CHAP 3

FEA for Nonlinear Elastic Problems

Nam-Ho Kim

Introduction

- Linear systems
  - Infinitesimal deformation: no significant difference between the deformed and undeformed shapes
  - Stress and strain are defined in the undeformed shape
  - The weak form is integrated over the undeformed shape
- Large deformation problem
  - The difference between the deformed and undeformed shapes is large enough that they cannot be treated the same
  - The definitions of stress and strain should be modified from the assumption of small deformation
  - The relation between stress and strain becomes nonlinear as deformation increases
- This chapter will focus on how to calculate the residual and tangent stiffness for a nonlinear elasticity model

Introduction

- **Frame of Reference**
  - The weak form must be expressed based on a frame of reference
  - Often initial (undeformed) geometry or current (deformed) geometry are used for the frame of reference
  - proper definitions of stress and strain must be used according to the frame of reference
- **Total Lagrangian Formulation**: initial (undeformed) geometry as a reference
- **Updated Lagrangian Formulation**: current (deformed) geometry
- Two formulations are theoretically identical to express the structural equilibrium, but numerically different because different stress and strain definitions are used

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Stress and Strain Measures

Goals – Stress & Strain Measures

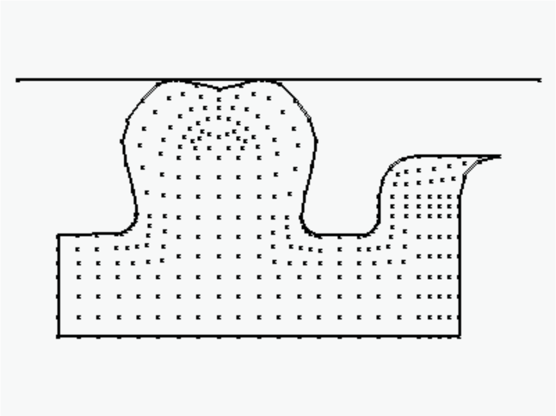
- Definition of a nonlinear elastic problem
- Understand the deformation gradient?
- What are Lagrangian and Eulerian strains?
- What is polar decomposition and how to do it?
- How to express the deformation of an area and volume
- What are Piola-Kirchhoff and Cauchy stresses?

Mild vs. Rough Nonlinearity

- **Mild** Nonlinear Problems (Chap 3)
  - Continuous, **history-independent** nonlinear relations between stress and strain
  - Nonlinear elasticity, Geometric nonlinearity, and deformation-dependent loads
- **Rough** Nonlinear Problems (Chap 4 & 5)
  - Equality and/or inequality constraints in constitutive relations
  - **History-dependent** nonlinear relations between stress and strain
  - Elastoplasticity and contact problems

What Is a Nonlinear Elastic Problem?

- **Elastic** (same for linear and nonlinear problems)
  - Stress-strain relation is elastic
  - Deformation disappears when the applied load is removed
  - Deformation is history-independent
  - Potential energy exists (function of deformation)
- **Nonlinear**
  - Stress-strain relation is nonlinear (**D** is not constant or do not exist)
  - Deformation is large



- Examples
  - Rubber material
  - Bending of a long slender member (small strain, large displacement)

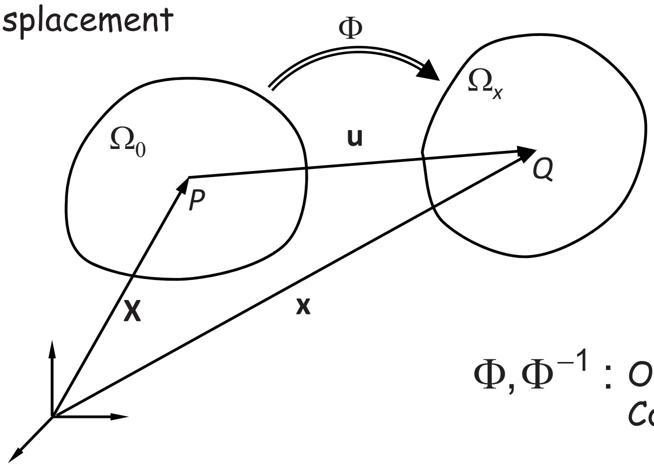
Reference Frame of Stress and Strain

- Force and displacement (vector) are independent of the configuration frame in which they are defined (**Reference Frame Indifference**)
- Stress and strain (tensor) depend on the configuration
- **Total Lagrangian or Material Stress/Strain**: when the reference frame is undeformed configuration
- **Updated Lagrangian or Spatial Stress/Strain**: when the reference frame is deformed configuration
- Question: What is the reference frame in linear problems?

Deformation and Mapping

- Initial domain Ω_0 is deformed to Ω_x
  - We can think of this as a **mapping** from Ω_0 to Ω_x
- $\mathbf{X}$: material point in Ω_0 $\qquad$ $\mathbf{x}$: material point in Ω_x
- Material point P in Ω_0 is deformed to Q in Ω_x

$$\mathbf{x} = \mathbf{X} + \underset{\substack{\uparrow \\ \text{displacement}}}{\mathbf{u}} \quad \Longrightarrow \quad \boxed{\mathbf{x} = \Phi(\mathbf{X},t) = \mathbf{X} + \mathbf{u}(\mathbf{X},t)}$$

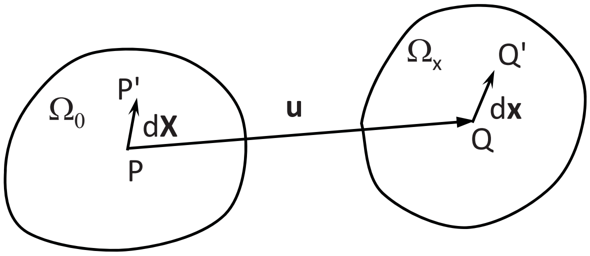


Φ, Φ^{-1} : One-to-one mapping
Continuously differentiable

Deformation Gradient

- Infinitesimal length d$\mathbf{X}$ in Ω_0 deforms to d$\mathbf{x}$ in Ω_x
- Remember that the mapping is **continuously differentiable**

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} d\mathbf{X} \quad \Rightarrow \quad d\mathbf{x} = \mathbf{F} d\mathbf{X}$$



- **Deformation gradient**:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \qquad \mathbf{F} = \mathbf{1} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{1} + \nabla_0 \mathbf{u}$$

$$\mathbf{1} = [\delta_{ij}],$$

$$\nabla_0 = \frac{\partial}{\partial \mathbf{X}}, \quad \nabla_x = \frac{\partial}{\partial \mathbf{x}}$$

  - gradient of mapping Φ
  - **Second-order tensor, Depend on both Ω_0 and Ω_x**
  - Due to one-to-one mapping: $\boxed{\det \mathbf{F} \equiv J > 0.}$

$$d\mathbf{X} = \mathbf{F}^{-1} d\mathbf{x}$$

  - **F includes both deformation and rigid-body rotation**

Example – Uniform Extension

- Uniform extension of a cube in all three directions

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3$$

- **Continuity requirement**: $\lambda_i > 0$ Why?
- Deformation gradient:

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = \lambda_3$: uniform expansion (**dilatation**) or contraction
- Volume change
  - Initial volume: $dV_0 = dX_1 dX_2 dX_3$
  - Deformed volume:

$$dV_x = dx_1 dx_2 dx_3 = \lambda_1 \lambda_2 \lambda_3 dX_1 dX_2 dX_3 = \lambda_1 \lambda_2 \lambda_3 dV_0$$

Green-Lagrange Strain

- Why different strains?
- Length change: $\|d\mathbf{x}\|^2 - \|d\mathbf{X}\|^2 = d\mathbf{x}^T d\mathbf{x} - d\mathbf{X}^T d\mathbf{X}$

$$= d\mathbf{X}^T\mathbf{F}^T\mathbf{F}d\mathbf{X} - d\mathbf{X}^T d\mathbf{X}$$

$$= d\mathbf{X}^T\underbrace{(\mathbf{F}^T\mathbf{F} - \mathbf{1})}_{\text{Ratio of length change}}d\mathbf{X}$$

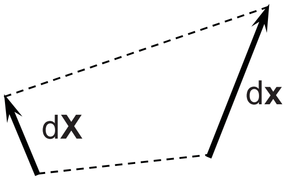
- **Right Cauchy-Green Deformation Tensor**

$$\boxed{\mathbf{C} = \mathbf{F}^T\mathbf{F}}$$

- **Green-Lagrange Strain Tensor**

$$\boxed{\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1})}$$

To match with infinitesimal strain



The effect of rotation is eliminated

Green-Lagrange Strain cont.

- Properties:
  - E is **symmetric**: $\mathbf{E}^T = \mathbf{E}$
  - No deformation: $\mathbf{F} = \mathbf{1}$, $\mathbf{E} = \mathbf{0}$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

$$\boxed{\begin{aligned}\mathbf{E} &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}^T}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}^T}{\partial \mathbf{X}}\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)\\ &= \tfrac{1}{2}\left(\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T + \nabla_0\mathbf{u}^T\nabla_0\mathbf{u}\right)\end{aligned}}$$

Displacement gradient

Higher-order term

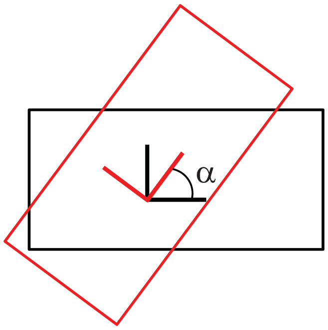
  - When $|\nabla_0\mathbf{u}| << 1$, $\mathbf{E} \approx \frac{1}{2}\left(\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T\right) = \varepsilon$

  - **E = 0 for a rigid-body motion**, but $\varepsilon \neq 0$

Example – Rigid-Body Rotation

- Rigid-body rotation

$$x_1 = X_1 \cos\alpha - X_2 \sin\alpha$$
$$x_2 = X_1 \sin\alpha + X_2 \cos\alpha$$
$$x_3 = X_3$$



- Approach 1: using deformation gradient

$$\mathbf{F} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \mathbf{0}$$

Green-Lagrange strain removes rigid-body rotation from deformation

Example – Rigid-Body Rotation cont.

- Approach 2: using displacement gradient

$$u_1 = x_1 - X_1 = X_1(\cos\alpha - 1) - X_2 \sin\alpha$$
$$u_2 = x_2 - X_2 = X_1 \sin\alpha + X_2(\cos\alpha - 1)$$
$$u_3 = x_3 - X_3 = 0$$

$$\nabla_0 \mathbf{u} = \begin{bmatrix} \cos\alpha - 1 & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\nabla_0 \mathbf{u}^T \nabla_0 \mathbf{u} = \begin{bmatrix} 2(1-\cos\alpha) & 0 & 0 \\ 0 & 2(1-\cos\alpha) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

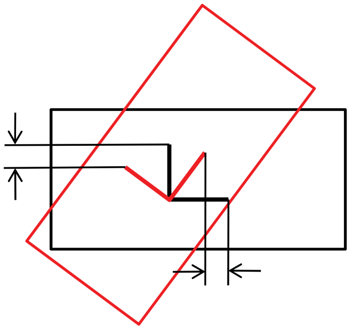
$$\mathbf{E} = \tfrac{1}{2}(\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T + \nabla_0\mathbf{u}^T\nabla_0\mathbf{u}) = \mathbf{0}$$

Example – Rigid-Body Rotation cont.

- What happens to engineering strain?

$$u_1 = x_1 - X_1 = X_1(\cos\alpha - 1) - X_2 \sin\alpha$$
$$u_2 = x_2 - X_2 = X_1 \sin\alpha + X_2(\cos\alpha - 1)$$
$$u_3 = x_3 - X_3 = 0$$

$$\varepsilon = \begin{bmatrix} \cos\alpha - 1 & 0 & 0 \\ 0 & \cos\alpha - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Engineering strain is unable to take care of rigid-body rotation

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Eulerian (Almansi) Strain Tensor

- Length change: $\|d\mathbf{x}\|^2 - \|d\mathbf{X}\|^2 = d\mathbf{x}^T d\mathbf{x} - d\mathbf{X}^T d\mathbf{X}$

$$= d\mathbf{x}^T d\mathbf{x} - d\mathbf{x}^T \mathbf{F}^{-T}\mathbf{F}^{-1} d\mathbf{x}$$
$$= d\mathbf{x}^T (\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}) d\mathbf{x}$$
$$= d\mathbf{x}^T (\mathbf{1} - \mathbf{b}^{-1}) d\mathbf{x}$$

- **Left Cauchy-Green Deformation Tensor**

$$\boxed{\mathbf{b} = \mathbf{F}\mathbf{F}^T}$$

$\mathbf{b}^{-1}$: Finger tensor

- **Eulerian (Almansi) Strain Tensor**

$$\boxed{\mathbf{e} = \frac{1}{2}(\mathbf{1} - \mathbf{b}^{-1})}$$

Reference is deformed (current) configuration

Eulerian Strain Tensor cont.

- Properties
  - Symmetric
  - Approach engineering strain when $\frac{\partial \mathbf{u}}{\partial \mathbf{x}} << 1$
  - In terms of displacement gradient

$$\mathbf{e} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}^T}{\partial \mathbf{x}} - \frac{\partial \mathbf{u}^T}{\partial \mathbf{x}}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)$$

$$= \frac{1}{2}\left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T - \nabla_x \mathbf{u}^T \nabla_x \mathbf{u} \right)$$

$$\nabla_x = \frac{\partial}{\partial \mathbf{x}}$$

Spatial gradient

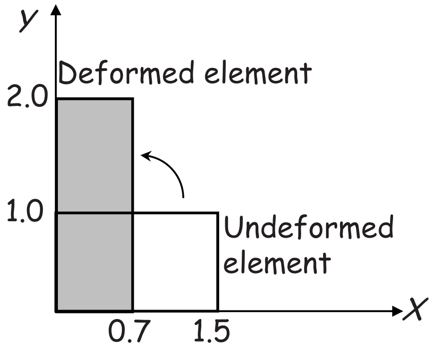
- Relation between **E** and **e**

$$\mathbf{E} = \mathbf{F}^T \mathbf{e} \mathbf{F}$$

Example – Lagrangian Strain

- Calculate **F** and **E** for deformation in the figure
- Mapping relation in Ω_0

$$\begin{cases} X = \sum_{I=1}^{4} N_I(s,t) X_I = \frac{3}{4}(s+1) \\ Y = \sum_{I=1}^{4} N_I(s,t) Y_I = \frac{1}{2}(t+1) \end{cases}$$



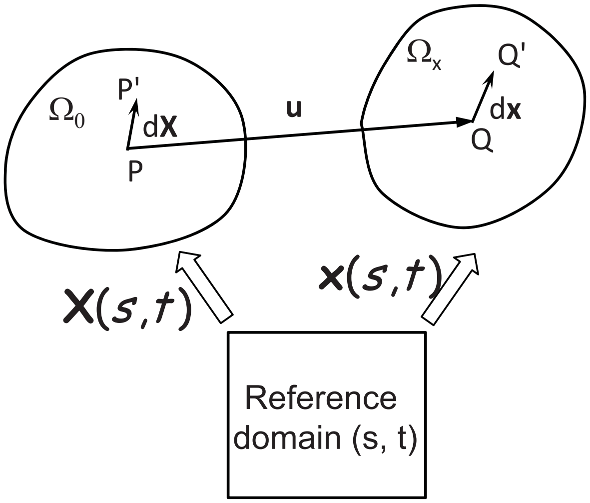
- Mapping relation in Ω_x

$$\begin{cases} x(s,t) = \sum_{I=1}^{4} N_I(s,t) x_I = 0.35(1-t) \\ y(s,t) = \sum_{I=1}^{4} N_I(s,t) y_I = s+1 \end{cases}$$

Example – Lagrangian Strain cont.

- Deformation gradient

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{s}}\frac{\partial \mathbf{s}}{\partial \mathbf{X}}$$
$$= \begin{bmatrix} 0 & -.35 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4/3 & 0 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -0.7 \\ 4/3 & 0 \end{bmatrix}$$



- Green-Lagrange Strain

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \begin{bmatrix} 0.389 & 0 \\ 0 & -0.255 \end{bmatrix}$$

Tension in X_1 dir.
Compression in X_2 dir.

Example – Lagrangian Strain cont.

- Almansi Strain

$$\mathbf{b} = \mathbf{F}\cdot\mathbf{F}^T = \begin{bmatrix} 0.49 & 0 \\ 0 & 1.78 \end{bmatrix}$$

$$\mathbf{e} = \tfrac{1}{2}\left(\mathbf{1} - \mathbf{b}^{-1}\right) = \begin{bmatrix} -0.52 & 0 \\ 0 & 0.22 \end{bmatrix}$$

Compression in x_1 dir.
Tension in x_2 dir.

- Engineering Strain

$$\nabla_0\mathbf{u} = \mathbf{F} - \mathbf{1} = \begin{bmatrix} -1 & -0.7 \\ 1.33 & -1 \end{bmatrix}$$

$$\varepsilon = \tfrac{1}{2}\left(\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T\right) = \begin{bmatrix} -1 & 0.32 \\ 0.32 & -1 \end{bmatrix}$$

Artificial shear deform.
Inconsistent normal deform.

Which strain is consistent with actual deformation?

