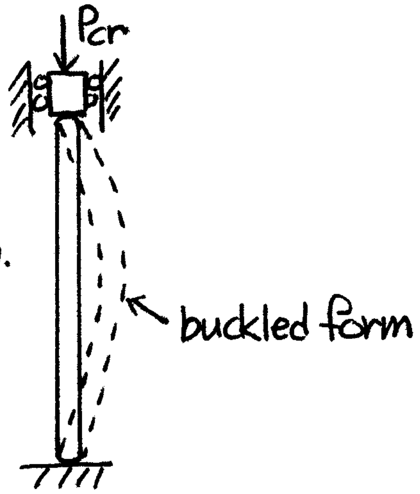
CH12. COLUMN BUCKLING

12.1. Introduction



- Straight column with pin joints fails with compressive load P_{cr} by elastic buckling.

- $P < P_c$, small lateral force, or eccentricity of P, provides stable elastic response.

- $P > P_c$, small lateral force causes large deflecton and eventually fracture.

12.2. Deflection of a Column

1. Ideal Slender Column

- pin jointed column

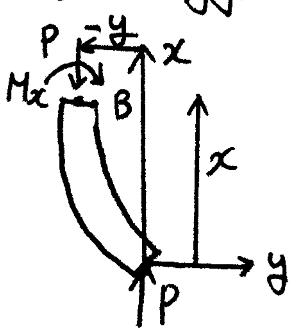
$$I = A r^2$$

radius of gyration

12.3. Euler Formula for Pin Joints

~ Equilibrium method, imperfection method, energy method.

1. Equilibrium Method



- Curvature $\kappa = \frac{1}{R} = \frac{M_x}{EI}$ R: radius of curvature

$M_x =$

$\Rightarrow M_x = EI\frac{d^2y}{dx^2} = -Py$

$\Rightarrow \boxed{\quad\quad\quad}$ where $\boxed{k^2 = \frac{P}{EI}}$

column buckling differential eq.

similar to vibration prob.

Boundary conditions

$y = 0$ at $x=0$, $x=L$.

- General Solution

$y(x) = A\sin kx + B\cos kx$

$y(0) = B = 0$

$y(L) = A\sin kL = 0$, $\sin kL = 0$., $kL = n\pi$

$k = \sqrt{\frac{P}{EI}} = \frac{n\pi}{L}$ $n = 1, 2, 3, \cdots$

$\therefore y(x) =$

Corresponding Euler load

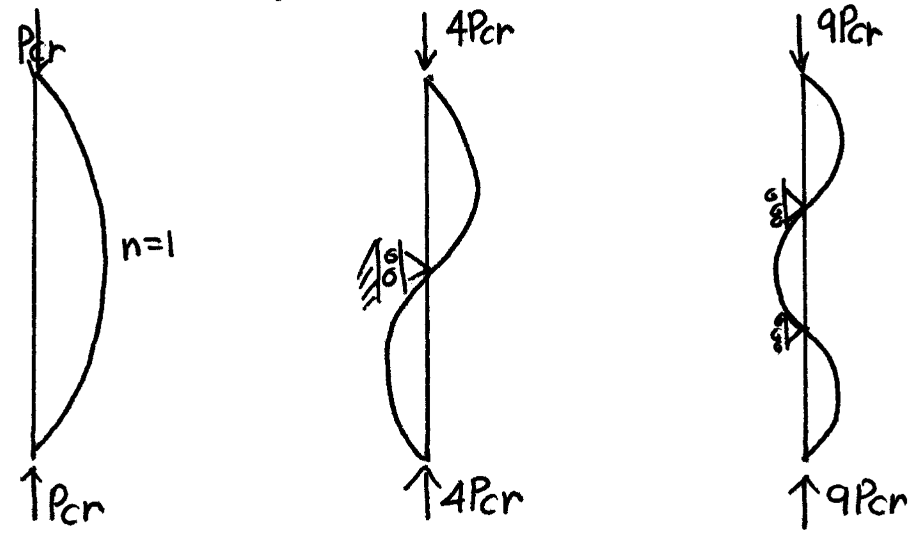
$P =$

$n=1 \Rightarrow$ minimum load $\Rightarrow$ critical load

$\boxed{\quad\quad\quad}$

A_1 cannot be determined. in this approach.

2. Higher Buckling Loads ; $n > 1$

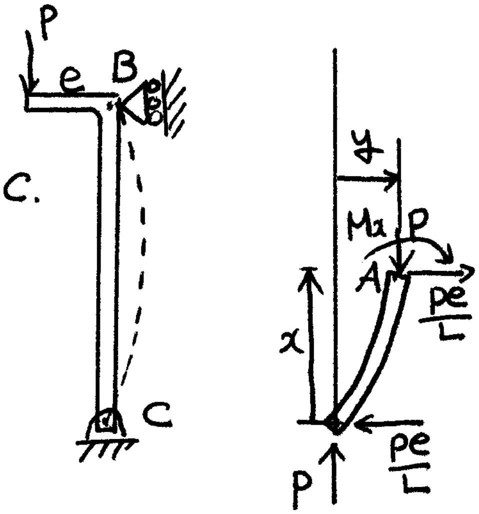


- when the column is prevented from Pcr load by fixing center location, then it buckles with $n=2$.

