
$$\begin{cases} \varepsilon_{rr} = \\ \varepsilon_{\theta\theta} = \end{cases}$$: plane stress

$$\begin{aligned} \varepsilon_{rr} \Rightarrow \; & EU = C_1\left[r(1-3\nu) + 2(1-\nu)(r\ln r - r)\right] + 2C_2(1-\nu)r \\ & \quad - C_3(1+\nu)\frac{1}{r} + C_5 \\ \varepsilon_{\theta\theta} \Rightarrow \; & EU = C_1\left[r(3-\nu) + 2r(1-\nu)\ln r\right] + 2C_2(1-\nu)r \\ & \quad - C_3(1+\nu)\frac{1}{r} \end{aligned}$$

By comparing two equations, $C_1 = 0$, $C_5 = 0$

※ $\sigma_{rr} + \sigma_{\theta\theta} = 4C_2$ = const. Since $\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta})$ = const, originally plane cross-section remains plane.

∘ Shrink Fit



radial displ due to $p = -\delta_1$

radial displ. due to $p = \delta_2$

P : shrink fit pressure

δ : interference at $r = b$ $\qquad \delta = |\delta_1| + |\delta_2|$

- For plane stress

plane strain

where $2C_2 = \dfrac{P_i a^2 - P_o b^2}{b^2 - a^2}$; $\qquad C_3 = (P_o - P_i)\dfrac{a^2 b^2}{b^2 - a^2}$

- For inner cylinder $(P_i = 0,\ P_o = P)$ $\quad r = b$

$\delta_1 = u = \qquad\qquad\qquad\qquad < 0$

- For outter cylinder $(P_i = P,\ P_o = 0)$ $\quad r = b$

$\delta_2 = u =$

$\therefore$ $\boxed{}$

If $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$

$\boxed{}$

If $a=0$ (solid axis inside)

- Stresses
  - Inner tube :
  - Outter tube :

$$\sigma_{rr}\big|_{max} = -P$$

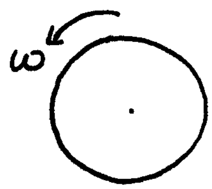
$$\sigma_{\theta\theta max} = \frac{c^2+b^2}{c^2-b^2}P \quad \text{at } r=b \text{ of the outter tube.}$$

11.4. Rotating Disk (A. Stodola)

- Equilibrium

$$\begin{cases} \dfrac{\partial\sigma_{rr}}{\partial r} + \dfrac{1}{r}\cancel{\dfrac{\partial\sigma_{r\theta}}{\partial\theta}} + \dfrac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta}) + F_r = 0 \\ \cancel{\dfrac{\partial\sigma_{r\theta}}{\partial r}} + \dfrac{1}{r}\cancel{\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta}} + \dfrac{2}{r}\cancel{\sigma_{r\theta}} + \cancel{F_\theta} = 0 \end{cases}$$

$$F_r = \rho r\omega^2$$



$$\Rightarrow \frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta}) + \rho\omega^2 r = 0$$

$$r\frac{d\sigma_{rr}}{dr} + \sigma_{rr} - \sigma_{\theta\theta} + \rho\omega^2 r^2 = 0$$

Equil. Eq.

If we choose ϕ, such that

then Equil. Eq. satisfies.

- **Compatibility**

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u}{r}$$

$$u = \varepsilon_{\theta\theta} r \qquad \frac{du}{dr} = \frac{d}{dr}(\varepsilon_{\theta\theta} r)$$

$$\therefore \frac{d}{dr}(\varepsilon_{\theta\theta} r) - \varepsilon_{rr} = 0$$

$$\Rightarrow \boxed{\qquad\qquad\qquad\qquad}$$ Compatibility eq.