Example – Uniaxial Tension

- Uniaxial tension of incompressible material ($\lambda_1 = \lambda > 1$)
- From incompressibility

$$\lambda_1\lambda_2\lambda_3 = 1 \quad \Rightarrow \quad \lambda_2 = \lambda_3 = \lambda^{-1/2}$$

$$x_1 = \lambda_1 X_1$$
$$x_2 = \lambda_2 X_2$$
$$x_3 = \lambda_3 X_3$$

- Deformation gradient and deformation tensor

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

- G-L Strain

$$\mathbf{E} = \frac{1}{2}\begin{bmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & \lambda^{-1} - 1 & 0 \\ 0 & 0 & \lambda^{-1} - 1 \end{bmatrix}$$

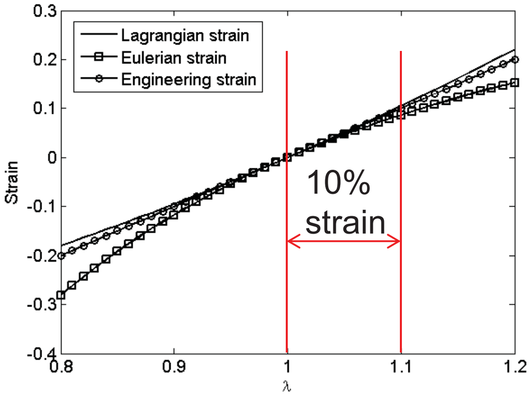
Example – Uniaxial Tension

- Almansi Strain (**b** = **C**)

$$\mathbf{b}^{-1} = \begin{bmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad \mathbf{e} = \frac{1}{2}\begin{bmatrix} 1 - \lambda^{-2} & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{bmatrix}$$

- Engineering Strain

$$\varepsilon = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda^{-1/2} - 1 & 0 \\ 0 & 0 & \lambda^{-1/2} - 1 \end{bmatrix}$$



- Difference

$$E_{11} = \frac{1}{2}(\lambda^2 - 1) \quad e_{11} = \frac{1}{2}(1 - \lambda^{-2}) \quad \varepsilon_{11} = \lambda - 1$$

Polar Decomposition

- Want to separate deformation from rigid-body rotation
- Similar to principal directions of strain
- Unique decomposition of deformation gradient

$$\boxed{\mathbf{F} = \mathbf{QU} = \mathbf{VQ}}$$

  - **Q**: **orthogonal tensor** (rigid-body rotation)
  - **U**, **V**: **right- and left-stretch tensor** (symmetric)
- **U** and **V** have the same eigenvalues (**principal stretches**), but different eigenvectors

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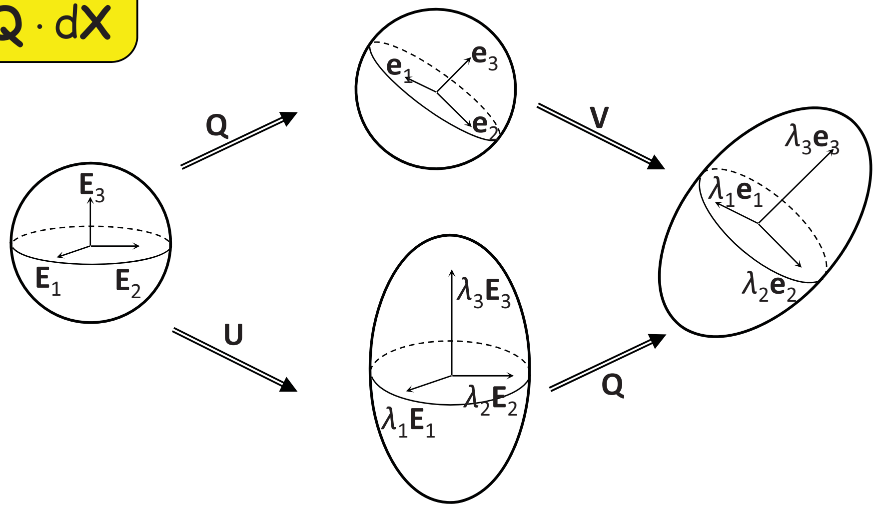
Polar Decomposition cont.

$$\boxed{\begin{aligned} d\mathbf{x} &= \mathbf{Q}\cdot\mathbf{U}\cdot d\mathbf{X} \\ &= \mathbf{V}\cdot\mathbf{Q}\cdot d\mathbf{X} \end{aligned}}$$

$$\mathbf{F} = \mathbf{VQ}$$



$$\mathbf{F} = \mathbf{QU}$$

- Eigenvectors of **U**: $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$
- Eigenvectors of **V**: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
- Eigenvalues of **U** and **V**: $\lambda_1, \lambda_2, \lambda_3$

Polar Decomposition cont.

- Relation between $\mathbf{U}$ and $\boldsymbol{C}$

$$\mathbf{U}^2 = \boldsymbol{C} \quad \mathbf{U} = \sqrt{\boldsymbol{C}}$$

  - $\mathbf{U}$ and $\boldsymbol{C}$ have the same eigenvectors.
  - Eigenvalue of $\mathbf{U}$ is the square root of that of $\boldsymbol{C}$
- How to calculate $\mathbf{U}$ from $\boldsymbol{C}$?
- Let eigenvectors of $\boldsymbol{C}$ be $\Phi = [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3]$
- Then, $\Lambda = \Phi^T \boldsymbol{C} \Phi$ where

$$\Lambda = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

Deformation tensor in principal directions

Polar Decomposition cont.

- And $\mathbf{U} = \Phi\sqrt{\Lambda}\,\Phi^T$

$$\sqrt{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- General Deformation

$$\mathbf{dx} = \mathbf{FdX} + \mathbf{b} = \mathbf{QUdX} + \mathbf{b}$$

1. Stretch in principal directions
2. Rigid-body rotation
3. Rigid-body translation

Useful formulas

$$\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{E}_i \otimes \mathbf{E}_i$$

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{E}_i \otimes \mathbf{E}_i$$

$$\mathbf{Q} = \sum_{i=1}^{3} \mathbf{e}_i \otimes \mathbf{E}_i$$

$$\mathbf{b} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{e}_i \otimes \mathbf{e}_i$$

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i$$

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{E}_i$$

Generalized Lagrangian Strain

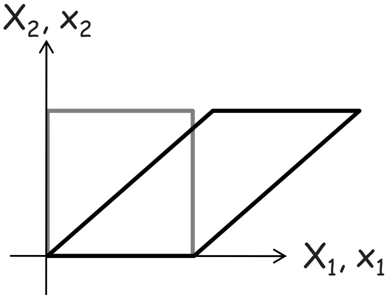
- G-L strain is a special case of general form of Lagrangian strain tensors (Hill, 1968)

$$\mathbf{E}_m = \frac{1}{2m}\left(\mathbf{U}^{2m} - \mathbf{1}\right)$$

Example – Polar Decomposition

- Simple shear problem

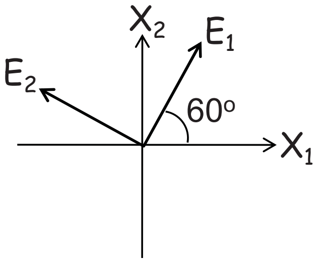
$$\begin{cases} x_1 = X_1 + kX_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases} \qquad k = \frac{2}{\sqrt{3}}$$



- Deformation gradient $\mathbf{F} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
- Deformation tensor $\boldsymbol{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & k \\ k & k^2+1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{7}{3} \end{bmatrix}$
- Find eigenvalues and eigenvectors of $\boldsymbol{C}$

$$\lambda_1 = 3, \quad \lambda_2 = 1/3$$

$$\mathbf{E}_1 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$



Example – Polar Decomposition cont.

- In $E_1 - E_2$ coordinates $\boldsymbol{C}' = \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}$
- Principal Direction Matrix $\Phi = [\mathbf{E}_1 \quad \mathbf{E}_2] = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$
- Deformation tensor in principal directions

$$\Lambda = \Phi^T \cdot \boldsymbol{C} \cdot \Phi$$

- Stretch tensor

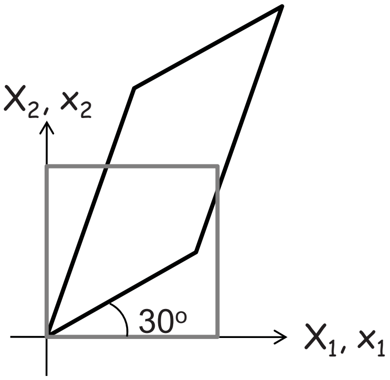
$$\sqrt{\Lambda} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix}$$

$$\mathbf{U} = \Phi \cdot \sqrt{\Lambda} \cdot \Phi^T = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & 5/2\sqrt{3} \end{bmatrix}$$

Example – Polar Decomposition cont.

- How **U** deforms a square?

$$\mathbf{U} \cdot \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sqrt{3}/2 \\ 1/2 \end{Bmatrix}, \quad \mathbf{U} \cdot \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1/2 \\ 5/2\sqrt{3} \end{Bmatrix}$$

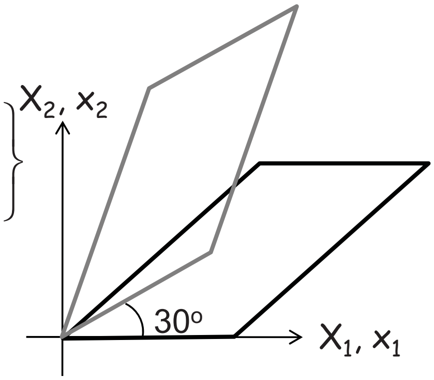


- Rotational Tensor

$$\mathbf{Q} = \mathbf{F} \cdot \mathbf{U}^{-1} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\mathbf{Q} \cdot \begin{Bmatrix} \sqrt{3}/2 \\ 1/2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \mathbf{Q} \cdot \begin{Bmatrix} 1/2 \\ 5/2\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} 1.15 \\ 1 \end{Bmatrix}$$

  - 30° clockwise rotation

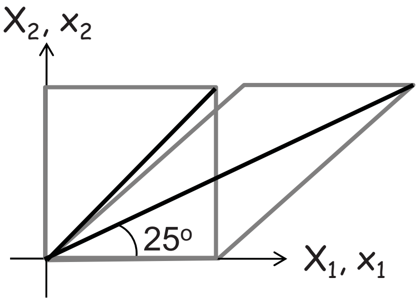


$$\mathbf{V} = \mathbf{F} \cdot \mathbf{Q}^T = \begin{bmatrix} 5\sqrt{3}/6 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

Example – Polar Decomposition cont.

- A straight line $X_2 = X_1 \tan\theta$ will deform to

$$X_1 = x_1 - kx_2, \qquad X_2 = x_2$$
$$\Rightarrow x_2 = (x_1 - kx_2)\tan\theta$$
$$\Rightarrow x_1 = \left(\tfrac{1}{\tan\theta} + k\right)x_2$$

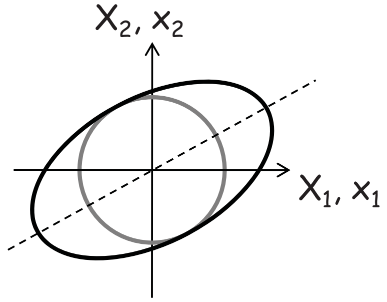


- Consider a diagonal line: θ = 45°

$$\tan\alpha = \frac{x_2}{x_1} = \frac{1}{1+k} \qquad \alpha = 24.9^\circ$$

- Consider a circle

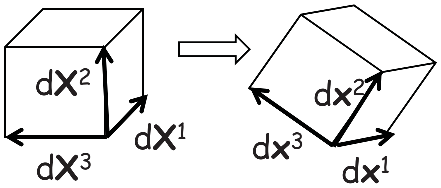
$$X_1^2 + X_2^2 = r^2$$
$$(x_1 - kx_2)^2 + x_2^2 = r^2$$
$$x_1^2 - 2kx_1x_2 + (1+k^2)x_2^2 = r^2$$



Equation of ellipse

Deformation of a Volume

- Infinitesimal volume by three vectors
  - Undeformed: $dV_0 = d\mathbf{X}^1 \cdot (d\mathbf{X}^2 \times d\mathbf{X}^3) = e_{rst}dX_r^1 dX_s^2 dX_t^3$
  - Deformed: $dV_x = d\mathbf{x}^1 \cdot (d\mathbf{x}^2 \times d\mathbf{x}^3) = e_{ijk}dx_i^1 dx_j^2 dx_k^3$



$$\begin{aligned} dV_x &= e_{ijk}dx_i^1 dx_j^2 dx_k^3 \\ &= e_{ijk}\left(\frac{\partial x_i}{\partial X_r}dX_r^1\right)\left(\frac{\partial x_j}{\partial X_s}dX_s^2\right)\left(\frac{\partial x_k}{\partial X_t}dX_t^3\right) \\ &= e_{ijk}\frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s}\frac{\partial x_k}{\partial X_t}dX_r^1 dX_s^2 dX_t^3 \\ &= \qquad e_{rst}J \qquad dX_r^1 dX_s^2 dX_t^3 \\ &= J\,dV_0 \end{aligned}$$

From Continuum Mechanics

$$J = \det\mathbf{F} = \lambda_1\lambda_2\lambda_3$$

$$e_{ijk}a_{ir}a_{js}a_{kt} = e_{rst}\det\mathbf{a}$$

Deformation of a Volume cont.

- Volume change

$$\boxed{dV_x = J\,dV_0}$$

- Volumetric Strain

$$\boxed{\frac{dV_x - dV_0}{dV_0} = J - 1}$$

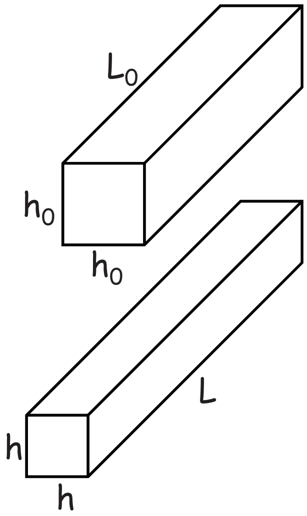
- Incompressible condition: J = 1
- Transformation of integral domain

$$\iiint_{\Omega_x} f\,d\Omega = \iiint_{\Omega_0} f J\,d\Omega$$

Example - Uniaxial Deformation of a Beam

- Initial dimension of $L_0 \times h_0 \times h_0$ deforms to $L \times h \times h$

$$\begin{aligned} x_1 &= \lambda_1 X_1 & \qquad \lambda_1 &= L / L_0 \\ x_2 &= \lambda_2 X_2 & \lambda_2 &= h / h_0 \\ x_3 &= \lambda_3 X_3 & \lambda_3 &= h / h_0 \end{aligned}$$



- Deformation gradient

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \begin{aligned} J &= \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3 \\ &= \frac{L}{L_0}\left(\frac{h}{h_0}\right)^2 = \frac{LA}{L_0 A_0} \end{aligned}$$

- Constant volume

$$J = 1 \quad \Rightarrow \quad h = h_0 \sqrt{\frac{L_0}{L}} \quad A = A_0 \frac{L_0}{L}$$

Deformation of an Area

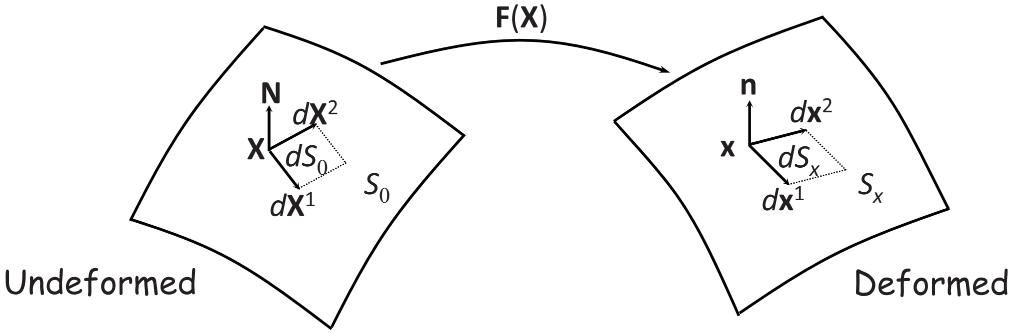
- Relationship between dS_0 and dS_x

$$\mathbf{N}dS_0 = d\mathbf{X}^1 \times d\mathbf{X}^2 \qquad N_i dS_0 = e_{ijk} dX_j^1 dX_k^2$$

$$\mathbf{n}dS_x = d\mathbf{x}^1 \times d\mathbf{x}^2 \qquad n_r dS_x = e_{rst} dx_s^1 dx_t^2$$

$$N_i dS_0 = e_{ijk} \frac{\partial X_j}{\partial x_s} \frac{\partial X_k}{\partial x_t} dx_s^1 dx_t^2$$

$$\times \frac{\partial X_i}{\partial x_r} \quad \Longrightarrow \quad \frac{\partial X_i}{\partial x_r} N_i dS_0 = e_{ijk} \frac{\partial X_i}{\partial x_r} \frac{\partial X_j}{\partial x_s} \frac{\partial X_k}{\partial x_t} dx_s^1 dx_t^2$$



Deformation of an Area cont..