For any n :

$P =$

3. Imperfection Method



- From moment equilibrium at B there must be a horizontal force at C.

- From the moment equilibrium at A

$$\Sigma M_A = M_x + \frac{Pex}{L} + Py = 0$$

$\Rightarrow$

- Differential Eq.

$$\frac{d^2y}{dx^2} + \frac{P}{EI} y = -\frac{Pex}{EIL} \qquad \text{Let } k = \sqrt{\frac{P}{EI}}$$

$\Rightarrow$

Boundary condition

$y(x) = 0$ at $x = 0$, $x = L$

- General Solution

$$y(x) = A \sin kx + B \cos kx - \frac{ex}{L}$$

$$y(0) = B - \cancel{\frac{ex}{L}}_{0} = 0 \qquad B = 0$$

$$y(L) = A \sin kL + B \cos kL - e = 0$$

$$A = \frac{e}{\sin kL}$$

$\therefore y(x) =$

Displacement $y \to \infty$ when $kL = n\pi$. when $n = 1$

$P = P_{cr} = \frac{\pi^2 EI}{L^2}$.

4. Energy Method

- Energy conservation

work done by external force | Heat energy input | increase in strain energy | increase in kinetic energy

Application to Column Buckling

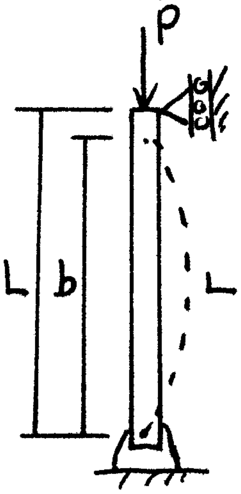
- infinitesimal motion : $\delta K \ll \delta U, \ \delta W,$
- adiabatic : $\delta H = 0$

$\Rightarrow$ $\boxed{}$

- It is assumed that the column may buckle when the load first reaches a value for which $\delta W = \delta U$
- Fourier series expansion

$\boxed{}$

: satisfy B.C. $y=0$ at $0, L$.



- Strain energy resulting from bending

$U_M =$

$$※ \ U_M = \int \frac{M^2}{2EI} dx \qquad M = EIy''.$$

$$y'' = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 a_n \sin\frac{n\pi x}{L}$$

$$\int_0^L \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

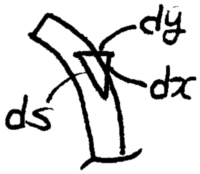
$$\Rightarrow U_M = \frac{1}{2} EI \quad \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^4 a_n^2 \cdot \frac{L}{2}$$

$U_M =$

$\boxed{}$

- Bar will buckle without changing its length L.

$$ds^2 = dx^2 + dy^2$$
$$= dx^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$



$$ds = \sqrt{1+(y')^2}\,dx$$

$$L = \int_0^b \sqrt{1+(y')^2}\,dx$$

binomial expansion
$$\sqrt{1+(y')^2} = 1 + \frac{1}{2}(y')^2 + \text{H.O.T.}$$

$$L \cong \int_0^b \left[1 + \frac{1}{2}(y')^2\right]dx$$
$$= b + \int_0^b \frac{1}{2}(y')^2\,dx$$

$$\therefore\ L - b \approx$$

- Work done by force P

$$\delta W = P(L-b) = \frac{P}{2}\int_0^L (y')^2\,dx.$$

$$\because\ y' = \sum_{n=1}^{\infty} \frac{n\pi}{L} a_n \cos\frac{n\pi x}{L}$$

$$\delta W = \sum_{n=1}^{\infty} \frac{P}{2}\cdot\left(\frac{n\pi}{L}\right)^2 a_n^2 \cdot \frac{L}{2}$$

$$\delta W = \delta U$$

$$\frac{\pi^2 P}{4L}\sum_{n=1}^{\infty} n^2 a_n^2 = \frac{\pi^4 EI}{4L^3}\sum_{n=1}^{\infty} n^4 a_n^2$$

- If $a_1 \neq 0$ & $a_2 = a_3 = \cdots = 0$

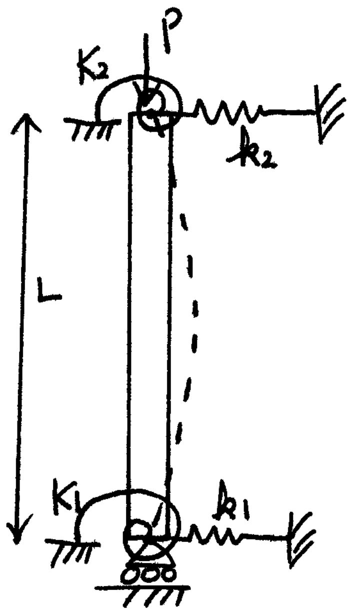
  $\Rightarrow$

- If $a_2 \neq 0$, $a_1 = a_3 = a_4 = \cdots = 0$

  $P =$: *second buckling load.*

12.4. Euler Buckling with Elastic Constraint

- Potential Energy



$V =$

$+$

$+$

- Stationary condition of V

$$\delta V = K_2 y_2' \delta y_2' + k_2 y_2 \delta y_2 + K_1 y_1' \delta y_1' + k_1 y_1 \delta y_1 + EI\int_0^L \underbrace{y''}_{u}\,\underbrace{\delta y''}_{v'}\, dx - P\int_0^L \underbrace{y'}_{u}\,\underbrace{\delta y'}_{v'}\, dx = 0.$$