- **Hooke's law (Plane stress)**

$$\begin{cases} \varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) \\ \varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) \end{cases}$$

. Comp. Eq. $\Rightarrow$

$$r\frac{d}{dr}\left[\frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr})\right] + \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) - \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) = 0$$

$$\Rightarrow r\frac{d}{dr}\left(\frac{d\phi}{dr} + \rho\omega^2 r^2 - \nu\frac{\phi}{r}\right) + (1+\nu)\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\phi}{r}\right] = 0$$

$$\Rightarrow r\frac{d^2\phi}{dr^2} + 2\rho\omega^2 r^2 - \nu\frac{d\phi}{dr} + \nu\frac{\phi}{r} + (1+\nu)\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 - \frac{\phi}{r}\right] = 0$$

$$\Rightarrow \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{\phi}{r^2} + (3+\nu)\rho\omega^2 r = 0$$

$\Rightarrow$

____

. Integration

$$\frac{1}{r}\frac{d}{dr}(r\phi) = -\frac{3+\nu}{2}\rho\omega^2 r^2 + C_1$$

$$r\phi = -\frac{3+\nu}{8}\rho\omega^2 r^4 + \frac{C_1}{2}r^2 + C_2$$

$\therefore$

- <u>Boundary conditions</u>

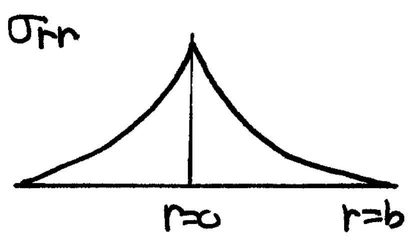
$$\sigma_{rr} = 0 \quad \text{at} \quad r = b$$

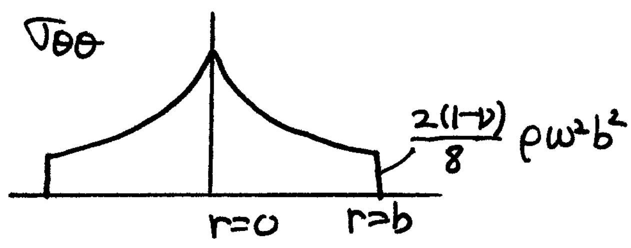
$$\sigma_{rr} = \text{finite} \quad \text{at} \quad r = 0 \quad \Rightarrow \quad C_2 = 0.$$

$$-\frac{3+\nu}{8}\rho\omega^2 b^2 + \frac{C_1}{2} = 0 \quad \Rightarrow \quad C_1 = \frac{3+\nu}{4}\rho\omega^2 b^2.$$

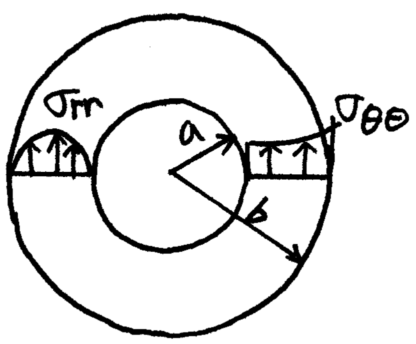
$\Rightarrow$

$$\sigma_{rr}\big|_{max} = \sigma_{\theta\theta}\big|_{max} = \frac{3+\nu}{8}\rho\omega^2 b^2 \quad \text{at} \quad r = 0$$





- <u>Boundary conditions for disk with hole</u>



$$\sigma_{rr} = 0 \quad \text{at} \quad r = a \;\&\; r = b$$

$$\Rightarrow \frac{C_1}{2} = \frac{3+\nu}{8}\rho\,\omega^2(a^2+b^2)$$

$$C_2 = -\frac{3+\nu}{8}\rho\,\omega^2 a^2 b^2$$

∴

$$\sigma_{\theta\theta}\big|_{max} = \frac{3+\nu}{4}\rho\omega^2 b^2\left(1+\frac{1-\nu}{3+\nu}\frac{a^2}{b^2}\right) \quad \text{at } r=a$$

$$\sigma_{rr}\big|_{max} = \frac{3+\nu}{8}\rho\omega^2 (b-a)^2 \quad \text{at } r=\sqrt{ab}$$

$$\sigma_{\theta\theta}\big|_{max} > \sigma_{rr}\big|_{max}.$$

$$\lim_{a\to 0} \sigma_{\theta}\big|_{r=0} =$$

Example 11.2. Hollow Cylinder (Closed End)

$E = 72$ GPa $\quad \nu = 0.33 \quad D_i = 200$ mm $\quad D_o = 800$ mm

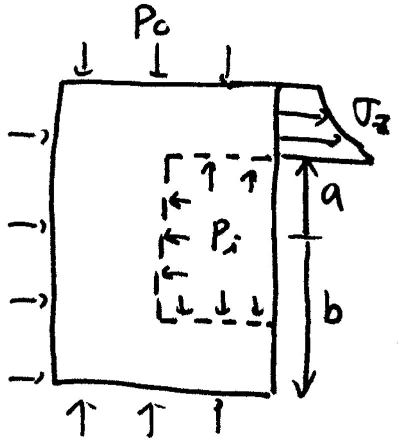
$P_i = 150$ MPa. Principal stresses, τ_{max}, u_i ? at $r=a$