- Results from Continuum Mechanics

$$e_{ijk} \left|\mathbf{F}\right| = e_{rst} \frac{\partial x_r}{\partial X_i} \frac{\partial x_s}{\partial X_j} \frac{\partial x_t}{\partial X_k}$$

$$e_{rst} \left|\mathbf{F}^{-1}\right| = e_{ijk} \frac{\partial X_i}{\partial x_r} \frac{\partial X_j}{\partial x_s} \frac{\partial X_k}{\partial x_t}.$$

- Use the second relation:

$$\frac{\partial X_i}{\partial x_r} N_i dS_0 = e_{ijk} \frac{\partial X_i}{\partial x_r} \frac{\partial X_j}{\partial x_s} \frac{\partial X_k}{\partial x_t} dx_s^1 dx_t^2 = \overbrace{e_{rst} \left|\mathbf{F}\right|^{-1} dx_s^1 dx_t^2}^{n_r dS_x}$$

$$\mathbf{n}dS_x = J\mathbf{F}^{-T} \cdot \mathbf{N}dS_0 \qquad \mathbf{n} \parallel \mathbf{F}^{-T} \cdot \mathbf{N} \quad \Rightarrow \quad \mathbf{n} = \frac{\mathbf{F}^{-T} \cdot \mathbf{N}}{\left\|\mathbf{F}^{-T} \cdot \mathbf{N}\right\|}$$

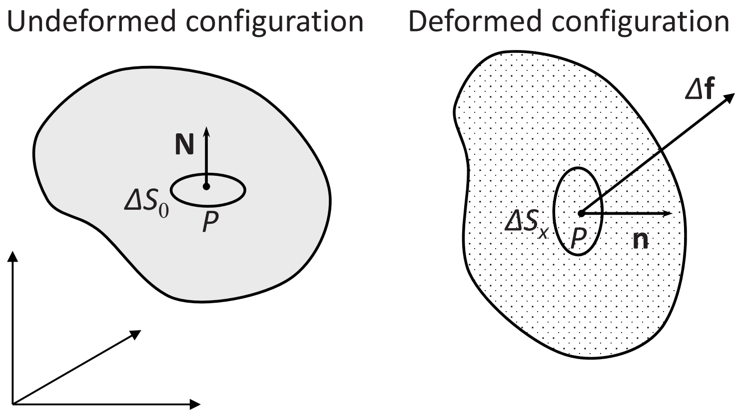
$$dS_x = J \left\|\mathbf{F}(\mathbf{x})^{-T} \mathbf{N}(\mathbf{X})\right\| dS_0$$

Stress Measures

- Stress and strain (tensor) depend on the configuration
- Cauchy (True) Stress: Force acts on the deformed config.
  - Stress vector at Ω_x: $\boxed{\mathbf{t} = \lim_{\Delta S_x \to 0} \frac{\Delta \mathbf{f}}{\Delta S_x} = \sigma \mathbf{n}}$

Cauchy Stress, sym

  - Cauchy stress refers to the current deformed configuration as a reference for both area and force (**true stress**)


Undeformed configuration

Deformed configuration

Stress Measures cont.

- The same force, but different area (undeformed area)

$$\boxed{\mathbf{T} = \lim_{\Delta S_0 \to 0} \frac{\Delta \mathbf{f}}{\Delta S_0} = \mathbf{P}^T \mathbf{N}}$$

First Piola-Kirchhoff Stress
Not symmetric

  - **P** refers to the force in the deformed configuration and the area in the undeformed configuration
- Make both force and area to refer to undeformed config.

$$d\mathbf{f} = \sigma \mathbf{n} dS_x = \mathbf{P}^T \mathbf{N} dS_0 \quad \Longleftarrow \quad \mathbf{n} dS_x = J\mathbf{F}^{-T} \cdot \mathbf{N} dS_0$$

$$d\mathbf{f} = \sigma(J\mathbf{F}^{-T}\mathbf{N} dS_0) = \mathbf{P}^T \mathbf{N} dS_0$$

$$\boxed{\mathbf{P} = J\mathbf{F}^{-1}\sigma}$$: Relation between σ and **P**

Stress Measures cont.

- Unsymmetric property of **P** makes it difficult to use
  - Remember we used the symmetric property of stress & strain several times in linear problems
- Make **P** symmetric by multiplying with $\mathbf{F}^{-T}$

$$\boxed{\mathbf{S} = \mathbf{P} \cdot \mathbf{F}^{-T} = J\mathbf{F}^{-1} \cdot \sigma \cdot \mathbf{F}^{-T}} \qquad \sigma = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T}$$

Second Piola-Kirchhoff Stress, symmetric

  - Just convenient mathematical quantities
- Further simplification is possible by handling J differently

$$\boxed{\tau = J\sigma = \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T}}$$

Kirchhoff Stress, symmetric

Stress Measures cont.

- Example

$$\iiint_{\Omega_x} \sigma : \bar{\varepsilon}\, d\Omega_x = \iiint_{\Omega_0} \sigma : \bar{\varepsilon} J\, d\Omega_0 = \iiint_{\Omega_0} \tau : \bar{\varepsilon}\, d\Omega_0$$

Integration can be done in Ω_0

- Observation
  - For linear problems (small deformation): $\varepsilon \approx \mathbf{E} \approx \mathbf{e}$
  - For linear problems (small deformation): $\sigma \approx \tau \approx \mathbf{P} \approx \mathbf{S}$
  - **S** and **E** are conjugate in energy
  - **S** and **E** are invariant in rigid-body motion

Example – Uniaxial Tension

- Cauchy (true) stress: $\sigma_{11} = \frac{F}{A}$, $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{13} = 0$
- Deformation gradient:

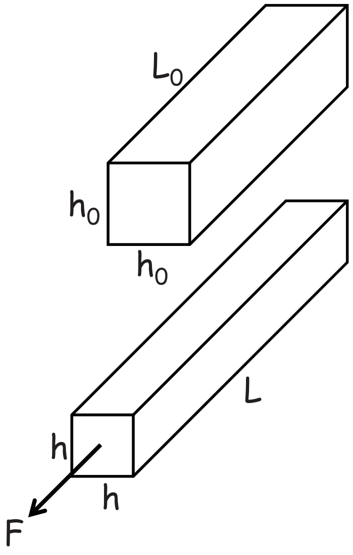
$$\mathbf{F}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{bmatrix}, \quad J = 1$$

- First P-K stress

$$P_{11} = (J\mathbf{F}^{-1}\sigma)_{11} = \frac{F}{A}\frac{1}{\lambda_1} = \frac{F}{A}\frac{A}{A_0} = \frac{F}{A_0}$$

- Second P-K stress

$$S_{11} = (J\mathbf{F}^{-1}\cdot\sigma\cdot\mathbf{F}^{-T})_{11} = \frac{F}{A}\frac{1}{\lambda_1^2} = \frac{F}{A}\frac{A^2}{A_0^2} = \frac{FA}{A_0^2} = \frac{F}{A_0\lambda_1}$$



No clear physical meaning

Summary

- Nonlinear elastic problems use different measures of stress and strain due to changes in the reference frame
- Lagrangian strain is independent of rigid-body rotation, but engineering strain is not
- Any deformation can be uniquely decomposed into rigid-body rotation and stretch
- The determinant of deformation gradient is related to the volume change, while the deformation gradient and surface normal are related to the area change
- Four different stress measures are defined based on the reference frame.
- All stress and strain measures are identical when the deformation is infinitesimal

3.3

Nonlinear Elastic Analysis

Goals

- Understanding the principle of minimum potential energy
  - Understand the concept of variation
- Understanding St. Venant-Kirchhoff material
- How to obtain the governing equation for nonlinear elastic problem
- What is the total Lagrangian formulation?
- What is the updated Lagrangian formulation?
- Understanding the linearization process

Numerical Methods for Nonlinear Elastic Problem

- We will obtain the variational equation using the **principle of minimum potential energy**
  - Only possible for elastic materials (potential exists)
- The N-R method will be used (need Jacobian matrix)
- **Total Lagrangian (material) formulation** uses the undeformed configuration as a reference, while the **updated Lagrangian (spatial)** uses the current configuration as a reference
- The total and updated Lagrangian formulations are **mathematically equivalent** but have different aspects in computation

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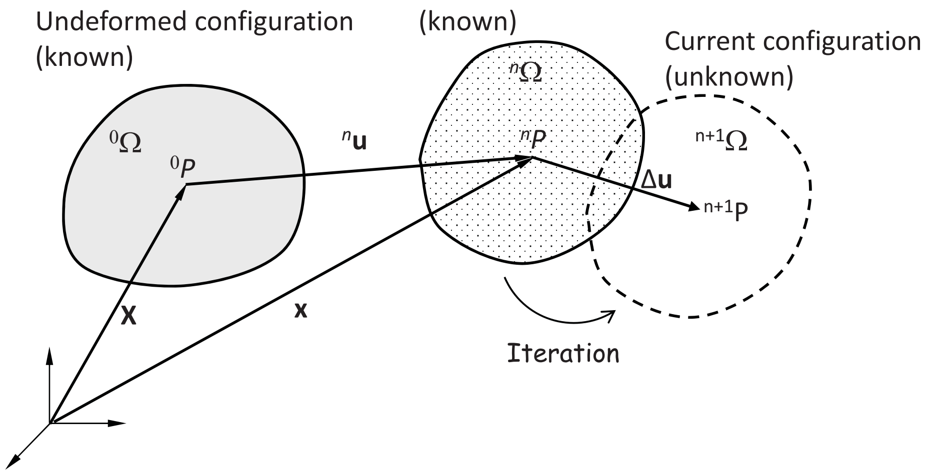
Total Lagrangian Formulation

- Using **incremental force method** and **N-R method**
  - Total No. of load steps (N), current load step (n)

$$^{n+1}\mathbf{f} = {}^{n}\mathbf{f} + \Delta\mathbf{f}^{n}$$

- Assume that the solution has converged up to t_n
- Want to find the equilibrium state at t_{n+1}



Total Lagrangian Formulation cont.

- In TL, the **undeformed configuration** is the reference
- 2^{nd} P-K stress (**S**) and G-L strain (**E**) are the natural choice
- In elastic material, **strain energy density W** exists, such that

$$\text{stress} = \frac{\partial W}{\partial \text{strain}}$$

- We need to express W in terms of **E**

Strain Energy Density and Stress Measures

- By differentiating strain energy density with respect to proper strains, we can obtain stresses
- When W(**E**) is given

$$\mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}}$$

Second P-K stress

- When W(**F**) is given

$$\frac{\partial W}{\partial \mathbf{F}} = \frac{\partial W}{\partial \mathbf{E}} : \frac{\partial \mathbf{E}}{\partial \mathbf{F}} = \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{E}} = \mathbf{F} \cdot \mathbf{S} = \mathbf{P}^T$$

First P-K stress

- It is difficult to have W(ε) because ε depends on rigid-body rotation. Instead, we will use **invariants** in Section 3.5

St. Venant-Kirchhoff Material

- Strain energy density for St. Venant-Kirchhoff material

$$\boxed{W(\mathbf{E}) = \tfrac{1}{2}\mathbf{E} : \mathbf{D} : \mathbf{E}}$$

Contraction operator: $\mathbf{a} : \mathbf{b} = a_{ij}b_{ij}$

- Fourth-order constitutive tensor (isotropic material)

$$\boxed{\mathbf{D} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}}$$

  - Lame's constants: $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$ $\quad \mu = \dfrac{E}{2(1+\nu)}$
  - Identity tensor (2nd order): $\mathbf{1} = [\delta_{ij}]$
  - Identity tensor (4th order): $\mathbf{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$$\mathbf{I} : \mathbf{a} = \mathbf{a}, \qquad \forall \text{2nd-order sym. } \mathbf{a}$$

$$\mathbf{1} : \mathbf{a} = \mathrm{tr}(\mathbf{a}) = a_{ii} = a_{11} + a_{22} + a_{33}$$

  - Tensor product: $\mathbf{a} \otimes \mathbf{a} = a_{ij}a_{kl}$ (4th-order)

St. Venant-Kirchhoff Material cont.

- Stress calculation
  - differentiate strain energy density

$$\boxed{\mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}} = \mathbf{D} : \mathbf{E} = \lambda \mathrm{tr}(\mathbf{E})\mathbf{1} + 2\mu \mathbf{E}}$$

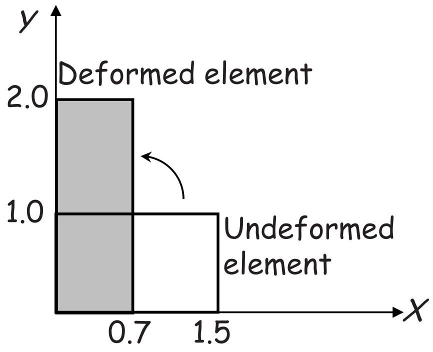
  - Limited to small strain but large rotation

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \tfrac{1}{2}(\mathbf{U}^T\mathbf{Q}^T\mathbf{Qu} - \mathbf{1}) = \tfrac{1}{2}(\mathbf{U}^2 - \mathbf{1})$$

  - Rigid-body rotation is removed and only the stretch tensor contributes to the strain
  - Can show $\mathbf{S} = \dfrac{\partial W}{\partial \mathbf{E}} = 2\dfrac{\partial W}{\partial \mathbf{C}}$

Deformation tensor

Example



- E = 30,000 and ν = 0.3
- G-L strain: $$\mathbf{E} = \begin{bmatrix} 0.389 & 0 \\ 0 & -0.255 \end{bmatrix}$$
- Lame's constants:

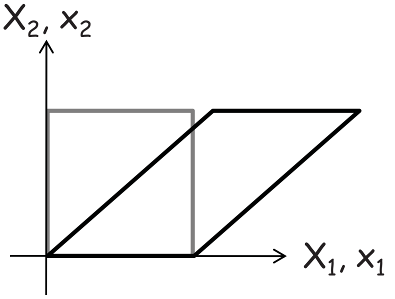
$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = 17{,}308 \qquad \mu = \frac{E}{2(1+\nu)} = 11{,}538$$

- 2^{nd} P-K Stress:

$$\mathbf{S} = \lambda \mathrm{tr}(\mathbf{E})\mathbf{1} + 2\mu\mathbf{E} = \lambda(.389 - .255)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\mu\begin{bmatrix} .389 & 0 \\ 0 & -.255 \end{bmatrix}$$
$$= \begin{bmatrix} 11{,}296 & 0 \\ 0 & -3{,}565 \end{bmatrix}$$

$$\sigma = \frac{1}{J}\mathbf{FSF}^{T} = \begin{bmatrix} -1{,}872 & 0 \\ 0 & 21{,}516 \end{bmatrix}$$

Example – Simple Shear Problem



- Deformation map

$$x_1 = X_1 + kX_2, \ \ x_2 = X_2, \ \ x_3 = X_3$$

$$\mathbf{F} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \qquad \mathbf{E} = \frac{1}{2}(\mathbf{F}^{T}\mathbf{F} - \mathbf{1}) = \frac{1}{2}\begin{bmatrix} 0 & k \\ k & k^2 \end{bmatrix}$$

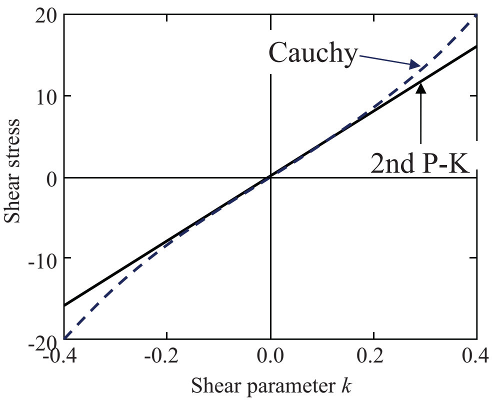
- Material properties

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = 40\text{MPa} \qquad \mu = \frac{E}{2(1+\nu)} = 40\text{MPa}$$

- 2nd P-K stress

$$\mathbf{S} = \lambda \mathrm{tr}(\mathbf{E})\mathbf{1} + 2\mu\mathbf{E} = 20\begin{bmatrix} k^2 & 2k \\ 2k & 3k^2 \end{bmatrix}\text{MPa}$$

$$\sigma = \frac{1}{J}\mathbf{FSF}^{T} = 20\begin{bmatrix} 5k^2 + 3k^4 & \enclose{circle}{2k} + 3k^3 \\ \enclose{circle}{2k} + 3k^3 & 3k^2 \end{bmatrix}\text{MPa}$$



Boundary Conditions

- Boundary Conditions

$\mathbf{u} = \mathbf{g},$	on Γ^h	Essential (displacement) boundary
$\mathbf{t} = \mathbf{P}^T\mathbf{N},$	on Γ^s	Natural (traction) boundary

↳ You can't use **S**

- Solution space (set)

$$\mathbb{V} = \left\{ \mathbf{u} \middle| \mathbf{u} \in [H^1(\Omega)]^3,\ \mathbf{u}|_{\Gamma^h} = \mathbf{g} \right\}$$

- Kinematically admissible space

$$\mathbb{Z} = \left\{ \bar{\mathbf{u}} \middle| \bar{\mathbf{u}} \in [H^1(\Omega)]^3,\ \bar{\mathbf{u}}|_{\Gamma^h} = 0 \right\}$$

Variational Formulation

- We want to minimize the potential energy (equilibrium)

  Π^{int}: stored internal energy

  Π^{ext}: potential energy of applied loads

$$\begin{aligned} \Pi(\mathbf{u}) &= \Pi^{int}(\mathbf{u}) + \Pi^{ext}(\mathbf{u}) \\ &= \iint_{\Omega_0} W(\mathbf{E})\, d\Omega - \iint_{\Omega_0} \mathbf{u}^T \mathbf{f}^b\, d\Omega - \int_{\Gamma_0^s} \mathbf{u}^T \mathbf{t}\ d\Gamma \end{aligned}$$

- Want to find $\mathbf{u} \in \mathbb{V}$ that minimizes the potential energy
  - Perturb **u** in the direction of $\bar{\mathbf{u}} \in \mathbb{Z}$ proportional to τ

$$\mathbf{u}_\tau = \mathbf{u} + \tau\bar{\mathbf{u}}$$

  - If **u** minimizes the potential, $\Pi(\mathbf{u})$ must be smaller than $\Pi(\mathbf{u}_\tau)$ for all possible $\bar{\mathbf{u}}$

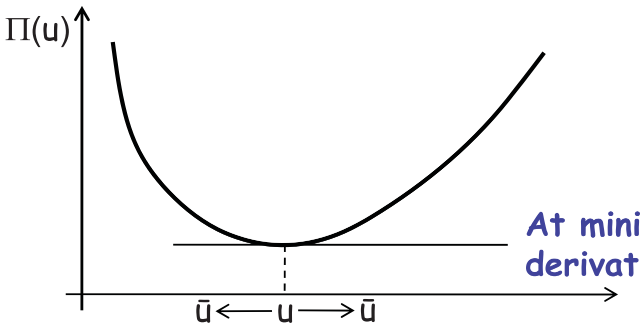
Variational Formulation cont.