- Integration by Parts

$$\Rightarrow (K_2 y_2' + EI y_2'')\delta y_2' + (K_1 y_1' - EI y_1'')\delta y_1' + (k_2 y_2 - P y_2')\delta y_2 + (k_1 y_1 + P y_1')\delta y_1 - EI\int_0^L \underbrace{y'''}_{u}\,\underbrace{\delta y'}_{v'}\, dx + P\int_0^L y''\, \delta y\, dx = 0$$

- Integration by Parts

$$(K_2 y_2' + EI y_2'')\delta y_2' + (K_1 y_1' - EI y_1'')\delta y_1' + (k_2 y_2 - P y_2' - EI y_2''')\delta y_2$$
$$+ (k_1 y_1 + P y_1' + EI y_1''')\delta y_1 + EI\int_0^L y''''\,\delta y\,dx + P\int_0^L y''\,\delta y\,dx = 0.$$

$\Rightarrow$

with

$1 \to x = 0$

$2 \to x = L$

Euler equation for column

- General solution

$y =$

$k = \frac{P}{EI}$.

For general case, substitute y into 4 B.C.s

$$\left[\ \underline{\underline{\Delta}}\ \right]\begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \underline{0}$$

$\Rightarrow |\Delta| = 0$ will lead to the critical load condition

- Specific B.C.s

pinned ends : $y_1 = 0$ $y_2 = 0$, $\delta y_1 = \delta y_2 = 0$

$K_1 = K_2 = 0$

B.C. $\Rightarrow \begin{cases} y_1'' = 0 \\ y_2'' = 0 \end{cases}$

$B = C = D = 0$. $A \sin kL = 0$ $kL = n\pi$ $P_{cr} = \frac{\pi^2 EI}{L^2}$

Example 12.1. Clamp-Free Column

Clamped one end ($x=0$) & free ($x=L$).

$y_1 = 0$, $y_1' = 0$, $\delta y_1 = \delta y_1' = 0$

$K_2 = k_2 = 0$ (free end)

B.C. (

$y_1 = 0$; $B + D = 0$, $y_1' = 0$; $kA + C = 0$

$y_2'' = 0$; $A \sin kL + B \cos kL = 0$,

$EI y_2''' + P y_2' = 0$; $y_2''' + k^2 y_2' = 0 \Rightarrow C = 0$

$\therefore A = C = 0$, $B = -D$, $\cos kL = 0$

$\therefore kL = \frac{(2n-1)\pi}{2}$ $n=1$ $kL = \frac{\pi}{2}$

$\therefore P =$

12.5. Local Buckling

~ Local buckling of flange or web before it fails as an Euler Beam.

~ $\frac{t}{b}$ large , buckle as an Euler column

$\frac{t}{b}$ small . local buckle . sheet buckling , plate buckling wrinkling

~ Local buckling may not cause immediate collapse.
It alters stress distribution , reduces compressive stiffness

Solve problems 12.8 12.12

12.8. In Figure P12.8, columns *AB* and *CD* have pinned ends, are made of an aluminum alloy ($E = 72.0$ GPa), and have equal rectangular cross sections of 20 mm by 30 mm. Determine the magnitude of P that will first cause one of the columns to buckle. Assume elastic conditions.

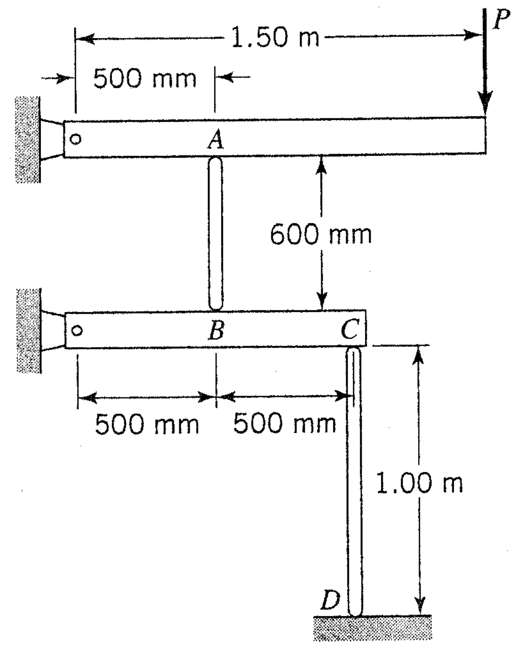


FIGURE P12.8

12.12. Determine the Euler load for the column shown in Figure 12.8*c*; see the discussion on the imperfection method in Section 12.3.