In axi-symmetric problems, σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz} are principal st.

$$\sigma_{rr} = \frac{P_i a^2}{b^2-a^2} - \frac{a^2 b^2 P_i}{b^2-a^2}\frac{1}{r^2} = \frac{a^2-b^2}{b^2-a^2}P_i = -150 \text{ MP}_a$$

$$\sigma_{\theta\theta} = \frac{a^2+b^2}{b^2-a^2}P_i = 170 \text{ MPa} > P_i \quad \text{This is } \sigma_{max}.$$

· For the closed-end cylinder



$$\int_a^b \sigma_{zz}\, 2\pi r dr = \pi(P_i a^2 - P_o b^2)$$

$$\sigma_{zz} = \frac{\pi(P_i a^2 - P_o b^2)}{\pi(b^2-a^2)}$$

$$= P_i\frac{a^2}{b^2-a^2} = 10 \text{ MPa}$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = 160 \text{ MPa}$$

Example 11.7. Rotating Solid Disk

Given : ρ, ν, r, b, ω

Instead of using stress function use displacement.

B.C. $\begin{cases} u = 0 & \text{at } r = 0 \\ \sigma_{rr} = 0 & \text{at } r = b \end{cases}$

For plane stress problem

$$\begin{cases} \sigma_{rr} = \frac{E}{1-\nu^2}(\varepsilon_{rr} + \nu\,\varepsilon_{\theta\theta}) = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \\ \sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\,\varepsilon_{rr}) = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) \end{cases}$$

Put these into equilibrium eq.

$$\frac{\partial}{\partial r}\left[\frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)\right] + \frac{1}{r}\left[\frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r} - \frac{u}{r} - \nu\frac{du}{dr}\right)\right] + \rho r\omega^2 = 0$$

$$\Rightarrow \frac{d^2u}{dr^2} - \nu\frac{u}{r^2} \underset{+\frac{\nu}{r}\frac{du}{dr}}{\wedge} + (1-\nu)\left(\frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) + \frac{1-\nu^2}{E}\rho\omega^2 r = 0$$

$$\Rightarrow \underbrace{\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}}_{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right]} = -\frac{1-\nu^2}{E}\rho\omega^2 r$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = -\frac{1-\nu^2}{E}\rho\omega^2 r$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}(ru) = -\frac{1}{2}\frac{1-\nu^2}{E}\rho\omega^2 r^2 + C_1$$

$$ru = -\frac{1}{8}\frac{1-\nu^2}{E}\rho\omega^2 r^4 + \frac{C_1}{2}r^2 + C_2$$

$$u = -\frac{1}{8}\frac{1-\nu^2}{E}\rho\omega^2 r^3 + \frac{C_1}{2}r + \frac{C_2}{r}$$

From B.C. $u|_{r=0} = 0 \Rightarrow C_2 = 0 \because$ finite.

$$u = -\frac{1-\nu^2}{8E}\rho\omega^2 r^3 + \frac{C_1}{2} r$$

Then,

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(-\frac{3(1-\nu^2)}{8E}\rho\omega^2 r^2 + \frac{C_1}{2} + -\frac{(1-\nu^2)\nu}{8E}\rho\omega^2 r^2 + \frac{C_1}{2}\nu\right)$$

$$= \frac{E}{1-\nu^2}\left[-\frac{(3+\nu)(1-\nu^2)}{8E}\rho\omega^2 r^2 + (1+\nu)\frac{C_1}{2}\right]$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[-\frac{1-\nu^2}{8}\rho\omega^2 r^2 + \frac{C_1}{2} - \frac{3(1-\nu^2)}{8}\nu\rho\omega^2 r^2 + \frac{C_1}{2}\nu\right]$$

$$= \frac{E}{1-\nu^2}\left[-\frac{(1+3\nu)(1-\nu^2)}{8E}\rho\omega^2 r^2 + (1+\nu)\frac{C_1}{2}\right]$$