- **Variation of Potential Energy** (Directional Derivative)

$$\overline{\Pi}(\mathbf{u},\bar{\mathbf{u}}) = \frac{d}{d\tau}\Pi(\mathbf{u}+\tau\bar{\mathbf{u}})\Big|_{\tau=0}$$

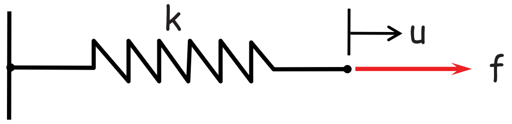
We will use "over-bar" for variation

  - Π depends on **u** only, but $\overline{\Pi}$ depends on both **u** and $\bar{\mathbf{u}}$
  - **Minimum potential energy happens when its variation becomes zero for every possible $\bar{u}$**
  - One-dimensional example



At minimum, all directional derivatives are zero

Example – Linear Spring



- Potential energy: $\Pi(u) = \frac{1}{2}k \cdot u^2 - f \cdot u$
- Perturbation: $\Pi(u+\tau\bar{u}) = \frac{1}{2}k \cdot (u+\tau\bar{u})^2 - f \cdot (u+\tau\bar{u})$
- Differentiation: $\frac{d}{d\tau}\left[\Pi(u+\tau\bar{u})\right] = k \cdot (u+\tau\bar{u}) \cdot \bar{u} - f \cdot \bar{u}$
- Evaluate at original state:

$$\frac{d}{d\tau}\left[\Pi(u+\tau\bar{u})\right]\Big|_{\tau=0} = k \cdot u \cdot \bar{u} - f \cdot \bar{u} = 0$$

Variation is similar to differentiation !!!

Variational Formulation cont.

- Variational Equation

$$\bar{\Pi}(\mathbf{u},\bar{\mathbf{u}}) = \iint_{\Omega_0} \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}} : \bar{\mathbf{E}}\, d\Omega - \iint_{\Omega_0} \bar{\mathbf{u}}^T \mathbf{f}^b\, d\Omega - \int_{\Gamma_0^s} \bar{\mathbf{u}}^T \mathbf{t}\;\, d\Gamma = 0$$

for all $\bar{\mathbf{u}} \in \mathbb{Z}$

  - From the definition of stress

$$\boxed{\iint_{\Omega_0} \mathbf{S} : \bar{\mathbf{E}}\, d\Omega = \iint_{\Omega_0} \bar{\mathbf{u}}^T \mathbf{f}^b\, d\Omega + \int_{\Gamma_0^s} \bar{\mathbf{u}}^T \mathbf{t}\;\, d\Gamma}$$

Variational equation in TL formulation

  - Note: load term is similar to linear problems
  - **Nonlinearity in the strain energy term**
- Need to write LHS in terms of **u** and $\bar{\mathbf{u}}$

Variational Formulation cont.

- How to express strain variation

$$\mathbf{E}(\mathbf{u}) = \tfrac{1}{2}(\mathbf{C} - \mathbf{1}) = \tfrac{1}{2}\left(\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T + \nabla_0\mathbf{u}^T\nabla_0\mathbf{u}\right)$$

$$\begin{aligned}
\bar{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}}) &= \frac{d}{d\tau}\mathbf{E}(\mathbf{u}+\tau\bar{\mathbf{u}})\Big|_{\tau=0} \\
&= \tfrac{1}{2}\left(\nabla_0\bar{\mathbf{u}} + \nabla_0\bar{\mathbf{u}}^T + \nabla_0\bar{\mathbf{u}}^T\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T\nabla_0\bar{\mathbf{u}}\right) \\
&= \tfrac{1}{2}\left((\mathbf{1} + \nabla_0\mathbf{u}^T)\nabla_0\bar{\mathbf{u}} + \nabla_0\bar{\mathbf{u}}^T(\mathbf{1} + \nabla_0\mathbf{u})\right) \\
&= \tfrac{1}{2}\left(\mathbf{F}^T\nabla_0\bar{\mathbf{u}} + \nabla_0\bar{\mathbf{u}}^T\mathbf{F}\right)
\end{aligned}$$

$$\boxed{\bar{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}}) = \mathrm{sym}(\nabla_0\bar{\mathbf{u}}^T\mathbf{F})}$$

Note: $\mathbf{E}(\mathbf{u})$ is nonlinear, but $\bar{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}})$ is linear

Variational Formulation cont.

- Variational Equation

$$\underbrace{\iint_{\Omega_0} \mathbf{S} : \bar{\mathbf{E}}\, d\Omega}_{a(\mathbf{u},\bar{\mathbf{u}})} = \underbrace{\iint_{\Omega_0} \bar{\mathbf{u}}^T \mathbf{f}^b\, d\Omega + \int_{\Gamma_0^s} \bar{\mathbf{u}}^T \mathbf{t}\, d\Gamma}_{\ell(\bar{\mathbf{u}})} \qquad \text{for all } \bar{\mathbf{u}} \in \mathbb{Z}$$

Energy form　　Load form

$$\boxed{a(\mathbf{u},\bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}), \quad \forall \bar{\mathbf{u}} \in \mathbb{Z}}$$

- Linear in terms of strain if St. Venant-Kirchhoff material is used
- Also linear in terms of $\bar{\mathbf{u}}$
- Nonlinear in terms of **u** because displacement-strain relation is nonlinear

Linearization (Increment)

- Linearization process is similar to variation and/or differentiation
  - First-order Taylor series expansion
  - Essential part of Newton-Raphson method
- Let $f(\mathbf{x}^{k+1}) = f(\mathbf{x}^k + \Delta\mathbf{u}^k)$, where we know $\mathbf{x}^k$ and want to calculate $\Delta\mathbf{u}^k$

$$f(\mathbf{x}^{k+1}) = f(\mathbf{x}^k) + \frac{df(\mathbf{x})}{d\mathbf{x}} \cdot \Delta\mathbf{u}^k + \text{H.O.T.}$$

- The first-order derivative is indeed linearization of $f(\mathbf{x})$

$$\boxed{L[f] \equiv \frac{d}{d\omega} f(\mathbf{x} + \omega\Delta\mathbf{u})\bigg|_{\omega=0} = \frac{\partial f}{\partial \mathbf{x}} \cdot \Delta\mathbf{u}} \qquad \text{Linearization}$$

$$\delta f = \bar{f} \equiv \frac{d}{d\tau} f(\mathbf{x} + \tau\bar{\mathbf{u}})\bigg|_{\tau=0} = \frac{\partial f}{\partial \mathbf{x}} \cdot \bar{\mathbf{u}} \qquad \text{Variation}$$

Linearization of Residual

- We are still in continuum domain (not discretized yet)
- Residual $\mathrm{R}(\mathbf{u}) = a(\mathbf{u}, \bar{\mathbf{u}}) - \ell(\bar{\mathbf{u}})$
- We want to linearize $\mathrm{R}(\mathbf{u})$ in the direction of $\Delta\mathbf{u}$
  - First, assume that $\mathbf{u}$ is perturbed in the direction of $\Delta\mathbf{u}$ using a variable τ. Then linearization becomes

$$L[\mathrm{R}(\mathbf{u})] = \left.\frac{\partial \mathrm{R}(\mathbf{u} + \tau\Delta\mathbf{u})}{\partial\tau}\right|_{\tau=0} = \left[\frac{\partial \mathrm{R}}{\partial \mathbf{u}}\right]^{T} \Delta\mathbf{u}$$

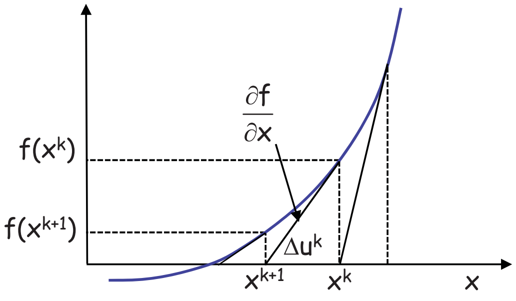
  - $\mathrm{R}(\mathbf{u})$ is nonlinear w.r.t. $\mathbf{u}$, but $L[\mathrm{R}(\mathbf{u})]$ is linear w.r.t. $\Delta\mathbf{u}$
  - Iteration k did not converged, and we want to make the residual at iteration k+1 zero

$$\mathrm{R}(\mathbf{u}^{k+1}) \approx \left[\frac{\partial \mathrm{R}(\mathbf{u}^{k})}{\partial \mathbf{u}}\right]^{T} \Delta\mathbf{u}^{k} + \mathrm{R}(\mathbf{u}^{k}) = 0$$

Newton-Raphson Iteration by Linearization

- This is N-R method (see Chapter 2)

$$\boxed{\left[\frac{\partial \mathrm{R}(\mathbf{u}^{k})}{\partial \mathbf{u}}\right]^{T} \Delta\mathbf{u}^{k} = -\mathrm{R}(\mathbf{u}^{k})}$$



- Update state $\mathbf{u}^{k+1} = \mathbf{u}^{k} + \Delta\mathbf{u}^{k}$

$$\mathbf{x}^{k+1} = \mathbf{X} + \mathbf{u}^{k+1}$$

- We know how to calculate $\mathrm{R}(\mathbf{u}^k)$, but how about $\left[\frac{\partial \mathrm{R}(\mathbf{u}^{k})}{\partial \mathbf{u}}\right]$?

$$\frac{\partial}{\partial \mathbf{u}}[\mathrm{R}(\mathbf{u})] = \frac{\partial}{\partial \mathbf{u}}[a(\mathbf{u}, \bar{\mathbf{u}}) - \cancel{\ell(\bar{\mathbf{u}})}]$$

  - Only linearization of energy form will be required
  - We will address displacement-dependent load later

Linearization cont.

- Linearization of energy form

$$L[a(\mathbf{u},\bar{\mathbf{u}})] = L\left[\iint_{{}^0\Omega} \mathbf{S} : \bar{\mathbf{E}}\, d\Omega\right] = \iint_{{}^0\Omega}[\Delta\mathbf{S} : \bar{\mathbf{E}} + \mathbf{S} : \Delta\bar{\mathbf{E}}]d\Omega$$

  - Note that the domain is fixed (undeformed reference)
  - Need to express in terms of displacement increment $\Delta\mathbf{u}$

- Stress increment (St. Venant-Kirchhoff material)

$$\Delta\mathbf{S} = \frac{\partial\mathbf{S}}{\partial\mathbf{E}} : \Delta\mathbf{E} = \mathbf{D} : \Delta\mathbf{E}$$

- Strain increment (Green-Lagrange strain)

$$\Delta\mathbf{E} = \tfrac{1}{2}(\Delta\mathbf{F}^T\mathbf{F} + \mathbf{F}^T\Delta\mathbf{F})$$

$$\Delta\mathbf{F} = \Delta\left(\frac{\partial\mathbf{x}}{\partial\mathbf{X}}\right) = \Delta\left(\frac{\partial(\mathbf{X}+\mathbf{u})}{\partial\mathbf{X}}\right) = \frac{\partial\Delta\mathbf{u}}{\partial\mathbf{X}} = \nabla_0\Delta\mathbf{u}$$

Linearization cont.

- Strain increment

$$\begin{aligned}\Delta\mathbf{E} &= \tfrac{1}{2}(\Delta\mathbf{F}^T\mathbf{F} + \mathbf{F}^T\Delta\mathbf{F})\\ &= \tfrac{1}{2}(\nabla_0\Delta\mathbf{u}^T\mathbf{F} + \mathbf{F}^T\nabla_0\Delta\mathbf{u})\\ &= \text{sym}(\nabla_0\Delta\mathbf{u}^T\mathbf{F})\end{aligned}$$

!!! Linear w.r.t. $\Delta\mathbf{u}$

- Inc. strain variation

$$\begin{aligned}\Delta\bar{\mathbf{E}} &= \Delta[\text{sym}(\nabla_0\bar{\mathbf{u}}^T\mathbf{F})]\\ &= \text{sym}(\nabla_0\bar{\mathbf{u}}^T\Delta\mathbf{F})\\ &= \text{sym}(\nabla_0\bar{\mathbf{u}}^T\nabla_0\Delta\mathbf{u})\end{aligned}$$

!!! Linear w.r.t. $\Delta\mathbf{u}$

- **Linearized energy form**

$$\boxed{L[a(\mathbf{u},\bar{\mathbf{u}})] = \iint_{{}^0\Omega}[\bar{\mathbf{E}} : \mathbf{D} : \Delta\mathbf{E} + \mathbf{S} : \Delta\bar{\mathbf{E}}]d\Omega \equiv a^*(\mathbf{u};\Delta\mathbf{u},\bar{\mathbf{u}})}$$

  - **Implicitly depends on u, but bilinear w.r.t. Δu and ū**
  - **First term: tangent stiffness**
  - **Second term: initial stiffness**

Linearization cont.

- N-R Iteration with Incremental Force
  - Let t_n be the current load step and (k+1) be the current iteration
  - Then, the N-R iteration can be done by

$$\boxed{a^*({}^n\mathbf{u}^k;\Delta\mathbf{u}^k,\bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}) - a({}^n\mathbf{u}^k,\bar{\mathbf{u}}), \quad \forall\bar{\mathbf{u}}\in\mathbb{Z}}$$

  - Update the total displacement

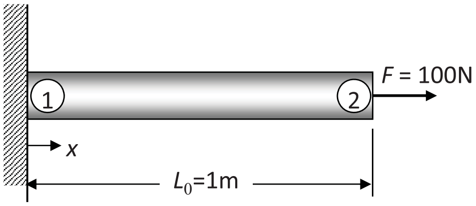
$${}^n\mathbf{u}^{k+1} = {}^n\mathbf{u}^k + \Delta\mathbf{u}^k$$

- In discrete form

$$\{\bar{\mathbf{d}}\}^T[{}^n\mathbf{K}_T^k]\{\Delta\mathbf{d}^k\} = \{\bar{\mathbf{d}}\}^T\{{}^n\mathbf{R}^k\}$$

- What are $[{}^n\mathbf{K}_T^k]$ and $\{{}^n\mathbf{R}^k\}$?

Example – Uniaxial Bar



- Kinematics $\dfrac{du}{dX} = u_2, \quad \dfrac{d\bar{u}}{dX} = \bar{u}_2$

$$E_{11} = \frac{du}{dX} + \frac{1}{2}\left(\frac{du}{dX}\right)^2 = u_2 + \frac{1}{2}(u_2)^2$$

- Strain variation

$$\bar{E}_{11} = \frac{d\bar{u}}{dX} + \frac{du}{dX}\frac{d\bar{u}}{dX} = \bar{u}_2(1+u_2)$$

- Strain energy density and stress

$$W(E_{11}) = \tfrac{1}{2}E\cdot(E_{11})^2 \qquad S_{11} = \frac{\partial W}{\partial E_{11}} = E\cdot E_{11} = E\left(u_2 + \frac{1}{2}(u_2)^2\right)$$

- Energy and load forms

$$a(u,\bar{u}) = \int_0^{L_0} S_{11}\bar{E}_{11}A\,dX = S_{11}AL_0(1+u_2)\bar{u}_2 \qquad \ell(\bar{u}) = \bar{u}_2F$$

- Variational equation $R = \bar{u}_2\left(S_{11}AL_0(1+u_2) - F\right) = 0, \quad \forall\bar{u}_2$

Example – Uniaxial Bar

- Linearization

$$\Delta S_{11} = E\Delta E_{11} = E(1+u_2)\Delta u_2 \qquad \Delta\bar{E}_{11} = \bar{u}_2\Delta u_2$$

$$a^*(u;\Delta u,\bar{u}) = \int_0^{L_0}\left(\bar{E}_{11}\cdot E\cdot\Delta E_{11} + S_{11}\cdot\Delta\bar{E}_{11}\right)A\,dX$$

$$= EAL_0(1+u_2)^2\bar{u}_2\Delta u_2 + S_{11}AL_0\bar{u}_2\Delta u_2$$

- N-R iteration

$$[E(1+u_2^k)^2 + S_{11}^k]AL_0\Delta u_2^k = F - S_{11}^k(1+u_2^k)AL_0$$

$$u_2^{k+1} = u_s^k + \Delta u_2^k$$

Example – Uniaxial Bar

(a) with initial stiffness

Iteration	*u*	Strain	Stress	conv
0	0.0000	0.0000	0.0000	9.999E−01
1	0.5000	0.6250	125.00	7.655E−01
2	0.3478	0.4083	81.664	1.014E−02
3	0.3252	0.3781	75.616	4.236E−06

(b) without initial stiffness

Iteration	*u*	Strain	Stress	conv
0	0.0000	0.0000	0.0000	9.999E−01
1	0.5000	0.6250	125.00	7.655E−01
2	0.3056	0.3252	70.448	6.442E−03
3	0.3291	0.3833	76.651	3.524E−04
4	0.3238	0.3762	75.242	1.568E−05
5	0.3250	0.3770	75.541	7.314E−07

Updated Lagrangian Formulation

- The current configuration is the reference frame
  - Remember it is unknown until we solve the problem
  - How are we going to integrate if we don't know integral domain?
- **What stress and strain should be used?**
  - For stress, we can use Cauchy stress (σ)
  - For strain, engineering strain is a pair of Cauchy stress
  - But, it must be defined in the current configuration

$$\varepsilon = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}^T + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) = \mathrm{sym}(\nabla_x \mathbf{u})$$

Variational Equation in UL

- Instead of deriving a new variational equation, we will convert from TL equation

$$\sigma = \frac{1}{J}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$$

$$\Rightarrow \quad \mathbf{S} = J\mathbf{F}^{-1}\cdot\sigma\cdot\mathbf{F}^{-T}$$

$$\bar{\mathbf{E}} = \frac{1}{2}\left(\frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{X}}^T \mathbf{F} + \mathbf{F}^T \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{X}}\right)$$

$$= \frac{1}{2}\mathbf{F}^T\left(\mathbf{F}^{-T}\frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{X}}^T + \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{X}}\mathbf{F}^{-1}\right)\mathbf{F}$$

$$= \frac{1}{2}\mathbf{F}^T\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}^T\frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{X}}^T + \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right)\mathbf{F}$$

$$= \frac{1}{2}\mathbf{F}^T\left(\frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{x}}^T + \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{x}}\right)\mathbf{F}$$

$$= \mathbf{F}^T\cdot\bar{\varepsilon}\cdot\mathbf{F}$$

Similarly

$$\Delta\mathbf{E} = \mathbf{F}^T\cdot\Delta\varepsilon\cdot\mathbf{F}$$

$$\Delta\varepsilon = \frac{1}{2}\left(\frac{\partial \Delta\mathbf{u}}{\partial \mathbf{x}}^T + \frac{\partial \Delta\mathbf{u}}{\partial \mathbf{x}}\right)$$

Variational Equation in UL cont.

