1. Solve Problem 2.5-3 using two beam elements. Write matrix equation after applying boundary conditions

At hinge, no rotational DOFs are connected. Thus, the global DOF should be $\{\mathbf{Q}_S\} = \{v_1, \theta_1, v_2, \theta_2^{(1)}, \theta_2^{(2)}, v_3, \theta_3\}^{\mathrm{T}}$. Since Node 1 and Node 3 are clamped, the free DOFs are $\{\mathbf{Q}\} = \{v_2, \theta_2^{(1)}, \theta_2^{(2)}\}^{\mathrm{T}}$

For Element 1 (only for free DOFs),

$$[\mathbf{k}^{(1)}] = \frac{EI_z}{a^3}\begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \end{bmatrix} \begin{matrix} v_2 \\ \theta_2^{(1)} \end{matrix}$$

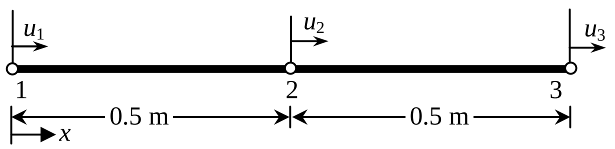
For Element 2 (only for free DOFs),

$$[\mathbf{k}^{(1)}] = \frac{EI_z}{b^3}\begin{bmatrix} 12 & -6b \\ -6b & 4b^2 \end{bmatrix} \begin{matrix} v_2 \\ \theta_2^{(2)} \end{matrix}$$

After assembly, the matrix equation becomes

$$\frac{EI_z}{a^3b^3}\begin{bmatrix} 12(a^3+b^3) & -6ab^3 & -6a^3b \\ -6ab^3 & 4a^2b^3 & 0 \\ -6a^3b & 0 & 4a^3b^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2^{(1)} \\ \theta_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}$$

2. Consider a bar element with three nodes, as shown in the figure. When the solution is approximated by $u(x) = N_1(x)u_1 + N_2(x)u_2 + N_3(x)u_3$, calculate interpolation functions $N_1(x), N_2(x), N_3(x)$. When a distributed load q0 is uniformly distributed on the element, calculate work-equivalent nodal forces.



Solution:

Since three nodes are available, we can use second-order approximation of the solution:

$$u(x) = a_0 + a_1x + a_2x^2$$

By imposing three nodal solutions, we have

$$\begin{aligned} u(0) &= u_1 = a_0 \\ u(\tfrac{1}{2}) &= u_2 = a_0 + \tfrac{1}{2}a_1 + \tfrac{1}{4}a_2 \\ u(1) &= u_3 = a_0 + a_1 + a_2 \end{aligned}$$

By solving a_1, a_2, and a_3 with respect to the nodal solution, we have

$$\begin{aligned} a_0 &= u_1 \\ a_1 &= -3u_1 + 4u_2 - u_3 \\ a_2 &= 2u_1 - 4u_2 + 2u_3 \end{aligned}$$

Thus, from the interpolation relation, we have

$$\begin{aligned} u(x) &= (1 - 3x + 2x^2)u_1 + 4(x - x^2)u_2 + (-x + 2x^2)u_3 \\ &= N_1u_1 + N_2u_2 + N_3u_3 \end{aligned}$$

where

$$\begin{aligned} N_1 &= 1 - 3x + 2x^2 \\ N_2 &= 4x - 4x^2 \\ N_3 &= -x + 2x^2 \end{aligned}$$

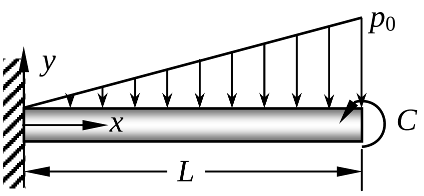
The work-equivalent load can be obtained from the expression of potential of applied load as

$$\begin{aligned} -V &= \int_0^1 q_0 u(x)\mathrm{d}x = u_1\left[q_0\int_0^1 N_1(x)\mathrm{d}x\right] + u_2\left[q_0\int_0^1 N_2(x)\mathrm{d}x\right] + u_3\left[q_0\int_0^1 N_3(x)\mathrm{d}x\right] \\ &= \tfrac{1}{6}q_0u_1 + \tfrac{2}{3}q_0u_2 + \tfrac{1}{6}q_0u_3 \end{aligned}$$

Thus, the work-equivalent load becomes

$$\{\mathbf{F}\} = \frac{q_0}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

3. Use the Rayleigh-Ritz method to determine the deflection $v(x)$, bending moment $M(x)$, and shear force $V_y(x)$ for the beam shown in the figure. Assume EI = 1,000 N-m^2, L = 1 m, and p_0 = 100 N/m, and C = 100 N-m. The displacement is expressed as $v(x) = c_0 + c_1x + c_2x^2 + c_3x^3$. Make sure the displacement boundary conditions are satisfied a priori. **Hint:** Potential energy of a couple is calculated as $V = -Cdv/dx$, where the rotation is calculated at the point of application of the couple.