From B.C. $\sigma_{rr}|_{r=b} = 0$

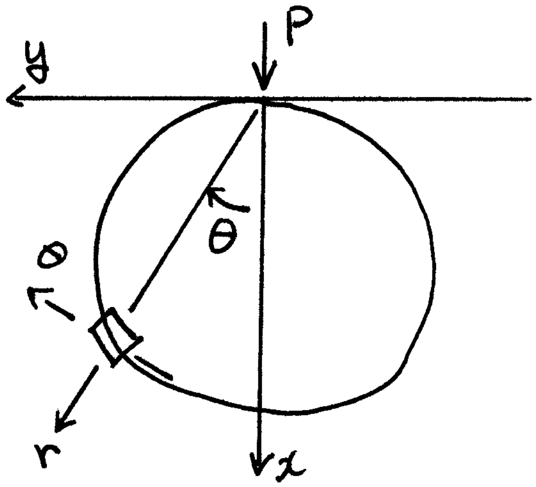
$$\frac{C_1}{2} = \frac{(3+\nu)(1-\nu)}{8E}\rho\omega^2 b^2$$

$$\Rightarrow \sigma_{rr} = \frac{(3+\nu)}{8}\rho b^2\omega^2\left[1 - \frac{r^2}{b^2}\right]$$

$$\sigma_{\theta\theta} = \frac{3+\nu}{8}\rho b^2\omega^2\left[1 - \frac{(1+3\nu)}{(3+\nu)}\frac{r^2}{b^2}\right]$$

same with pp. 135

11.5 Concentrated Force on Straight Boundary



- **Boundary Conditions**

$\sigma_{\theta\theta} = 0 \quad$ at $\quad \theta = \pm\frac{\pi}{2}$

$\sigma_{r\theta} = 0 \quad$ at $\quad \theta = \pm\frac{\pi}{2}$

- **Stress function**

$$\boxed{\qquad\qquad\qquad}$$

$$\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)$$

$$= \frac{1}{r}f(\theta) + \frac{1}{r}f''(\theta)$$

$$= \frac{1}{r}\left(f''(\theta) + f(\theta)\right) \equiv \frac{1}{r}F(\theta) \quad , \quad F(\theta) = f''(\theta) + f(\theta)$$

- **Compatibility condition**

$$\nabla^2\nabla^2\phi = \nabla^2\left(\frac{1}{r}F(\theta)\right) = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial r^3} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}F(\theta)\right) = 0$$

$$\Rightarrow \frac{2}{r^3}F(\theta) - \frac{1}{r^3}F(\theta) + \frac{1}{r^3}\frac{\partial^2}{\partial\theta^2}F(\theta) = 0$$

$$\Rightarrow \boxed{\qquad\qquad\qquad\qquad} \qquad \text{O.D.E.}$$

$$F(\theta) = 2C_1\cos\theta + 2C_2\sin\theta$$

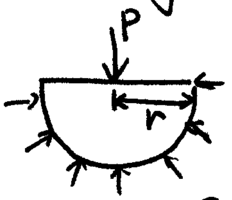
$$\Rightarrow \quad f''(\theta) + f(\theta) = 2C_1\cos\theta + 2C_2\sin\theta \qquad \text{O.D.E}$$

$$\Rightarrow \quad f(\theta) = C_1\theta\sin\theta + C_2\theta\cos\theta + C_3\cos\theta + C_4\sin\theta$$

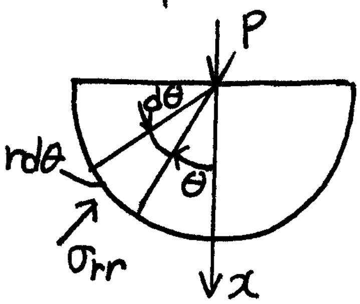
- Try only the first term.

$$\begin{cases} \sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{r}C_1\theta\sin\theta + \frac{1}{r^2}(2C_1 r\cos\theta - C_1 r\theta\sin\theta) \\ \quad = 2C_1\frac{1}{r}\cos\theta \\ \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = 0 \\ \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0 \end{cases}$$

- <u>Boundary conditions</u>



$P = \Sigma$ (Vertical component of stress) $\times$ area



$$\Sigma F_x = 2\int_0^{\frac{\pi}{2}} \sigma_{rr}\cdot\cos\theta\cdot r\,d\theta + P = 0$$