- Energy Form

$$a(\mathbf{u},\bar{\mathbf{u}}) = \iint_{\Omega_0} \mathbf{S}:\bar{\mathbf{E}}\,d\Omega = \iint_{\Omega_0} (\mathbf{J}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}):(\mathbf{F}^{T}\bar{\boldsymbol{\varepsilon}}\mathbf{F})\,d\Omega$$

$$F_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}F_{mi}\bar{\varepsilon}_{mn}F_{nj} = \delta_{mk}\delta_{nl}\sigma_{kl}\bar{\varepsilon}_{mn} = \sigma_{mn}\bar{\varepsilon}_{mn}$$

$$\iint_{\Omega_0} \mathbf{S}:\bar{\mathbf{E}}\,d\Omega = \iint_{\Omega_0} \boldsymbol{\sigma}:\bar{\boldsymbol{\varepsilon}}\,J\,d\Omega = \iint_{\Omega_x} \boldsymbol{\sigma}:\bar{\boldsymbol{\varepsilon}}\,d\Omega$$

  - We just showed that material and spatial forms are mathematically equivalent
- Although they are equivalent, we use different notation:

$$\boxed{a(\mathbf{u},\bar{\mathbf{u}}) = \iint_{\Omega_x} \boldsymbol{\sigma}:\bar{\boldsymbol{\varepsilon}}\,d\Omega}$$

Is this linear or nonlinear?

- Variational Equation

$$\boxed{a(\mathbf{u},\bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}), \quad \forall\bar{\mathbf{u}} \in \mathbb{Z}}$$

What happens to load form?

Linearization of UL

- Linearization of $a_x(\mathbf{u},\bar{\mathbf{u}})$ will be challenging because we don't know the current configuration (it is function of **u**)
- Similar to the energy form, we can convert the linearized energy form of TL
- Remember $a^*(\mathbf{u};\Delta\mathbf{u},\bar{\mathbf{u}}) = \iint_{{}^0\Omega}[\bar{\mathbf{E}}:\mathbf{D}:\Delta\mathbf{E} + \mathbf{S}:\Delta\bar{\mathbf{E}}]\,d^0\Omega$
- Initial stiffness term

$$\mathbf{S}:\Delta\bar{\mathbf{E}} = J(\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}):\frac{1}{2}\left(\frac{\partial\bar{\mathbf{u}}}{\partial\mathbf{X}}^{T}\frac{\partial\Delta\mathbf{u}}{\partial\mathbf{X}} + \frac{\partial\Delta\mathbf{u}}{\partial\mathbf{X}}^{T}\frac{\partial\bar{\mathbf{u}}}{\partial\mathbf{X}}\right)$$

$$= JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}\,\frac{1}{2}\left(\frac{\partial\bar{u}_m}{\partial X_i}\frac{\partial\Delta u_m}{\partial X_j} + \frac{\partial\Delta u_m}{\partial X_i}\frac{\partial\bar{u}_m}{\partial X_j}\right)$$

$$\equiv J\sigma_{kl}\,\frac{1}{2}\boxed{\left(\frac{\partial\bar{u}_m}{\partial x_k}\frac{\partial\Delta u_m}{\partial x_l} + \frac{\partial\Delta u_m}{\partial x_k}\frac{\partial\bar{u}_m}{\partial x_l}\right)} \longrightarrow \eta_{kl}(\Delta\mathbf{u},\bar{\mathbf{u}})$$

Linearization of UL cont.

- Initial stiffness term

$$\boxed{\mathbf{S} : \Delta\bar{\mathbf{E}} = J\boldsymbol{\sigma} : \boldsymbol{\eta}(\Delta\mathbf{u}, \bar{\mathbf{u}})} \qquad \boldsymbol{\eta}(\Delta\mathbf{u}, \bar{\mathbf{u}}) = \mathrm{sym}(\nabla_x \bar{\mathbf{u}}^T \nabla_x \Delta\mathbf{u})$$

- Tangent stiffness term

$$(\bar{\mathbf{E}} : \mathbf{D} : \Delta\mathbf{E}) = (\mathbf{F}^T \cdot \bar{\varepsilon} \cdot \mathbf{F}) : \mathbf{D} : (\mathbf{F}^T \cdot \Delta\varepsilon \cdot \mathbf{F})$$

$$= F_{ki}\bar{\varepsilon}_{kl}F_{lj}D_{ijmn}F_{pm}\Delta\varepsilon_{pq}F_{qn}$$

$$= J\bar{\varepsilon}_{kl}\left[\boxed{\frac{1}{J}F_{ki}F_{lj}D_{ijmn}F_{pm}F_{qn}}\right]\Delta\varepsilon_{pq}$$

→ 4th-order spatial constitutive tensor

$$\boxed{\bar{\mathbf{E}} : \mathbf{D} : \Delta\mathbf{E} = J\bar{\varepsilon} : \mathbf{c} : \Delta\varepsilon}$$

where $c_{ijkl} = \dfrac{1}{J}F_{ir}F_{js}F_{km}F_{ln}D_{rsmn}$

Spatial Constitutive Tensor

- For St. Venant-Kirchhoff material

$$\mathbf{D} = \lambda(\mathbf{1} \otimes \mathbf{1}) + 2\mu\mathbf{I} \qquad D_{rsmn} = \lambda\delta_{rs}\delta_{mn} + \mu(\delta_{rm}\delta_{sn} + \delta_{rn}\delta_{sm})$$

- It is possible to show

$$c_{ijkl} = \frac{1}{J}\left[\lambda b_{ij}b_{kl} + \mu(b_{ik}b_{jl} + b_{il}b_{jk})\right].$$

- Observation
  - **D** (material) is constant, but **c** (spatial) is not
  - $\mathbf{S} = \mathbf{D} : \mathbf{E}, \quad \sigma \neq \mathbf{c} : \varepsilon$

Linearization of UL cont.

- From equivalence, the energy form is linearized in TL and converted to UL

$$L[a(\mathbf{u},\bar{\mathbf{u}})] = \iint_{\Omega_0} [\bar{\varepsilon} : \mathbf{c} : \Delta\varepsilon + \sigma : \eta] J \, d\Omega$$

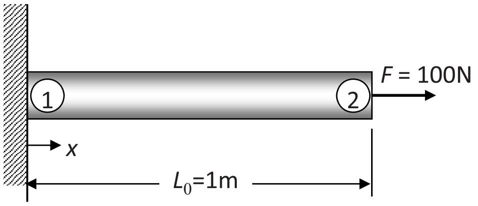
$$\boxed{a^*(\mathbf{u};\Delta\mathbf{u},\bar{\mathbf{u}}) = \iint_{\Omega_x} [\bar{\varepsilon} : \mathbf{c} : \Delta\varepsilon + \sigma : \eta] d\Omega}$$

- N-R Iteration

$$\boxed{a^*({}^n\mathbf{u}^k;\Delta\mathbf{u}^k,\bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}) - a({}^n\mathbf{u}^k,\bar{\mathbf{u}}), \quad \forall\bar{\mathbf{u}} \in \mathbb{Z}}$$

- Observations
  - Two formulations are theoretically identical with different expression
  - Numerical implementation will be different
  - Different constitutive relation

Example – Uniaxial Bar



- Kinematics

$$\frac{du}{dx} = \frac{u_2}{1+u_2}, \quad \frac{d\bar{u}}{dx} = \frac{\bar{u}_2}{1+u_2}$$

- Deformation gradient: $F_{11} = \dfrac{dx}{dX} = 1 + u_2, \quad J = 1 + u_2$

- Cauchy stress: $\sigma_{11} = \dfrac{1}{J} F_{11} S_{11} F_{11} = E(u_2 + \dfrac{1}{2} u_2^2)(1 + u_2)$

- Strain variation: $\varepsilon_{11}(\bar{u}) = F_{11}^{-T} \bar{E}_{11} F_{11}^{-1} = \dfrac{\bar{u}_2}{1 + u_2}$

- Energy & load forms: $a(u,\bar{u}) = \int_0^L \sigma_{11}\varepsilon_{11}(\bar{u}) A \, dx = \sigma_{11} A \bar{u}_2 \qquad \ell(\bar{u}) = \bar{u}_2 F$

- Residual: $R = \bar{u}_2 \left(\sigma_{11} A - F \right) = 0, \quad \forall \bar{u}_2$

Example – Uniaxial Bar

- Spatial constitutive relation: $c_{1111} = \frac{1}{J}F_{11}F_{11}F_{11}F_{11}E = (1+u_2)^3 E$
- Linearization: $\int_0^L \varepsilon_{11}(\bar{u})c_{1111}\varepsilon_{11}(\Delta u)A\,dx = EA(1+u_2)^2\bar{u}_2\Delta u_2$

$$\int_0^L \sigma_{11}\eta_{11}(\Delta u,\bar{u})A\,dx = \frac{\sigma_{11}A}{1+u_2}\bar{u}_2\Delta u_2$$

$$a^*(u;\Delta u,\bar{u}) = \int_0^L \left(\varepsilon_{11}(\bar{u})c_{1111}\varepsilon_{11}(\Delta u) + \sigma_{11}\eta(\Delta u,\bar{u})\right)A\,dx$$

$$= EA(1+u_2)^2\bar{u}_2\Delta u_2 + \frac{\sigma_{11}}{1+u_2}A\bar{u}_2\Delta u_2$$

Iteration	*u*	Strain	Stress	conv
0	0.0000	0.0000	0.000	9.999E−01
1	0.5000	0.3333	187.500	7.655E−01
2	0.3478	0.2581	110.068	1.014E−02
3	0.3252	0.2454	100.206	4.236E−06

Section 3.5

Hyperelastic Material Model

Goals

- Understand the definition of hyperelastic material
- Understand strain energy density function and how to use it to obtain stress
- Understand the role of invariants in hyperelasticity
- Understand how to impose incompressibility
- Understand mixed formulation and perturbed Lagrangian formulation
- Understand linearization process when strain energy density is written in terms of invariants

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What Is Hyperelasticity?

- Hyperelastic material - stress-strain relationship derives from a strain energy density function
  - Stress is a function of total strain (independent of history)
  - Depending on strain energy density, different names are used, such as Mooney-Rivlin, Ogden, Yeoh, or polynomial model
- Generally comes with incompressibility ($J = 1$)
  - The volume preserves during large deformation
  - Mixed formulation – completely incompressible hyperelasticity
  - Penalty formulation - nearly incompressible hyperelasticity
- Example: rubber, biological tissues
  - nonlinear elastic, isotropic, incompressible and generally independent of strain rate
- Hypoelastic material: relation is given in terms of stress and strain rates

Strain Energy Density

- We are interested in isotropic materials
  - Material frame indifference: no matter what coordinate system is chosen, the response of the material is identical
  - The components of a deformation tensor depends on coord. system
  - Three invariants of **C** are independent of coord. system
- Invariants of **C**

$$I_1 = \mathrm{tr}(\boldsymbol{C}) = C_{11} + C_{22} + C_{33} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \tfrac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2) \right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$

$$I_3 = \det \boldsymbol{C} = \lambda_1^2\lambda_2^2\lambda_3^2$$

No deformation
$I_1 = 3$
$I_2 = 3$
$I_3 = 1$

  - In order to be material frame indifferent, material properties must be expressed using invariants
  - For incompressibility, $I_3 = 1$

Strain Energy Density cont.

- Strain Energy Density Function
  - Must be zero when **C** = **1**, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$$W(I_1, I_2, I_3) = \sum_{m+n+k=1}^{\infty} A_{mnk}(I_1 - 3)^m (I_2 - 3)^n (I_3 - 1)^k$$

  - For incompressible material

$$W(I_1, I_2) = \sum_{m+n=1}^{\infty} A_{mn}(I_1 - 3)^m (I_2 - 3)^n$$

  - Ex: Neo-Hookean model

$$W(I_1) = A_{10}(I_1 - 3) \qquad A_{10} = \frac{\mu}{2}$$

  - Mooney-Rivlin model

$$W(I_1, I_2) = A_{10}(I_1 - 3) + A_{01}(I_2 - 3)$$

Strain Energy Density cont.

- Strain Energy Density Function
  - Yeoh model

$$W_1(I_1) = A_{10}(I_1 - 3) + A_{20}(I_1 - 3)^2 + A_{30}(I_1 - 3)^3$$

  - Ogden model

$$W_1(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}\left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right)$$

Initial shear modulus

$$\mu = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \mu_i$$

  - When N = 1 and a_1 = 1, Neo-Hookean material
  - When N = 2, α_1 = 2, and $\alpha_2 = -2$, Mooney-Rivlin material

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Example – Neo-Hookean Model

- Uniaxial tension with incompressibility

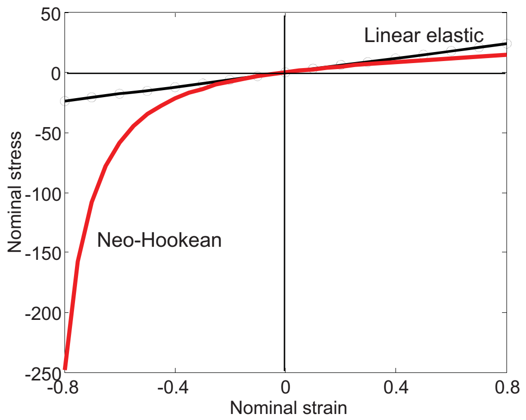
$$\lambda_1 = \lambda \qquad \lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$$

- Energy density

$$W = A_{10}(I_1 - 3) = A_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) = A_{10}(\lambda^2 + \frac{2}{\lambda} - 3)$$

- Nominal stress

$$P = \frac{\partial W}{\partial \lambda} = 2A_{10}\left(\lambda - \frac{1}{\lambda^2}\right) = \mu\left(1 + \varepsilon - \frac{1}{(1+\varepsilon)^2}\right)$$



Example – St. Venant Kirchhoff Material

- Show that St. Venant-Kirchhoff material has the following strain energy density

$$W(\mathbf{E}) = \frac{\lambda}{2}\left[\,\mathrm{tr}(\mathbf{E})\,\right]^2 + \mu\,\mathrm{tr}(\mathbf{E}^2)$$

$$\mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}} = \lambda\,\mathrm{tr}(\mathbf{E})\frac{\partial \mathrm{tr}(\mathbf{E})}{\partial \mathbf{E}} + \mu\frac{\partial \mathrm{tr}(\mathbf{E}^2)}{\partial \mathbf{E}}$$

- First term

$$\mathrm{tr}(\mathbf{E}) = \mathbf{1} : \mathbf{E} \qquad \frac{\partial \mathrm{tr}(\mathbf{E})}{\partial \mathbf{E}} = \mathbf{1}$$

$$\lambda\,\mathrm{tr}(\mathbf{E})\frac{\partial \mathrm{tr}(\mathbf{E})}{\partial \mathbf{E}} = \lambda\mathbf{1}(\mathbf{1} : \mathbf{E}) = \lambda(\mathbf{1}\otimes\mathbf{1}) : \mathbf{E}$$

- Second term

$$\frac{\partial E_{ij}E_{ji}}{\partial E_{kl}} = \delta_{ik}\delta_{jl}E_{ji} + E_{ij}\delta_{jk}\delta_{il} = E_{lk} + E_{lk} = 2E_{lk}$$

Example – St. Venant Kirchhoff Material cont.

