1. Solve Problem 1.3-3(c)

(c)
$$\left.\begin{aligned}\phi_1 &= a_1 - a_2 a\\ \phi_2 &= a_1 + a_2 a\end{aligned}\right\}\quad \begin{aligned}a_1 &= (\phi_1+\phi_2)/2\\ a_2 &= (\phi_2-\phi_1)/2a\end{aligned}$$

$$\phi_3 = a_1 + a_3 b$$

$$a_3 = \frac{\phi_3 - a_1}{b} = \frac{2\phi_3 - \phi_1 - \phi_2}{2b}$$

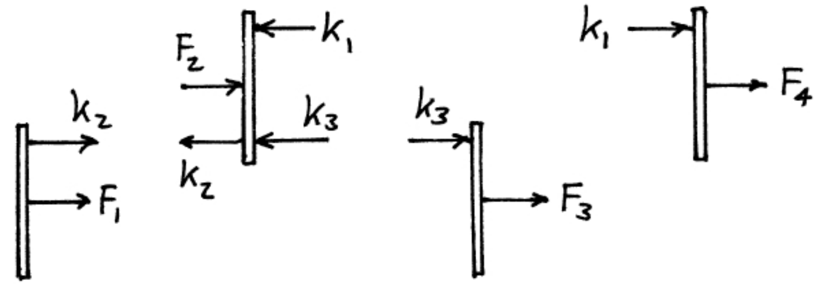
$$\phi = \frac{\phi_1+\phi_2}{2} + \frac{\phi_2-\phi_1}{2a}x + \frac{2\phi_3-\phi_1-\phi_2}{2b}y$$

$$\phi = \left(\frac{1}{2} - \frac{x}{2a} - \frac{y}{2b}\right)\phi_1 + \left(\frac{1}{2} + \frac{x}{2a} - \frac{y}{2b}\right)\phi_2 + \frac{y}{b}\phi_3$$

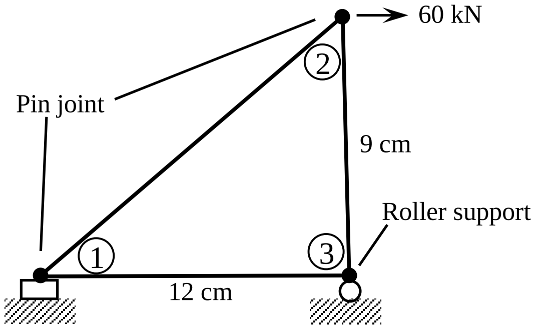
2. Solve Problem 2.2-2(b)

(b)
$$\begin{bmatrix} k_2+k_5+k_6 & -k_2 & -k_6 & 0 \\ -k_2 & k_1+k_2+k_3 & -k_3 & -k_1 \\ -k_6 & -k_3 & k_3+k_4+k_6 & -k_4 \\ 0 & -k_1 & -k_4 & k_1+k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Example — column 2:

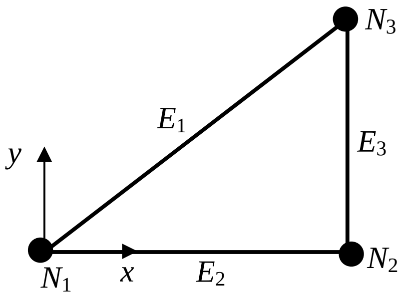


3. A two–dimensional truss shown in the figure is made of aluminum with Young's modulus E = 80 GPa and failure stress σ_Y = 150 MPa. Determine the minimum cross-sectional area of each member so that the truss is safe with safety factor 1.5.



Solution:

Let's assume the cross-sectional areas be 0.05 cm^2. We will modify them later, if necessary. Element numbers, node numbers and the coordinate system used here are referred to the following figure.



Connectivity table

Elem	LN #1	LN #2	E (N/cm^2)	A(cm^2)	L (cm)	θ	*l*
1	1	3	30×10^6	.05	15	36.87°	.80
2	1	2	30×10^6	.05	12	0°	1
3	2	3	30×10^6	.05	9	90°	0

Element stiffness matrix after applying transformation to the global coordinates:

$$[\mathbf{k}] = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Element 1:

$$[\mathbf{k}] = 10^3\begin{bmatrix} 64 & 48 & -64 & -48 \\ 48 & 36 & -48 & -36 \\ -64 & -48 & 64 & 48 \\ -48 & -36 & 48 & 36 \end{bmatrix}\begin{matrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{matrix}$$