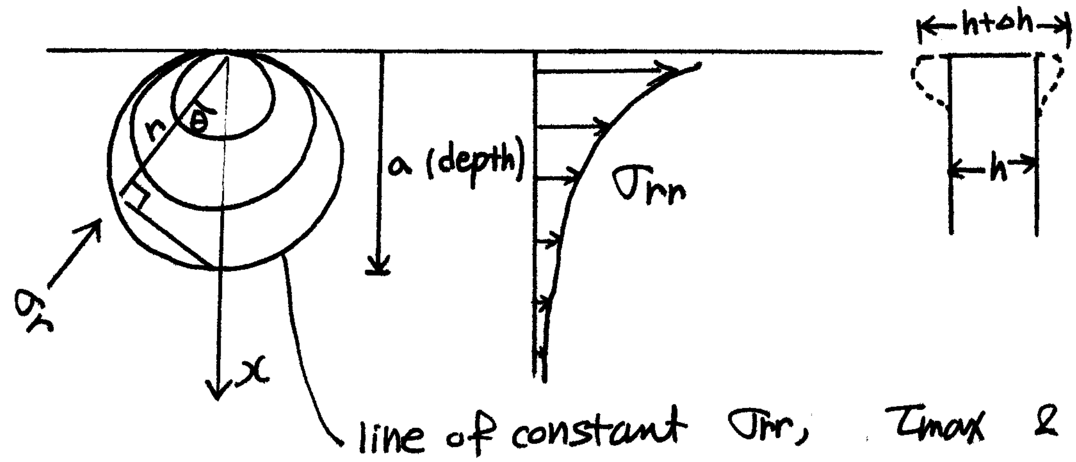
$$P + 2\int_0^{\frac{\pi}{2}} \frac{1}{r}2C_1\cos\theta\cdot r\cos\theta\,d\theta = 0$$

$$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\Rightarrow \quad C_1 = -\frac{P}{\pi}$$

$\therefore$

at $r=0 \quad \sigma_{rr} \to -\infty$



$r = a\cos\theta$

$\sigma_{rr} =$ constant

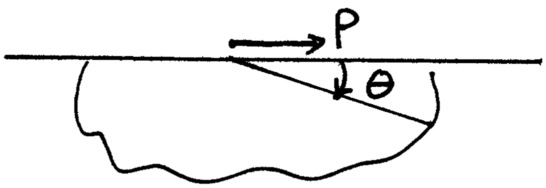
$\sigma_{\theta\theta} = 0$

$\tau_{max} =$

$\therefore$ $r-\theta$ directions are principal direction

$$\Delta h = h\,\varepsilon_{zz} = \frac{h}{E}\left(\cancel{\sigma_{zz}} - \nu\sigma_{rr}\right) = -\frac{\nu h}{E}\cdot\left(-\frac{2P}{\pi a}\right) = \frac{2\nu h P}{\pi a E}.$$

✱ Concentrated Horizontal Force

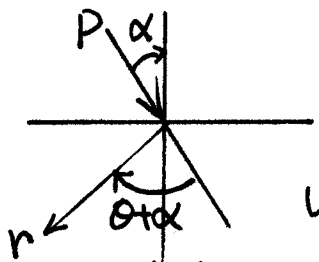


Angle starts from horizonal axis

$$\phi = -\frac{P}{\pi} r\theta\sin\theta$$

$$\sigma_{rr} = -\frac{2P}{\pi r}\cos\theta$$

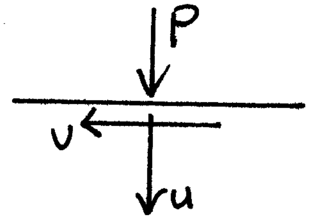
✱



use superposition

$$\begin{cases} \sigma_{rr} = -\frac{2P}{\pi}\frac{1}{r}\cos(\alpha+\theta) \\ \sigma_{\theta\theta} = 0 \\ \sigma_{r\theta} = 0 \end{cases}$$

∘ Displacement (Vertical load)



$$\begin{cases} \varepsilon_{rr} = \dfrac{\partial u}{\partial r} = \dfrac{1}{E}\left(\sigma_{rr} - \nu \cancel{\sigma_{\theta\theta}}\right) \\ \varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = \dfrac{1}{E}\left(\cancel{\sigma_{\theta\theta}} - \nu\sigma_{rr}\right) \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} + \dfrac{\partial v}{\partial r} - \dfrac{v}{r} = \dfrac{1}{G}\cancel{\tau_{r\theta}} = 0 \end{cases}$$