- Therefore

$$\begin{aligned}\mathbf{S} &= \lambda\,\mathrm{tr}(\mathbf{E})\frac{\partial \mathrm{tr}(\mathbf{E})}{\partial \mathbf{E}} + \mu\frac{\partial \mathrm{tr}(\mathbf{E}^2)}{\partial \mathbf{E}}\\ &= \lambda(\mathbf{1}\otimes\mathbf{1}) : \mathbf{E} + 2\mu\mathbf{E}\\ &= \underbrace{\left[\,\lambda(\mathbf{1}\otimes\mathbf{1}) + 2\mu\mathbf{I}\,\right]}_{\mathbf{D}} : \mathbf{E}\end{aligned}$$

Nearly Incompressible Hyperelasticity

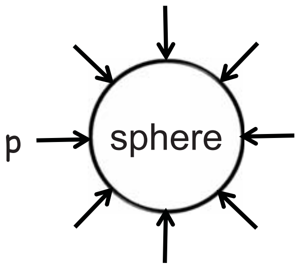
- Incompressible material
  - Cannot calculate stress from strain. Why?
- Nearly incompressible material
  - Many material show nearly incompressible behavior
  - We can use the bulk modulus to model it
- Using I_1 and I_2 enough for incompressibility?
  - No, I_1 and I_2 actually vary under hydrostatic deformation
  - We will use reduced invariants: J_1, J_2, and J_3

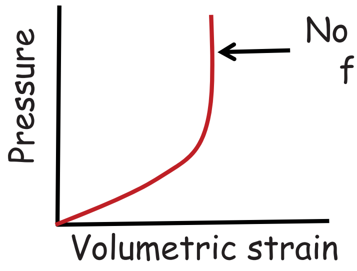
$$J_1 = I_1 I_3^{-1/3} \quad J_2 = I_2 I_3^{-2/3} \quad J_3 = J = I_3^{1/2}$$

- Will J_1 and J_2 be constant under dilatation?

Locking

- What is locking
  - Elements do not want to deform even if forces are applied
  - Locking is one of the most common modes of failure in NL analysis
  - It is very difficult to find and solutions show strange behaviors
- Types of locking
  - Shear locking: shell or beam elements under transverse loading
  - Volumetric locking: large elastic and plastic deformation
- Why does locking occur?
  - Incompressible sphere under hydrostatic pressure

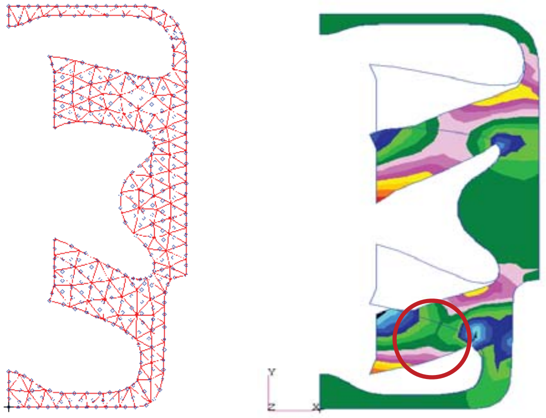


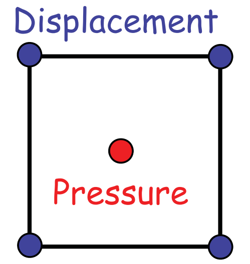


No unique pressure
for given displ.

How to solve locking problems?

- Mixed formulation (incompressibility)
  - Can't interpolate pressure from displacements
  - Pressure should be considered as an independent variable
  - Becomes the Lagrange multiplier method
  - The stiffness matrix becomes positive semi-definite

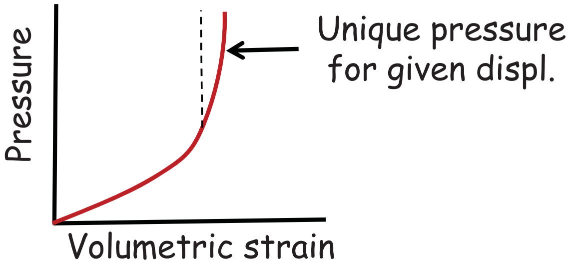




4x1 formulation

Penalty Method

- Instead of incompressibility, the material is assumed to be nearly incompressible
- This is closer to actual observation
- Use a large bulk modulus (penalty parameter) so that a small volume change causes a large pressure change
- Large penalty term makes the stiffness matrix ill-conditioned
- Ill-conditioned matrix often yields excessive deformation
- Temporarily reduce the penalty term in the stiffness calculation
- Stress calculation use the penalty term as it is



$$[K] = \begin{bmatrix} 1 & & & \\ & 10^7 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Example – Hydrostatic Tension (Dilatation)

$$\begin{cases} x_1 = \alpha X_1 \\ x_2 = \alpha X_2 \\ x_3 = \alpha X_3 \end{cases} \qquad \mathbf{F} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

- Invariants

$$I_1 = 3\alpha^2 \quad I_2 = 3\alpha^4 \quad I_3 = \alpha^6$$

I_1 and I_2 are not constant

- Reduced invariants

$$\begin{aligned} J_1 &= I_1 I_3^{-1/3} = 3 \\ J_2 &= I_2 I_3^{-2/3} = 3 \\ J_3 &= I_3^{1/2} = \alpha^3 \end{aligned}$$

J_1 and J_2 are constant

Strain Energy Density

- Using reduced invariants

$$W(J_1, J_2, J_3) = W_D(J_1, J_2) + W_H(J_3)$$

  - $W_D(J_1, J_2)$: Distortional strain energy density
  - $W_H(J_3)$: Dilatational strain energy density
- The second terms is related to nearly incompressible behavior

$$W_H(J_3) = \frac{K}{2}(J_3 - 1)^2$$

  - K: bulk modulus $= \lambda + \frac{2}{3}\mu$ for linear elastic material

Abaqus: $W_H(J_3) = \frac{1}{2D}(J_3 - 1)^2$

Mooney-Rivlin Material

- Most popular model
  - (not because accuracy, but because convenience)

$$\begin{aligned} W(J_1, J_2, J_3) &= W_D(J_1, J_2) + W_H(J_3) \\ &= A_{10}(J_1 - 3) + A_{01}(J_2 - 3) + \frac{K}{2}(J_3 - 1)^2 \end{aligned}$$

  - Initial shear modulus ~ $2(A_{10} + A_{01})$
  - Initial Young's modulus ~ $6(A_{10} + A_{01})$ (3D) or $8(A_{10} + A_{01})$ (2D)
  - Bulk modulus = K
- Hydrostatic pressure

$$p = \frac{\partial W}{\partial J_3} = \frac{\partial W_H}{\partial J_3} = K(J_3 - 1)$$

  - Numerical instability for large K (volumetric locking)
  - Penalty method with K as a penalty parameter

Mooney-Rivlin Material cont.

- Second P-K stress

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \frac{\partial W}{\partial J_1}\frac{\partial J_1}{\partial \mathbf{E}} + \frac{\partial W}{\partial J_2}\frac{\partial J_2}{\partial \mathbf{E}} + \frac{\partial W}{\partial J_3}\frac{\partial J_3}{\partial \mathbf{E}}$$

$$\mathbf{S} = A_{10}J_{1,\mathbf{E}} + A_{01}J_{2,\mathbf{E}} + K(J_3 - 1)J_{3,\mathbf{E}}$$

$$a_{,\mathbf{E}} = \frac{\partial a}{\partial \mathbf{E}}$$

  - Use chain rule of differentiation

$$\begin{aligned} J_1 &= I_1 I_3^{-1/3} \\ J_2 &= I_2 I_3^{-2/3} \\ J_3 &= I_3^{1/2} \end{aligned}$$

$$\begin{aligned} J_{1,\mathbf{E}} &= (I_3^{-1/3})I_{1,\mathbf{E}} - \tfrac{1}{3}I_1(I_3^{-4/3})I_{3,\mathbf{E}} \\ J_{2,\mathbf{E}} &= (I_3^{-2/3})I_{2,\mathbf{E}} - \tfrac{2}{3}I_2(I_3^{-5/3})I_{3,\mathbf{E}} \\ J_{3,\mathbf{E}} &= \tfrac{1}{2}(I_3^{-1/2})I_{3,\mathbf{E}} \end{aligned}$$

$$\begin{aligned} I_{1,\mathbf{E}} &= 2\mathbf{1} \\ I_{2,\mathbf{E}} &= 4(1 + \mathrm{tr}\mathbf{E})\mathbf{1} - 4\mathbf{E} \\ I_{3,\mathbf{E}} &= (2 + 4\mathrm{tr}\mathbf{E})\mathbf{1} - 4\mathbf{E} + [\tfrac{9}{4}e_{imn}e_{jrs}E_{mr}E_{ns}] \end{aligned}$$

$$\begin{aligned} I_{1,\mathbf{E}} &= 2\mathbf{1} \\ I_{2,\mathbf{E}} &= 2(I_1\mathbf{1} - \mathbf{C}) \\ I_{3,\mathbf{E}} &= 2I_3\mathbf{C}^{-1} \end{aligned}$$

Example

- Show $I_{1,E} = 2\mathbf{1}, \quad I_{2,E} = 2(I_1\mathbf{1} - \mathbf{C}), \quad I_{3,E} = 2I_3\mathbf{C}^{-1}$
- Let $\bar{I}_1 = \mathrm{tr}(\mathbf{C}), \quad \bar{I}_2 = \tfrac{1}{2}\mathrm{tr}(\mathbf{CC}), \quad \bar{I}_3 = \tfrac{1}{3}\mathrm{tr}(\mathbf{CCC})$
- Then $I_1 = \bar{I}_1, \quad I_2 = \tfrac{1}{2}\bar{I}_1^2 - \bar{I}_2, \quad I_3 = \bar{I}_3 + \tfrac{1}{6}\bar{I}_1^3 - \bar{I}_1\bar{I}_2$
- Derivatives

$$\frac{\partial \bar{I}_1}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial \bar{I}_2}{\partial C_{ij}} = C_{ji}, \quad \frac{\partial \bar{I}_3}{\partial C_{ij}} = C_{jk}C_{ki}$$

$$\frac{\partial I_1}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial I_2}{\partial C_{ij}} = I_1\delta_{ij} - C_{ji}, \quad \frac{\partial \bar{I}_3}{\partial C_{ij}} = I_3 C_{ji}^{-1}$$

and

$$\frac{\partial}{\partial \mathbf{C}} = 2\frac{\partial}{\partial \mathbf{E}}$$

Mixed Formulation

- Using bulk modulus often causes instability
  - Selectively reduced integration (Full integration for deviatoric part, reduced integration for dilatation part)
- Mixed formulation: Independent treatment of pressure

$$W_H(J_3, p) = p(J_3 - 1)$$

  - Pressure p is additional unknown (pure incompressible material)
  - Advantage: No numerical instability
  - Disadvantage: system matrix is not positive definite
- Perturbed Lagrangian formulation

$$W_H(J_3, p) = p(J_3 - 1) - \frac{1}{2K}p^2$$

  - Second term make the material nearly incompressible and the system matrix positive definite

Variational Equation (Perturbed Lagrangian)

- Stress calculation

$$W(J_1, J_2, J_3) = A_{10}(J_1 - 3) + A_{01}(J_2 - 3) + p(J_3 - 1) + \frac{1}{2K}p^2$$

$$\mathbf{S} = A_{10}J_{1,\mathbf{E}} + A_{01}J_{2,\mathbf{E}} + pJ_{3,\mathbf{E}}$$

- Variation of strain energy density

$$\bar{W} = W_{,\mathbf{E}}\bar{\mathbf{E}} + W_{,p}\bar{p}$$

$$= \mathbf{S} : \bar{\mathbf{E}} + (J_3 - 1 - \frac{p}{K})\bar{p}$$

- Introduce a vector of unknowns: $\mathbf{r} = (\mathbf{u}, p)$

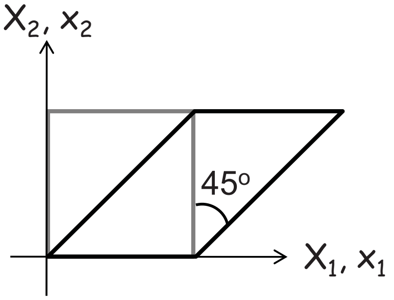
$$a(\mathbf{r}, \bar{\mathbf{r}}) = \iint_{\Omega_0} \left[\mathbf{S} : \bar{\mathbf{E}} + \bar{p}H \right] d\Omega$$

$$H = J_3 - 1 - \frac{p}{K}$$ Volumetric strain

Example – Simple Shear

- Calculate 2nd P-K stress for the simple shear deformation
  - material properties (A_{10}, A_{01}, K)

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$I_1 = 4, \quad I_2 = 4, \quad I_3 = 1$$

$$I_{1,\mathbf{E}} = 2\mathbf{1}$$

$$I_{2,\mathbf{E}} = 2(I_1\mathbf{1} - \mathbf{C}) = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$I_{3,\mathbf{E}} = 2I_3\mathbf{C}^{-1} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example – Simple Shear cont.

$$J_1 = I_1 I_3^{-1/3} = 4 \qquad J_{1,E} = I_{1,E} - \frac{4}{3} I_{3,E} = \frac{2}{3}\begin{bmatrix} -5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$J_2 = I_2 I_3^{-2/3} = 4$$

$$J_3 = I_3^{-1/2} = 1 \qquad J_{2,E} = I_{2,E} - \tfrac{8}{3} I_{3,E} = \frac{2}{3}\begin{bmatrix} -7 & 5 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S} = A_{10} J_{1,E} + A_{01} J_{2,E} + K(J_3 - 1) J_{3,E}$$

$$= \frac{2}{3}\begin{bmatrix} -5A_{10} - 7A_{01} & 4A_{10} + 5A_{01} & 0 \\ 4A_{10} + 5A_{01} & -A_{10} - 2A_{01} & 0 \\ 0 & 0 & -A_{10} + A_{01} \end{bmatrix}$$

Note: S_{11}, S_{22} and S_{33} are not zero

Stress Calculation Algorithm

- Given: $\{\mathbf{E}\} = \{E_{11}, E_{22}, E_{33}, E_{12}, E_{23}, E_{13}\}^T$, $\{p\}$, (A_{10}, A_{01})

$$\{\mathbf{1}\} = \{1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0\}^T \qquad \{\mathbf{C}\} = 2\{\mathbf{E}\} + \{\mathbf{1}\}$$

$$I_1 = C_1 + C_2 + C_3$$
$$I_2 = C_1C_2 + C_1C_3 + C_2C_3 - C_4C_4 - C_5C_5 - C_6C_6$$
$$I_3 = (C_1C_2 - C_4C_4)C_3 + (C_4C_6 - C_1C_5)C_5 + (C_4C_5 - C_2C_6)C_6$$

$$\{I_{1,E}\} = 2\{1 \;\; 1 \;\; 1 \;\; 0\}$$
$$\{I_{2,E}\} = 2\{C_2 + C_3 \quad C_3 + C_1 \quad C_1 + C_2 \quad -C_4 \quad -C_5 \quad -C_6\}$$
$$\{I_{3,E}\} = 2\{C_2C_3 - C_5^2 \quad C_3C_1 - C_6^2 \quad C_1C_2 - C_4^2$$
$$C_5C_6 - C_3C_4 \quad C_6C_4 - C_1C_5 \quad C_4C_5 - C_2C_6\}$$

$$\{J_{1,E}\} = I_3^{-1/3}\{I_{1,E}\} - \tfrac{1}{3} I_1 I_3^{-4/3}\{I_{3,E}\}$$
$$\{J_{2,E}\} = I_3^{-2/3}\{I_{2,E}\} - \tfrac{2}{3} I_2 I_3^{-5/3}\{I_{3,E}\}$$
$$\{J_{3,E}\} = \tfrac{1}{2} I_3^{-1/2}\{I_{3,E}\},$$

$$\{\mathbf{S}\} = A_{10}\{J_{1,E}\} + A_{01}\{J_{2,E}\} + p\{J_{3,E}\}$$

For penalty method, use $K(J_3 - 1)$ instead of p

Linearization (Penalty Method)

- Stress increment

$$\Delta\mathbf{S} = W_{,\mathbf{E}\mathbf{E}} : \Delta\mathbf{E} = \mathbf{D} : \Delta\mathbf{E}$$

- Material stiffness

$$\mathbf{D} = \frac{\partial\mathbf{S}}{\partial\mathbf{E}} = A_{10}J_{1,\mathbf{E}\mathbf{E}} + A_{01}J_{2,\mathbf{E}\mathbf{E}} + K(J_3 - 1)J_{3,\mathbf{E}\mathbf{E}} + KJ_{3,\mathbf{E}} \otimes J_{3,\mathbf{E}}$$

- Linearized energy form

$$a^{*}(\mathbf{u};\Delta\mathbf{u},\bar{\mathbf{u}}) = \iint_{\Omega_0}\left[\bar{\mathbf{E}} : \mathbf{D} : \Delta\mathbf{E} + \mathbf{S} : \Delta\bar{\mathbf{E}}\right]d\Omega$$

Linearization cont.