Element 2: $$[\mathbf{k}] = 10^3 \begin{bmatrix} 125 & 0 & -125 & 0 \\ 0 & 0 & 0 & 0 \\ -125 & 0 & 125 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}$$

Element 3: $$[\mathbf{k}] = 10^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 166.7 & 0 & -166.7 \\ 0 & 0 & 0 & 0 \\ 0 & -166.7 & 0 & 166.7 \end{bmatrix} \begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix}$$

The three element stiffness matrices are assembled to the global stiffness matrix:

$$[\mathbf{K}] = 10^3 \begin{bmatrix} 189 & 48 & -125 & 0 & -64 & -48 \\ 48 & 36 & 0 & 0 & -48 & -36 \\ -125 & 0 & 125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 166.7 & 0 & -166.7 \\ -64 & -48 & 0 & 0 & 64 & 48 \\ -48 & -36 & 0 & -166.7 & 48 & 202.7 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix}$$

Node 1 is fixed ($u_1 = v_1 = 0$), and Node 2 is fixed in the vertical direction ($v_2 = 0$). The known boundary conditions are denoted in the following matrix equation:

$$10^3 \begin{bmatrix} 189 & 48 & -125 & 0 & -64 & -48 \\ 48 & 36 & 0 & 0 & -48 & -36 \\ -125 & 0 & 125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 166.7 & 0 & -166.7 \\ -64 & -48 & 0 & 0 & 64 & 48 \\ -48 & -36 & 0 & -166.7 & 48 & 202.7 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ 0 \\ R_{1y} \\ 60000 \\ 0 \end{Bmatrix}$$

Deleting those rows and columns corresponding to zero displacements, we have

$$10^3 \begin{bmatrix} 125 & 0 & 0 \\ 0 & 64 & 48 \\ 0 & 48 & 202.7 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60000 \\ 0 \end{Bmatrix}$$

The above equation can be solved for unknown displacements, as

$$\{u_2 \quad u_3 \quad v_3\} = \{0 \quad 1.14 \quad -0.27\}\,\text{cm}$$

Using the displacements and solving for the forces in each member using Eq. (2.45):

$$P^{(e)} = \frac{AE}{L}\Delta L = \frac{AE}{L}\left(\left(\frac{x_j - x_i}{L}\right)(u_j - u_i) + \left(\frac{y_j - y_i}{L}\right)(v_j - v_i)\right)$$

$$P^{(1)} = 75 \text{ kN}; \quad P^{(2)} = 0 \text{ kN}; \quad P^{(3)} = -45 \text{ kN}$$

Now, the element force will remain constant even if the cross-sectional areas are changed because the system is statically determinate. Thus, the minimum cross-sectional areas can be obtained from given safety factor and failure stress.

$$\sigma_{\max} = \frac{\sigma_Y}{S_F} = 100\,\text{MPa} = 10^4\,\frac{\text{N}}{\text{cm}^2}$$

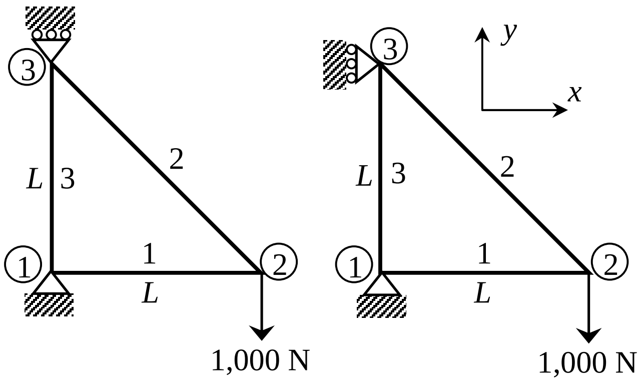
Element 1: $$A^{(1)} = \frac{\left|P^{(1)}\right|}{\sigma_{\max}} = \frac{75000}{10^4} = 7.5\,\text{cm}^2$$

Element 2: $$A^{(2)} = \frac{\left|P^{(2)}\right|}{\sigma_{\max}} = \frac{0}{10^4} = 0\,\text{cm}^2$$

Element 3: $$A^{(3)} = \frac{\left|P^{(3)}\right|}{\sigma_{\max}} = \frac{45000}{10^4} = 4.5\,\text{cm}^2$$

Note that Element 2 is a zero-force member. Thus, theoretically it can be removed. However, the truss will be unstable if member 2 is removed. In many designs, the minimum cross-sectional area is required.