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} = \frac{\sigma_{rr}}{E} = -\frac{2P}{\pi E}\frac{1}{r}\cos\theta$$

$$\underline{u = -\frac{2P}{\pi E}\cos\theta \ln r + f_1(\theta)}$$

$$\frac{\partial v}{\partial \theta} = -u - \frac{\nu}{E} r\left(-\frac{2P}{\pi}\frac{1}{r}\cos\theta\right)$$

$$= \frac{2P}{\pi E}\cos\theta \ln r - f_1(\theta) + \frac{2P\nu}{\pi E}\cos\theta$$

$$\underline{v = \frac{2P\nu}{\pi E}\sin\theta + \frac{2P}{\pi E}\ln r \sin\theta - \int f_1(\theta)d\theta + f_2(r).}$$

· Put u & v into $\gamma_{r\theta}$.

$$r\cdot\gamma_{r\theta} = \frac{2P}{\pi E}\cancel{\sin\theta \ln r} + f_1'(\theta) + \frac{2P}{\pi E}\sin\theta + r f_2'(r)$$

$$- \frac{2P\nu}{\pi E}\sin\theta - \frac{2P}{\pi E}\cancel{\ln r \sin\theta} + \int f_1(\theta)d\theta - f_2(r) = 0$$

$$\Rightarrow \begin{cases} f_1'(\theta) + \int f_1(\theta)d\theta + \dfrac{2P(1-\nu)}{\pi E}\sin\theta = 0 \\ r f_2'(r) - f_2(r) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} f_1(\theta) = A\sin\theta + B\cos\theta - \dfrac{P(1-\nu)}{\pi E}\theta\sin\theta \\ f_2(r) = C\cdot r \end{cases}$$

$$\therefore\ u = -\frac{2P}{\pi E}\ln r\cos\theta + A\sin\theta + B\cos\theta - \frac{P(1-\nu)}{\pi E}\theta\sin\theta$$

$$v = \frac{2P\nu}{\pi E}\sin\theta + \frac{2P}{\pi E}\ln r\sin\theta + A\cos\theta - B\sin\theta$$

$$-\frac{P(1-\nu)}{\pi E}\theta\cos\theta + \frac{P(1-\nu)}{\pi E}\sin\theta + c\cdot r$$

- Boundary conditions

1) $v|_{\theta=0} = 0 = A + cr$ for all r $\Rightarrow$ $A = c = 0.$

2) $u|_{\substack{\theta=0 \\ r=d}} = 0 = B - \frac{2P}{\pi E}\ln d$ $\quad\therefore\ B = \frac{2P}{\pi E}\ln d$

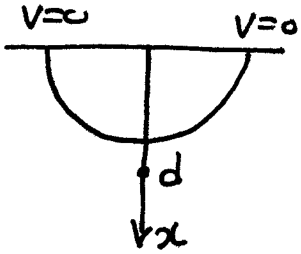
※ vertical displacement is zero at distand d.

- Vertical displacement along x-axis.

$$u|_{\theta=0} = -\frac{2P}{\pi E}\ln r + \frac{2P}{\pi E}\ln d = \frac{2P}{\pi E}\ln\frac{d}{r}$$

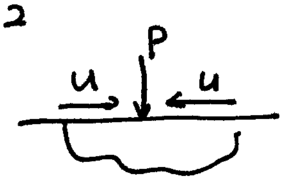
- surface change

$$v|_{\theta=\pm\frac{\pi}{2}} = -\frac{P}{\pi E}\left[2\ln\frac{d}{r} - (1+\nu)\right]$$



$$v = 0 \text{ at } \ln\frac{d}{r} = \frac{1+\nu}{2}$$

- $$u|_{\theta=\pm\frac{\pi}{2}} = -\frac{1-\nu}{2E}P.$$



Summary

$$\boxed{\begin{aligned} u &= -\frac{2P}{\pi E}\ln r\cos\theta + \frac{2P}{\pi E}\ln d\cos\theta - \frac{P(1-\nu)}{\pi E}\theta\sin\theta \\ &= \frac{2P}{\pi E}\ln\frac{d}{r}\cos\theta - \frac{P(1-\nu)}{\pi E}\theta\sin\theta \\ v &= \frac{P(1+\nu)}{\pi E}\sin\theta - \frac{2P}{\pi E}\ln\frac{d}{r}\sin\theta - \frac{P(1-\nu)}{\pi E}\theta\cos\theta \end{aligned}}$$

- Cartesian coord.