- Second-order derivatives of reduced invariants

$$J_{1,\mathbf{E}\mathbf{E}} = I_{1,\mathbf{E}\mathbf{E}}I_3^{-\frac{1}{3}} - \frac{1}{3}I_3^{-\frac{4}{3}}(I_{1,\mathbf{E}} \otimes I_{3,\mathbf{E}} + I_{3,\mathbf{E}} \otimes I_{1,\mathbf{E}}) + \frac{4}{9}I_1I_3^{-\frac{7}{3}}I_{3,\mathbf{E}} \otimes I_{3,\mathbf{E}} - \frac{1}{3}I_1I_3^{-\frac{4}{3}}I_{3,\mathbf{E}\mathbf{E}}$$

$$J_{2,\mathbf{E}\mathbf{E}} = I_{2,\mathbf{E}\mathbf{E}}I_3^{-\frac{2}{3}} - \frac{2}{3}I_3^{-\frac{5}{3}}(I_{2,\mathbf{E}} \otimes I_{3,\mathbf{E}} + I_{3,\mathbf{E}} \otimes I_{2,\mathbf{E}}) + \frac{10}{9}I_2I_3^{-\frac{8}{3}}I_{3,\mathbf{E}} \otimes I_{3,\mathbf{E}} - \frac{2}{3}I_2I_3^{-\frac{5}{3}}I_{3,\mathbf{E}\mathbf{E}}$$

$$J_{3,\mathbf{E}\mathbf{E}} = -\frac{1}{4}I_3^{-\frac{3}{2}}I_{3,\mathbf{E}} \otimes I_{3,\mathbf{E}} + \frac{1}{2}I_3^{-\frac{1}{2}}I_{3,\mathbf{E}\mathbf{E}}$$

$$I_{1,\mathbf{E}\mathbf{E}} = \mathbf{0}$$

$$I_{2,\mathbf{E}\mathbf{E}} = 4\mathbf{1} \otimes \mathbf{1} - \mathbf{I}$$

$$I_{3,\mathbf{E}\mathbf{E}} = 4I_3\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - I_3\mathbf{C}^{-1}\mathbf{I}\mathbf{C}^{-1}$$

MATLAB Function Mooney

- Calculates **S** and **D** for a given deformation gradient

```
%
% 2nd PK stress and material stiffness for Mooney-Rivlin material
%
function [Stress D] = Mooney(F, A10, A01, K, ltan)
% Inputs:
%  F = Deformation gradient [3x3]
%  A10, A01, K = Material constants
%  ltan = 0 Calculate stress alone;
%         1 Calculate stress and material stiffness
% Outputs:
%  Stress = 2nd PK stress [S11, S22, S33, S12, S23, S13];
%  D = Material stiffness [6x6]
%
```

Summary

- Hyperelastic material: strain energy density exists with incompressible constraint
- In order to be material frame indifferent, material properties must be expressed using invariants
- Numerical instability (volumetric locking) can occur when large bulk modulus is used for incompressibility
- Mixed formulation is used for purely incompressibility (additional pressure variable, non-PD tangent stiffness)
- Perturbed Lagrangian formulation for nearly incompressibility (reduced integration for pressure term)

Section 3.6

Finite Element Formulation for Nonlinear Elasticity

Voigt Notation

- We will use the **Voigt notation** because the tensor notation is not convenient for implementation
  - 2^{nd}-order tensor $\Rightarrow$ vector
  - 4^{th}-order tensor $\Rightarrow$ matrix
- Stress and strain vectors (Voigt notation)

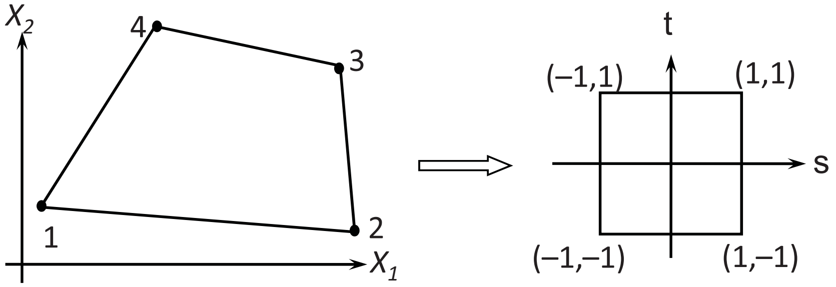
$$\{\mathbf{S}\} = \{S_{11} \quad S_{22} \quad S_{12}\}^T$$

$$\{\mathbf{E}\} = \{E_{11} \quad E_{22} \quad 2E_{12}\}^T$$

  - Since stress and strain are symmetric, we don't need 21 component

4-Node Quadrilateral Element in TL

- We will use plane-strain, 4-node quadrilateral element to discuss implementation of nonlinear elastic FEA
- We will use TL formulation
- UL formulation will be discussed in Chapter 4



Finite Element at undeformed domain

Reference Element

Interpolation and Isoparametric Mapping

- Displacement interpolation

$$\mathbf{u} = \sum_{I=1}^{N_e} N_I(\mathbf{s})\mathbf{u}_I$$

Nodal displacement vector (u_I, v_I)

Interpolation function

- **Isoparametric mapping**
  - The same interpolation function is used for geometry mapping

$$\mathbf{X} = \sum_{I=1}^{N_e} N_I(\mathbf{s})\mathbf{X}_I$$

Nodal coordinate (X_I, Y_I)

$$N_1 = \tfrac{1}{4}(1-s)(1-t)$$
$$N_2 = \tfrac{1}{4}(1+s)(1-t)$$
$$N_3 = \tfrac{1}{4}(1+s)(1+t)$$
$$N_4 = \tfrac{1}{4}(1-s)(1+t)$$

Interpolation (shape) function

- Same for all elements
- Mapping depends of geometry

Displacement and Deformation Gradients

- Displacement gradient

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \sum_{I=1}^{N_e} \frac{\partial N_I(\mathbf{s})}{\partial \mathbf{X}} \mathbf{u}_I \qquad u_{i,j} = \sum_{I=1}^{N_e} N_{I,j}(\mathbf{s}) u_{Ii}$$

$$\nabla_0 \mathbf{u} = \{u_{1,1} \quad u_{1,2} \quad u_{2,1} \quad u_{2,2}\}^T$$

  - How to calculate $\dfrac{\partial N_I(\mathbf{s})}{\partial \mathbf{X}}$?

- Deformation gradient

$$\{\mathbf{F}\} = \{F_{11} \quad F_{12} \quad F_{21} \quad F_{22}\}^T = \{1 + u_{1,1} \quad u_{1,2} \quad u_{2,1} \quad 1 + u_{2,2}\}^T$$

  - **Both displacement and deformation gradients are not symmetric**

Green-Lagrange Strain

- Green-Lagrange strain

$$\{\mathbf{E}\} = \begin{Bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{Bmatrix} = \begin{Bmatrix} u_{1,1} + \frac{1}{2}(u_{1,1}u_{1,1} + u_{2,1}u_{2,1}) \\ u_{2,2} + \frac{1}{2}(u_{1,2}u_{2,1} + u_{2,2}u_{2,2}) \\ u_{1,2} + u_{2,1} + u_{1,2}u_{1,1} + u_{2,1}u_{2,2} \end{Bmatrix}$$

  - Due to nonlinearity, $\{\mathbf{E}\} \neq [\mathbf{B}]\{\mathbf{d}\}$
  - For St. Venant-Kirchhoff material, $\{\mathbf{S}\} = [\mathbf{D}]\{\mathbf{E}\}$

$$[\mathbf{D}] = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Variation of G-R Strain

- Although **E**(**u**) is nonlinear, $\bar{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}})$ is linear

$$\bar{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}}) = \mathrm{sym}(\nabla_0\bar{\mathbf{u}}^T\mathbf{F}) \qquad \boxed{\{\bar{\mathbf{E}}\} = [\mathbf{B}_N]\{\bar{\mathbf{d}}\}}$$

$$[\mathbf{B}_N] = \begin{bmatrix} F_{11}N_{1,1} & F_{21}N_{1,1} & F_{11}N_{2,1} & F_{21}N_{2,1} & \cdots & F_{11}N_{4,1} & F_{21}N_{4,1} \\ F_{12}N_{1,2} & F_{22}N_{1,2} & F_{12}N_{2,2} & F_{22}N_{2,2} & \cdots & F_{12}N_{4,2} & F_{22}N_{4,2} \\ \begin{matrix} F_{11}N_{1,2} \\ +F_{12}N_{1,1} \end{matrix} & \begin{matrix} F_{21}N_{1,2} \\ +F_{22}N_{1,1} \end{matrix} & \begin{matrix} F_{11}N_{2,2} \\ +F_{12}N_{2,1} \end{matrix} & \begin{matrix} F_{21}N_{2,2} \\ +F_{22}N_{2,1} \end{matrix} & \cdots & \begin{matrix} F_{11}N_{4,2} \\ +F_{12}N_{4,1} \end{matrix} & \begin{matrix} F_{21}N_{4,2} \\ +F_{22}N_{4,1} \end{matrix} \end{bmatrix}$$

Function of **u**
Different from linear strain-displacement matrix

Variational Equation

- Energy form

$$\begin{aligned} a(\mathbf{u},\bar{\mathbf{u}}) &= \iint_{\Omega_0} \mathbf{S} : \bar{\mathbf{E}}\, d\Omega \\ &\approx \{\bar{\mathbf{d}}\}^T \iint_{\Omega_0} [\mathbf{B}_N]^T\{\mathbf{S}\}\, d\Omega \\ &\equiv \{\bar{\mathbf{d}}\}^T\{\mathbf{F}^{int}\} \end{aligned}$$

- Load form

$$\begin{aligned} \ell(\bar{\mathbf{u}}) &= \iint_{\Omega_0} \bar{\mathbf{u}}^T\mathbf{f}^b\, d\Omega + \int_{\Gamma_0^s} \bar{\mathbf{u}}^T\mathbf{t}\, d\Gamma \\ &\approx \sum_{I=1}^{N_e} \bar{\mathbf{u}}_I^T \left\{ \iint_{\Omega_0} N_I(\mathbf{s})\mathbf{f}^b\, d\Omega + \int_{\Gamma_0^s} N_I(\mathbf{s})\mathbf{t}\, d\Gamma \right\} \\ &\equiv \{\bar{\mathbf{d}}\}^T\{\mathbf{F}^{ext}\} \end{aligned}$$

- Residual

$$\{\bar{\mathbf{d}}\}^T\{\mathbf{F}^{int}(\mathbf{d})\} = \{\bar{\mathbf{d}}\}^T\{\mathbf{F}^{ext}\}, \quad \forall\{\bar{\mathbf{d}}\} \in \mathbb{Z}_h$$

Linearization – Tangent Stiffness

- Incremental strain $\boxed{\{\Delta\mathbf{E}\} = [\mathbf{B}_N]\{\Delta\mathbf{d}\}}$
- Linearization

$$\iint_{\Omega_0} \bar{\mathbf{E}} : \mathbf{D} : \Delta\mathbf{E}\, d\Omega = \{\bar{\mathbf{d}}\}^T \left[\iint_{\Omega_0} [\mathbf{B}_N]^T [\mathbf{D}][\mathbf{B}_N]\, d\Omega \right] \{\Delta\mathbf{d}\}$$

$$\iint_{\Omega_0} \mathbf{S} : \Delta\bar{\mathbf{E}}\, d\Omega = \{\bar{\mathbf{d}}\}^T \left[\iint_{\Omega_0} [\mathbf{B}_G]^T [\Sigma][\mathbf{B}_G]\, d\Omega \right] \{\Delta\mathbf{d}\}$$

$$[\Sigma] = \begin{bmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{12} & S_{22} & 0 & 0 \\ 0 & 0 & S_{11} & S_{12} \\ 0 & 0 & S_{12} & S_{22} \end{bmatrix}$$

$$[\mathbf{B}_G] = \begin{bmatrix} N_{1,1} & 0 & N_{2,1} & 0 & N_{3,1} & 0 & N_{4,1} & 0 \\ N_{1,2} & 0 & N_{2,2} & 0 & N_{3,2} & 0 & N_{4,2} & 0 \\ 0 & N_{1,1} & 0 & N_{2,1} & 0 & N_{3,1} & 0 & N_{4,1} \\ 0 & N_{2,1} & 0 & N_{2,2} & 0 & N_{3,2} & 0 & N_{4,2} \end{bmatrix}$$

Linearization – Tangent Stiffness

- Tangent stiffness

$$\boxed{[\mathbf{K}_T] = \iint_{\Omega_0} \left[[\mathbf{B}_N]^T [\mathbf{D}][\mathbf{B}_N] + [\mathbf{B}_G]^T [\Sigma][\mathbf{B}_G] \right] d\Omega_0}$$

- Discrete incremental equation (N-R iteration)

$$\boxed{\{\bar{\mathbf{d}}\}^T [\mathbf{K}_T]\{\Delta\mathbf{d}\} = \{\bar{\mathbf{d}}\}^T \{\mathbf{F}^{ext} - \mathbf{F}^{int}\}, \quad \forall \{\bar{\mathbf{d}}\} \in \mathbb{Z}_h}$$

  - $[K_T]$ changes according to stress and strain
  - Solved iteratively until the residual term vanishes

Summary

- For elastic material, the variational equation can be obtained from the principle of minimum potential energy
- St. Venant-Kirchhoff material has linear relationship between 2nd P-K stress and G-L strain
- In TL, nonlinearity comes from nonlinear strain-displacement relation
- In UL, nonlinearity comes from constitutive relation and unknown current domain (Jacobian of deformation gradient)
- TL and UL are mathematically equivalent, but have different reference frames
- TL and UL have different interpretation of constitutive relation.

Section 3.7

MATLAB Code for Hyperelastic Material Model

HYPER3D.m

- Building the tangent stiffness matrix, [**K**], and the residual force vector, {**R**}, for hyperelastic material
- Input variables for HYPER3D.m

Variable	Array size	Meaning
MID	Integer	Material Identification No. (3) (Not used)
PROP	(3,1)	Material properties (A10, A01, K)
UPDATE	Logical variable	If true, save stress values
LTAN	Logical variable	If true, calculate the global stiffness matrix
NE	Integer	Total number of elements
NDOF	Integer	Dimension of problem (3)
XYZ	(3,NNODE)	Coordinates of all nodes
LE	(8,NE)	Element connectivity

```
function HYPER3D(MID, PROP, UPDATE, LTAN, NE, NDOF, XYZ, LE)
%***********************************************************************
%  MAIN PROGRAM COMPUTING GLOBAL STIFFNESS MATRIX AND RESIDUAL FORCE FOR
%  HYPERELASTIC MATERIAL MODELS
%***********************************************************************
%%
  global DISPTD FORCE GKF SIGMA
  %
  % Integration points and weights
  XG=[-0.57735026918963D0, 0.57735026918963D0];
  WGT=[1.00000000000000D0, 1.00000000000000D0];
  %
  % Index for history variables (each integration pt)
  INTN=0;
  %
  %LOOP OVER ELEMENTS, THIS IS MAIN LOOP TO COMPUTE K AND F
  for IE=1:NE
    % Nodal coordinates and incremental displacements
    ELXY=XYZ(LE(IE,:),:);
    % Local to global mapping
    IDOF=zeros(1,24);
    for I=1:8
      II=(I-1)*NDOF+1;
      IDOF(II:II+2)=(LE(IE,I)-1)*NDOF+1:(LE(IE,I)-1)*NDOF+3;
    end
    DSP=DISPTD(IDOF);
    DSP=reshape(DSP,NDOF,8);
 %
    %LOOP OVER INTEGRATION POINTS
    for LX=1:2, for LY=1:2, for LZ=1:2
      E1=XG(LX); E2=XG(LY); E3=XG(LZ);
      INTN = INTN + 1;
      %
      % Determinant and shape function derivatives
      [~, SHPD, DET] = SHAPEL([E1 E2 E3], ELXY);
      FAC=WGT(LX)*WGT(LY)*WGT(LZ)*DET;
```