4. It is desired to use FEM to solve the two plane truss problems shown in the figure below. Assume $AE = 10^6$ N, L = 1 m. Before solving the global equations, $[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$, find the determinant of $[\mathbf{K}]$. Does $[\mathbf{K}]$ have an inverse? Explain your answer.



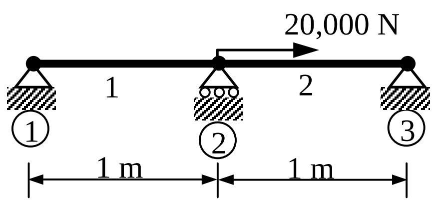
Solution:
After deleting rows and columns corresponding to the fixed DOFs, the above two trusses have the following global matrix equations:

Truss 1:
$$10^5 \begin{bmatrix} 14 & -3.5 & -3.5 \\ -3.5 & 3.5 & 3.5 \\ -3.5 & 3.5 & 3.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \\ 0 \end{Bmatrix}$$

Truss 2:
$$10^5 \begin{bmatrix} 14 & -3.5 & 3.5 \\ -3.5 & 3.5 & -3.5 \\ 3.5 & -3.5 & 14 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \\ 0 \end{Bmatrix}$$

After finding the determinant of the two stiffness matrices, the first set of boundary conditions are not valid for a static problem because the determinant of [**K**] is zero, and therefore the matrix cannot be inverted. The roller boundary condition of Truss 1 allows rotation of the entire truss as a rigid-body. Thus no unique solution for displacements can be obtained.

5. In the 1D bar shown below, the temperature of **Element 2** is **<u>100 °C above</u>** the reference temperature, while Element 1 is in the reference temperature. An external force of 20,000 N is applied in the *x*-direction (horizontal direction) at Node 2. Assume $E = 10^{11}$Pa, $A = 10^{-4}$ m^2, and $\alpha = 10^{-5}$ /°C.



(a) Write down the stiffness matrices and thermal force vectors for each element.

(b) Write down the global matrix equations.

(c) Solve the global equations to determine the displacement at Node 2.

(d) Determine the forces in each element. State whether it is tension or compression.

(e) Show that force equilibrium is satisfied at Node 2

Solution:
(a) Since the truss joints (nodes) are allowed to move only in the horizontal direction, we can consider the two members as uniaxial bar elements, which have two DOFs. The element stiffness matrices and thermal load vectors become

$$\left[\mathbf{k}^{(1)}\right] = 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \end{matrix}, \qquad \left[\mathbf{k}^{(2)}\right] = 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_2 \\ u_3 \end{matrix}$$

$$\left\{\mathbf{f}_T^{(1)}\right\} = AE\alpha\Delta T\begin{Bmatrix}-1\\1\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}\begin{matrix}u_1\\u_2\end{matrix}, \quad \left\{\mathbf{f}_T^{(2)}\right\} = AE\alpha\Delta T\begin{Bmatrix}-1\\1\end{Bmatrix} = 10^4\begin{Bmatrix}-1\\1\end{Bmatrix}\begin{matrix}u_2\\u_3\end{matrix}$$

(b) The global structural stiffness matrix is

$$\left[\mathbf{K}_s\right] = 10^7\begin{bmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\begin{matrix}u_1\\u_2\\u_3\end{matrix}$$

Then, the FE equations are

$$10^7\begin{bmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1 = 0\\u_2\\u_{3=0}\end{Bmatrix} = \begin{Bmatrix}R_1\\F_2 = 20,000\\R_3\end{Bmatrix} + \begin{Bmatrix}0\\-10,000\\10,000\end{Bmatrix}$$

(c) By deleting the first and last rows and columns, we have a scalar equation for u_2, as

$$10^7[2]\{u_2\} = \{10,000\}$$

The solution of the above equation is u_2 = 0.5 mm.

(d) Changes in element length: $\Delta L^{(1)} = 0.5\,\text{mm},\ \Delta L^{(2)} = -0.5\,\text{mm}$. Using $P = EA(\Delta L / L - \alpha\Delta T)$, the element forces can be found, as

$$P^{(1)} = AE(\frac{\Delta L}{L} - \alpha\Delta T) = 10^7 \times 5\times 10^{-4} = 5000\,\text{N(tension)}$$

$$P^{(2)} = AE(\frac{\Delta L}{L} - \alpha\Delta T) = 10^7(0.0005 - 0.001) = -15,000\,\text{N(compression)}$$

(e) At Node 2: $\sum F_x = -5000 - 15000 + 20000 = 0$