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r}, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{r\theta} = 0$$

$$\sigma_{xx} = \frac{\sigma_{rr}+\sigma_{\theta\theta}}{2} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{2}\cos 2\theta + \cancel{\sigma_{r\theta}}\sin 2\theta = \sigma_{rr}\frac{1+\cos 2\theta}{2} = \sigma_{rr}\cos^2\theta$$

$$\sigma_{yy} = \frac{\sigma_{rr}+\sigma_{\theta\theta}}{2} - \frac{\sigma_{rr}-\sigma_{\theta\theta}}{2}\cos 2\theta + \cancel{\sigma_{r\theta}}\sin 2\theta = \sigma_{rr}\sin^2\theta$$

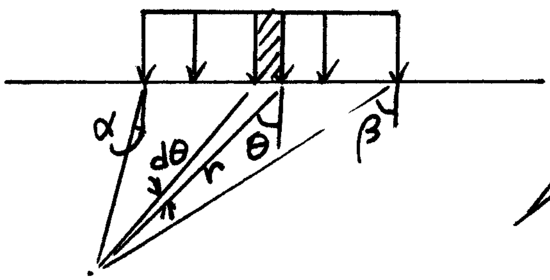
$$\sigma_{xy} = -\frac{\sigma_{rr}-\sigma_{\theta\theta}}{2}\sin 2\theta + \cancel{\sigma_{r\theta}}\cos 2\theta = \sigma_{rr}\sin\theta\cos\theta$$

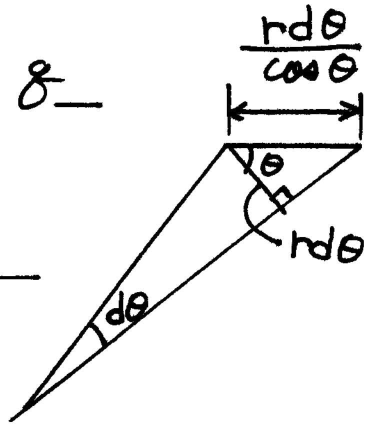
$$\therefore \boxed{\begin{aligned} \sigma_{xx} &= \sigma_{rr}\cos^2\theta = -\frac{2P}{\pi a}\cos^4\theta \\ \sigma_{yy} &= \sigma_{rr}\sin^2\theta = -\frac{2P}{\pi a}\sin^2\theta\cos^2\theta \\ \sigma_{xy} &= -\frac{2P}{\pi a}\sin\theta\cos^3\theta \end{aligned}}$$

where $a = r\cos\theta$

$r = \dfrac{a}{\cos\theta}$

11.6. Distributed Load q





- load on the segment $= q\dfrac{r\,d\theta}{\cos\theta}$

$$\sigma_{xx} = -\frac{2P}{\pi}\frac{\cos^3\theta}{r}$$

$$\sigma_{xx} = -\int_{\alpha}^{\beta}\frac{2q}{\pi}\frac{r\,d\theta}{\cos\theta}\frac{\cos^3\theta}{r}$$

· Assume $q = \text{const}$

$$\sigma_{xx} = -\frac{2q}{\pi}\int_{\alpha}^{\beta}\cos^2 d\theta = -\frac{2q}{\pi}\int_{\alpha}^{\beta}\frac{1+\cos 2\theta}{2}d\theta$$

$$= -\frac{q}{\pi}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\alpha}^{\beta} = -\frac{q}{\pi}\left[\beta - \alpha + \frac{1}{2}\sin 2\beta - \frac{1}{2}\sin 2\alpha\right]$$

$$\sigma_{yy} = -\frac{q}{\pi}\left[\beta - \alpha - \frac{1}{2}\sin 2\beta + \frac{1}{2}\sin 2\alpha\right]$$

$$\sigma_{xy} = \frac{q}{\pi}\left[\frac{1}{2}\cos 2\beta - \frac{1}{2}\cos 2\alpha\right]$$