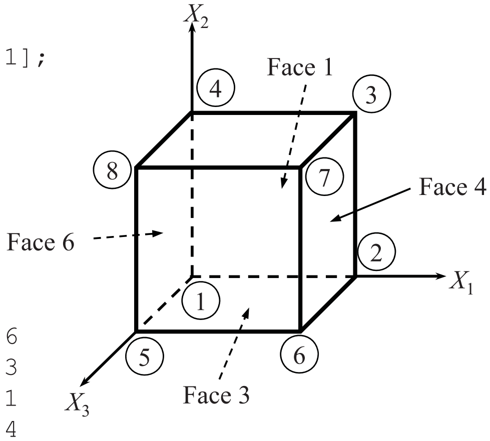
```
      % Deformation gradient
      F=DSP*SHPD' + eye(3);
      %
      % Computer stress and tangent stiffness
      [STRESS DTAN] = Mooney(F, PROP(1), PROP(2), PROP(3), LTAN);
      %
      % Store stress into the global array
      if UPDATE
        SIGMA(:,INTN)=STRESS;
        continue;
      end
      %
      % Add residual force and tangent stiffness matrix
      BM=zeros(6,24); BG=zeros(9,24);
      for I=1:8
        COL=(I-1)*3+1:(I-1)*3+3;
        BM(:,COL)=[SHPD(1,I)*F(1,1) SHPD(1,I)*F(2,1) SHPD(1,I)*F(3,1);
                   SHPD(2,I)*F(1,2) SHPD(2,I)*F(2,2) SHPD(2,I)*F(3,2);
                   SHPD(3,I)*F(1,3) SHPD(3,I)*F(2,3) SHPD(3,I)*F(3,3);
                   SHPD(1,I)*F(1,2)+SHPD(2,I)*F(1,1)
SHPD(1,I)*F(2,2)+SHPD(2,I)*F(2,1) SHPD(1,I)*F(3,2)+SHPD(2,I)*F(3,1);
                   SHPD(2,I)*F(1,3)+SHPD(3,I)*F(1,2)
SHPD(2,I)*F(2,3)+SHPD(3,I)*F(2,2) SHPD(2,I)*F(3,3)+SHPD(3,I)*F(3,2);
                   SHPD(1,I)*F(1,3)+SHPD(3,I)*F(1,1)
SHPD(1,I)*F(2,3)+SHPD(3,I)*F(2,1) SHPD(1,I)*F(3,3)+SHPD(3,I)*F(3,1)];
        %
        BG(:,COL)=[SHPD(1,I) 0         0;
                   SHPD(2,I) 0         0;
                   SHPD(3,I) 0         0;
                   0         SHPD(1,I) 0;
                   0         SHPD(2,I) 0;
                   0         SHPD(3,I) 0;
                   0         0         SHPD(1,I);
                   0         0         SHPD(2,I);
                   0         0         SHPD(3,I)];
      end
```

```
 %
      % Residual forces
      FORCE(IDOF) = FORCE(IDOF) - FAC*BM'*STRESS;
      %
      % Tangent stiffness
      if LTAN
        SIG=[STRESS(1) STRESS(4) STRESS(6);
             STRESS(4) STRESS(2) STRESS(5);
             STRESS(6) STRESS(5) STRESS(3)];
        SHEAD=zeros(9);
        SHEAD(1:3,1:3)=SIG;
        SHEAD(4:6,4:6)=SIG;
        SHEAD(7:9,7:9)=SIG;
        %
        EKF = BM'*DTAN*BM + BG'*SHEAD*BG;
        GKF(IDOF,IDOF)=GKF(IDOF,IDOF)+FAC*EKF;
      end
    end; end; end;
  end
end
```

Example Extension of a Unit Cube

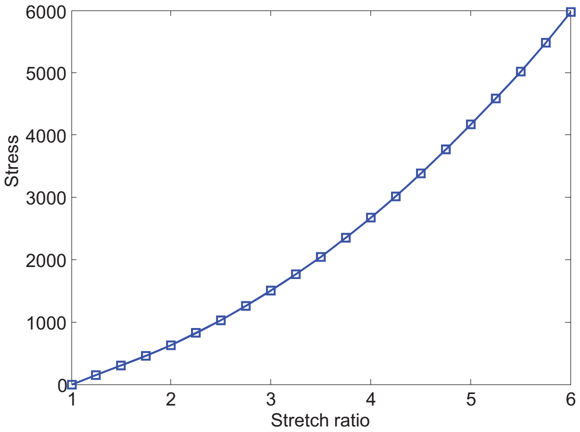
- Face 4 is extended with a stretch ratio λ = 6.0
- BC: u_1 = 0 at Face 6, u_2 = 0 at Face 3, and u_3 = 0 at Face 1
- Mooney-Rivlin: A_{10} = 80MPa, A_{01} = 20MPa, and K = 10^7

```
% Nodal coordinates
XYZ=[0 0 0;1 0 0;1 1 0;0 1 0;0 0 1;1 0 1;1 1 1;0 1 1];
%
% Element connectivity
LE=[1 2 3 4 5 6 7 8];
%
% No external force
EXTFORCE=[];
%
% Prescribed displacements [Node, DOF, Value]
SDISPT=[1 1 0;4 1 0;5 1 0;8 1 0;   % u1=0 for Face 6
        1 2 0;2 2 0;5 2 0;6 2 0;   % u2=0 for Face 3
        1 3 0;2 3 0;3 3 0;4 3 0;   % u3=0 for Face 1
        2 1 5;3 1 5;6 1 5;7 1 5];  % u1=5 for Face 4
%
% Load increments [Start End Increment InitialFactor FinalFactor]
TIMS=[0.0 1.0 0.05 0.0 1.0]';
%
% Material properties
MID=-1;
PROP=[80   20   1E7];
```



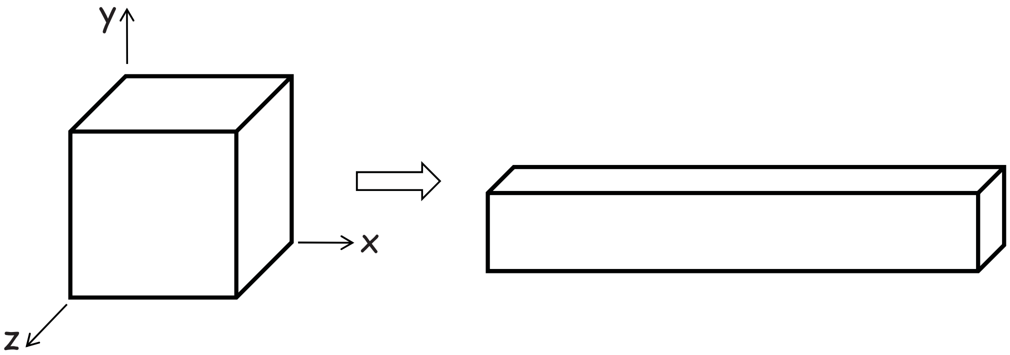
Example Extension of a Unit Cube

```
     Time   Time step    Iter        Residual
   0.05000  5.000e-02      2     1.17493e+05
Not converged. Bisecting load increment   2

     Time   Time step    Iter        Residual
   0.02500  2.500e-02      2     2.96114e+04
                           3     2.55611e+02
                           4     1.84747e-02
                           5     1.51867e-10

     Time   Time step    Iter        Residual
   0.05000  2.500e-02      2     2.48106e+04
                           3     1.69171e+02
                           4     7.67766e-03
                           5     2.39898e-10

     Time   Time step    Iter        Residual
   0.10000  5.000e-02      2     8.45251e+04
                           3     1.88898e+03
                           4     8.72537e-01
                           5     1.86783e-07
...
     Time   Time step    Iter        Residual
   1.00000  5.000e-02      2     8.55549e+03
                           3     8.98726e+00
                           4     9.88176e-06
                           5     1.66042e-09
```



Hyperelastic Material Analysis Using ABAQUS

- *ELEMENT,TYPE=C3D8RH,ELSET=ONE
  - 8-node linear brick, reduced integration with hourglass control, hybrid with constant pressure
- *MATERIAL,NAME=MOONEY
  *HYPERELASTIC, MOONEY-RIVLIN
  80., 20.,
  - Mooney-Rivlin material with A_{10} = 80 and A_{01} = 20
- *STATIC,DIRECT
  - Fixed time step (no automatic time step control)



Hyperelastic Material Analysis Using ABAQUS

```
*HEADING
 - Incompressible hyperelasticity (Mooney-
   Rivlin) Uniaxial tension
*NODE,NSET=ALL
1,
2,1.
3,1.,1.,
4,0.,1.,
5,0.,0.,1.
6,1.,0.,1.
7,1.,1.,1.
8,0.,1.,1.
*NSET,NSET=FACE1
1,2,3,4
*NSET,NSET=FACE3
1,2,5,6
*NSET,NSET=FACE4
2,3,6,7
*NSET,NSET=FACE6
4,1,8,5
*ELEMENT,TYPE=C3D8RH,ELSET=ONE
1,1,2,3,4,5,6,7,8
*SOLID SECTION, ELSET=ONE,
   MATERIAL= MOONEY
```

```
*MATERIAL,NAME=MOONEY
*HYPERELASTIC, MOONEY-RIVLIN
80., 20.,
*STEP,NLGEOM,INC=20
UNIAXIAL TENSION
*STATIC,DIRECT
1.,20.
*BOUNDARY,OP=NEW
FACE1,3
FACE3,2
FACE6,1
FACE4,1,1,5.
*EL PRINT,F=1
S,
E,
*NODE PRINT,F=1
U,RF
*OUTPUT,FIELD,FREQ=1
*ELEMENT OUTPUT
S,E
*OUTPUT,FIELD,FREQ=1
*NODE OUTPUT
U,RF
*END STEP
```

Hyperelastic Material Analysis Using ABAQUS

- Analytical solution procedure
  - Gradually increase the principal stretch λ from 1 to 6
  - Deformation gradient

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix}$$

  - Calculate $J_{1,E}$ and $J_{2,E}$
  - Calculate 2nd P-K stress

$$\mathbf{S} = A_{10}J_{1,E} + A_{01}J_{2,E}$$

  - Calculate Cauchy stress

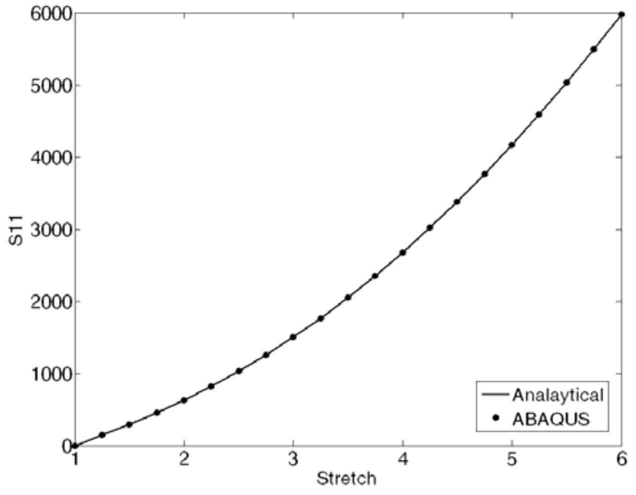
$$\sigma = \frac{1}{J}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$$

  - Remove the hydrostatic component of stress

$$\sigma_{11} = \sigma_{11} - \sigma_{22}$$

Hyperelastic Material Analysis Using ABAQUS

- Comparison with analytical stress vs. numerical stress

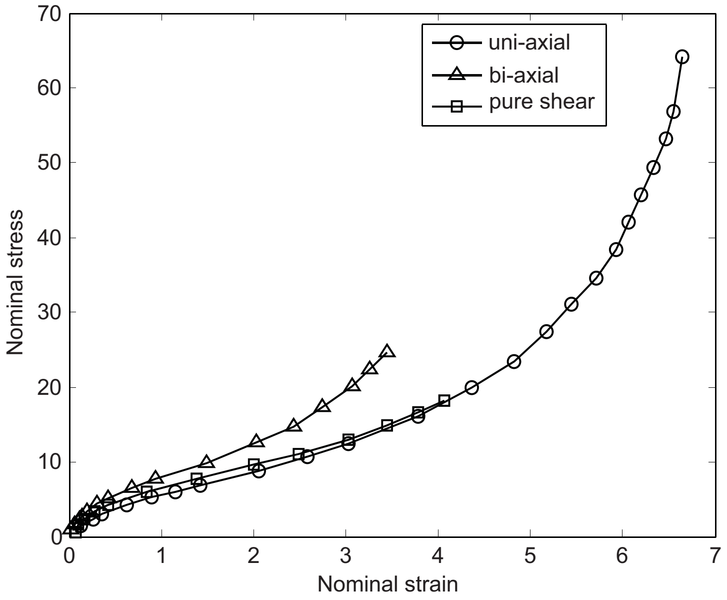


Section 3.9

Fitting Hyperelastic Material Parameters from Test Data

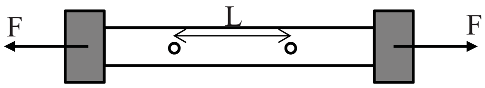
Elastomer Test Procedures

- Elastomer tests
  - simple tension, simple compression, equi-biaxial tension, simple shear, pure shear, and volumetric compression

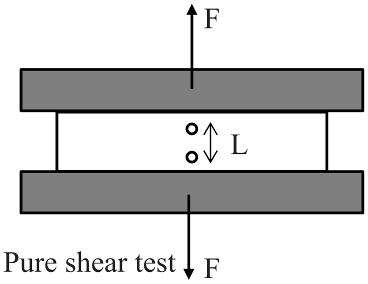


Elastomer Tests

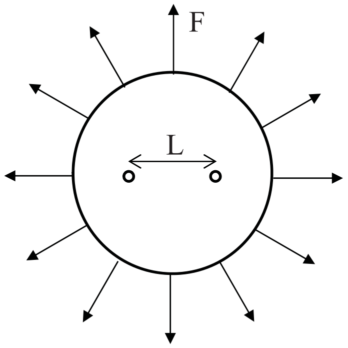
- Data type: Nominal stress vs. principal stretch



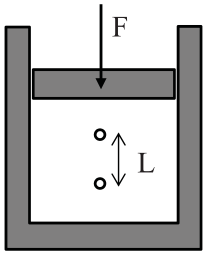
Simple tension test



Pure shear test



Equal biaxial test



Volumetric compression test

Data Preparation

- Need enough number of independent experimental data
  - No rank deficiency for curve fitting algorithm
- All tests measure principal stress and principle stretch

Experiment Type	Stretch	Stress
Uniaxial tension	Stretch ratio λ = L/L_0	Nominal stress T^E = F/A_0
Equi-biaxial tension	Stretch ratio λ = L/L_0 in y-direction	Nominal stress T^E = F/A_0 in y-direction
Pure shear test	Stretch ratio λ = L/L_0	Nominal stress T^E = F/A_0
Volumetric test	Compression ratio λ = L/L_0	Pressure T^E = F/A_0

Data Preparation cont.

- Uni-axial test $\lambda_1 = \lambda, \quad \lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$

$$T = \frac{\partial U}{\partial \lambda} = 2(1 - \lambda^{-3})(A_{10}\lambda + A_{01})$$

$$T(A_{10}, A_{01}, \lambda) = \boxed{\{x\}^T\{b\}} = \begin{bmatrix} 2(\lambda - \lambda^{-2}) & 2(1 - \lambda^{-3}) \end{bmatrix} \begin{Bmatrix} A_{10} \\ A_{01} \end{Bmatrix}$$

- Equi-biaxial test $\lambda_1 = \lambda_2 = \lambda, \quad \lambda_3 = 1/\lambda^2$

$$T = \frac{1}{2}\frac{\partial U}{\partial \lambda} = 2(\lambda - \lambda^{-5})(A_{10} + \lambda^2 A_{01})$$

- Pure shear test $\lambda_1 = \lambda, \quad \lambda_2 = 1, \quad \lambda_3 = 1/\lambda$

$$T = \frac{\partial U}{\partial \lambda} = 2(\lambda - \lambda^{-3})(A_{10} + A_{01})$$

Data Preparation cont.

- Data Preparation

Type	1	1	1	…	4	4	…	4
λ	λ_1	λ_2	λ_3	…	λ_i	λ_{i+1}	…	λ_{NDT}
T^E	T_1^E	T_2^E	T_3^E	…	T_i^E	T_{i+1}^E	…	T_{NDT}^E

- For Mooney-Rivlin material model, nominal stress is a linear function of material parameters (A_{10}, A_{01})

Curve Fitting for Mooney-Rivlin Material

- Need to determine A_{10} and A_{01} by minimizing error between test data and model

$$\underset{A_{10},A_{01}}{\text{minimize}} \quad \sum_{k=1}^{NDT} \left(T_k^E - T(A_{10}, A_{01}, \lambda_k) \right)^2$$

- For Mooney-Rivlin, $T(A_{10}, A_{01}, \lambda_k)$ is linear function
  - Least-squares can be used

$$\{b\} = \begin{Bmatrix} A_{10} \\ A_{01} \end{Bmatrix}$$

$$\{T^E\} = \begin{Bmatrix} T_1^E \\ T_2^E \\ \vdots \\ T_{NDT}^E \end{Bmatrix} \qquad \{T\} = \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{NDT} \end{Bmatrix} = \begin{bmatrix} \mathbf{x}(\lambda_1)^T \\ \mathbf{x}(\lambda_1)^T \\ \vdots \\ \mathbf{x}(\lambda_{NDT})^T \end{bmatrix} \{\mathbf{b}\} = [\mathbf{X}]\{\mathbf{b}\}$$

Curve Fitting cont.

- Minimize error(square)

$$\begin{aligned} \{\mathbf{e}\}^T\{\mathbf{e}\} &= \{\mathbf{T}^E - \mathbf{T}\}^T\{\mathbf{T}^E - \mathbf{T}\} \\ &= \{\mathbf{T}^E - \mathbf{Xb}\}^T\{\mathbf{T}^E - \mathbf{Xb}\} \\ &= \{\mathbf{T}^E\}^T\{\mathbf{T}^E\} - 2\{\mathbf{b}\}^T[\mathbf{X}]^T\{\mathbf{T}^E\} + \{\mathbf{b}\}^T[\mathbf{X}]^T[\mathbf{X}]\{\mathbf{b}\} \end{aligned}$$

- Minimization → Linear regression equation

$$\boxed{[\mathbf{X}]^T[\mathbf{X}]\{\mathbf{b}\} = [\mathbf{X}]^T\{\mathbf{T}^E\}}$$

Stability of Constitutive Model

- Stable material: the slope in the stress-strain curve is always positive (**Drucker stability**)
- Stability requirement (Mooney-Rivlin material)

$$\boxed{d\varepsilon : \mathbf{D} : d\varepsilon > 0}$$

- Stability check is normally performed at several specified deformations (principal directions)

$$d\sigma_1 d\varepsilon_1 + d\sigma_2 d\varepsilon_2 > 0$$

$$\begin{Bmatrix} d\varepsilon_1 & d\varepsilon_2 \end{Bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \end{Bmatrix} > 0$$

- In order to be P.D.

$$D_{11} + D_{22} > 0$$

$$D_{11}D_{22} - D_{12}D_{21} > 0$$