Solution:

$$v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$$\frac{dv}{dx} = c_1 + 2c_2 x + 3c_3 x^2$$

Displacement boundary condition: $v(0) = 0,\ v'(0) = 0 \quad \Rightarrow c_0 = c_1 = 0$

$$v(x) = c_2 x^2 + c_3 x^3$$

Now we only need to determine c_2 and c_3.

$$\frac{dv}{dx} = 2c_2 x + 3c_3 x^2$$

$$\frac{d^2v}{dx^2} = 2c_2 + 6c_3 x$$

$$\frac{d^3v}{dx^3} = 6c_3$$

Potential energy:

$$\pi = \int_0^L \tfrac{1}{2} EI \left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L v p_y dx - \frac{dv}{dx}(1)C$$

$$= \int_0^L \tfrac{1}{2} EI \left(2c_2 + 6c_3 x\right)^2 dx - \int_0^L \left(c_2 x^2 + c_3 x^3\right)\left(-100x\right) dx - \left(2c_2 + 3c_3\right)C$$

$$= \int_0^1 2000\left(c_2^2 + 6c_2 c_3 x + 9{c_3}^2 x^2\right) dx + \int_0^1 \left(100c_2 x^3 + 100c_3 x^4\right) dx - 100\left(2c_2 + 3c_3\right)$$
$$= 2000\left(c_2^2 + 3c_2 c_3 + 3c_3^2\right) + \left(25c_2 + 20c_3\right) - 100\left(2c_2 + 3c_3\right)$$
$$= 2000\left(c_2^2 + 3c_2 c_3 + 3c_3^2\right) - 175c_2 - 280c_3$$

$$\begin{cases} \dfrac{\partial \pi}{\partial c_2} = 0 \Rightarrow 2000\left(2c_2 + 3c_3\right) - 175 = 0 \\ \dfrac{\partial \pi}{\partial c_3} = 0 \Rightarrow 2000\left(3c_2 + 6c_3\right) - 280 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_2 = 0.035 \\ c_3 = 0.005833 \end{cases}$$

Final results:

$$v(x) = 0.035x^2 + 0.005833x^3$$

$$M(x) = EI\,\frac{d^2v}{dx^2} = 1000\big(2\times 0.035 + 6\times 0.005833x\big) = 70 + 35x$$

$$V_y(x) = -EI\,\frac{d^3v}{dx^3} = -1000(6\times 0.005833) = -35$$

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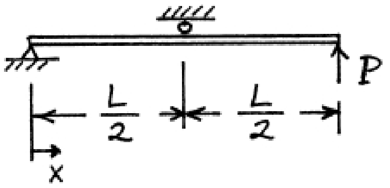
4. Solve problem 4.5-8.

4.5-8



$$v = a_0 + a_1 x + a_2 x^2$$

$v = 0$ at $x = 0$; $a_0 = 0$

$v = 0$ at $x = \frac{L}{2}$; $a_1\frac{L}{2} + a_2\frac{L^2}{4}$; $a_2 = -\frac{2}{L}a_1$

$$\therefore\ v = a_1\left(x - \frac{2}{L}x^2\right)$$

$$v_{,x} = a_1\left(1 - \frac{4}{L}x\right)$$

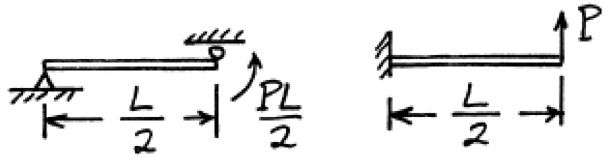
$$v_{,xx} = -a_1\frac{4}{L}$$

$$\Pi_p = \int_0^L \frac{EI}{2}\left(-a_1\frac{4}{L}\right)^2 dx - a_1\left(L - \frac{2}{L}L^2\right) = \frac{EI}{2}a_1^2\frac{16}{L^2}L + a_1 LP$$

$$\frac{d\Pi_p}{da_1} = 0 = \frac{16EI}{L}a_1 + PL\ ,\quad a_1 = -\frac{PL^2}{16EI}$$

At $x = L$, $v = -a_1 L = \frac{PL^3}{16EI}$

Elementary beam theory:



$$v = \theta \frac{L}{2} + \frac{P(L/2)^3}{3EI}$$

$$v = \frac{(PL/2)(L/2)}{3EI}\frac{L}{2} + \frac{PL^3}{24EI} = \frac{PL^3}{3EI}\left(\frac{1}{8}+\frac{1}{8}\right) = \frac{PL^3}{12EI}$$

$$\frac{1/16 - 1/12}{1/12}\,100\% = \left(\frac{3}{4} - 1\right)100\% = -25\%$$

$$M = EIv_{,xx} = EI\left(-\frac{4}{L}\right)\left(-\frac{PL^2}{16EI}\right) = \frac{PL}{4}$$

(for all x)

error is −50% at $x = \frac{L}{2}$

5. A space frame structure as shown in the figure consists of 25 truss members. All members have the same circular cross-sections with diameter d = 0.5 in. At nodes 1 and 2, a constant force F = 60,000 lb is applied in the y–direction. Four nodes (7, 8, 9, and 10) are fixed on the ground. The frame structure is made of a steel material whose properties are Young's modulus $E = 3\times10^7$ psi, Poisson's ratio $\nu = 0.3$. Calculate displacements of all nodes and stress of all members using finite element software. Provide a plot that shows labels for elements and nodes along with boundary conditions. Provide deformed geometry of the structure and a table of stress in each element.