· Principal stress & direction

$$\tan 2\theta_p = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \tan(\beta + \alpha)$$

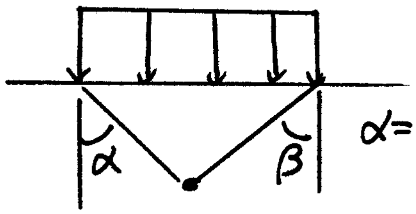
$$\therefore \theta_p = \frac{\beta + \alpha}{2}$$

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\sigma_1 = -\frac{q}{\pi}\left[\beta - \alpha - \sin(\beta - \alpha)\right]$$

$$\sigma_2 = -\frac{q}{\pi}\left[\beta - \alpha + \sin(\beta - \alpha)\right]$$

$$\tau_{max} = \frac{q}{\pi}\sin(\beta - \alpha).$$



$\alpha = -\beta$

$$\tau_{max} = \frac{q}{\pi}\sin 2\beta$$

at $\beta = 45°$, $\tau_{max} = \frac{q}{\pi}$

※ Shear failure under the surface.

at surface $\beta = \frac{\pi}{2} \Rightarrow \sigma_1 = \sigma_2 = -q$

at $\beta = -\alpha$ $\quad \sigma_1 = -\frac{q}{\pi}(2\beta - \sin 2\beta)$

$$\sigma_2 = -\frac{q}{\pi}(2\beta + \sin 2\beta)$$

$$\tau_{max})_{max} = \frac{q}{\pi} \text{ at } \beta = 45°$$

Solve problems 11.3 11.10 11.23 11.36

11.3. An open thick-wall cylinder of inner radius $a = 100$ mm and outer radius $b = 200$ mm is subjected to an internal pressure $p_1 = 200$ MPa.

a. Determine the stress components σ_{rr} and $\sigma_{\theta\theta}$ at $r = 100$ mm, $r = 150$ mm, and $r = 200$ mm.

b. Sketch the distribution of σ_{rr} and $\sigma_{\theta\theta}$ through the wall of the cylinder.

11.10. An aluminum composite cylinder ($E = 72$ GPa and $\nu = 0.33$) is made by shrinking an outer cylinder onto an inner cylinder (Figure P11.10). Initially the outer radius c of the inner cylinder is larger than the inner radius of the outer cylinder by an amount $\delta = 0.125$ mm (see Problem 11.8). The cylinder is subjected to an internal pressure $p_i = 200$ MPa. Determine the stress $\sigma_{\theta\theta}$ in the inner cylinder at $r = 150$ mm and in the outer cylinder at $r = 150$ mm.

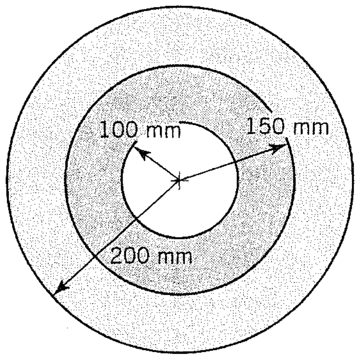


FIGURE P11.10

11.23. Two cylinders are slip-fitted together to form a composite open cylinder. Both cylinders are made of a steel having a yield stress $Y = 700$ MPa. The inner cylinder has inner and outer diameters of 100 and 150+ mm, respectively. The outer cylinder has inner and outer diameters of 150 and 300 mm, respectively.

a. Determine the shrink pressure p_s and maximum internal pressure p_1 that can be applied to the cylinder if it has been designed with a factor of safety of $SF = 1.85$ for simultaneous initiation of yielding at the inner radii of the inner and outer cylinders. Use the maximum shear-stress criterion of failure.

b. Determine the outer diameter of the inner cylinder required for the design. For the steel $E = 200$ GPa and $\nu = 0.29$.

11.36. A solid disk of radius b is subjected to an angular velocity ω [rad/s]. The disk has mass density ρ, modulus of elasticity E, Poisson's ratio ν, and yield strength Y. Temperature effects are negligible.

a. Determine the angular velocity ω_Y at which the disk yields initially. Assume that $\sigma_{zz} = 0$ and that the maximum shear-stress criterion applies.

b. Determine the angular velocity ω_P at which the disk becomes fully plastic. Compare ω_P to ω_Y.

c. After the disk becomes fully plastic, it is returned to rest. Determine the resulting residual stresses in the disk.