1. Solve Problem 4.5-11 with a linear polynomial for u(x) and a quadratic polynomial for v(x). Hint: After applying for PMPE, you may get the following relation: bi * (expression) = 0, where bi is the unknown coefficient of polynomial. In order to get non-zero coefficient, you need to set expression = 0, which will give you the expression of buckling load.

$\boxed{4.5-11}$

$$u = a_1 x \qquad v = b_1 x^2$$

$$u_{,x} = a_1 \qquad v_{,x} = 2b_1 x$$

$$v_{,xx} = 2b_1$$

$$\Pi_p = \int_0^L \frac{AE}{L} a_1^2 dx + \int_0^L \frac{P}{2} 4b_1^2 x^2 dx + \int_0^L \frac{EI}{2} 4b_1^2 dx - P(a_1 L)$$

$$= \frac{AEL}{2} a_1^2 + \frac{2PL^3}{3} b_1^2 + 2EIL\, b_1^2 - PLa_1$$

$$\frac{\partial \Pi_p}{\partial a_1} = 0 = AELa_1 - PL \longrightarrow a_1 = \frac{P}{AE}, \quad \therefore \sigma_x = Eu_{,x} = \frac{P}{A}$$

$$\frac{\partial \Pi_p}{\partial b_1} = 0 = \frac{4PL^3}{3} b_1 + 4EILb_1$$

$$0 = b_1 \underbrace{\left(\frac{4PL^3}{3} + 4EIL\right)}_{\text{must vanish if } b_1 \neq 0} \longrightarrow P = -\frac{3EI}{L^2}$$

Exact P_{cr} is $P_{cr} = -\frac{\pi^2 EI}{4L^2} = -2.467\frac{EI}{L^2}$

2. Solve Problem 3.3-3

$\boxed{3.3-3}$ Eqs. 3.2-7: $x_1 = 0,\ x_2 = \frac{L}{2},\ x_3 = L$

$$N_1 = \frac{\left(\frac{L}{2}-x\right)(L-x)}{\frac{L}{2}L} = \frac{1}{L^2}\left(L^2-3Lx+2x^2\right)$$

$$N_2 = \frac{(-x)(L-x)}{-\frac{L}{2}\left(\frac{L}{2}\right)} = \frac{4}{L^2}\left(Lx-x^2\right)$$

$$N_3 = \frac{-x\left(\frac{L}{2}-x\right)}{-L\left(-\frac{L}{2}\right)} = \frac{1}{L^2}\left(2x^2-Lx\right)$$

$$\lfloor \underset{\sim}{B} \rfloor = \frac{d}{dx}\lfloor \underset{\sim}{N} \rfloor = \frac{1}{L^2}\lfloor -3L+4x \quad 4(L-2x) \quad 4x-L \rfloor$$

$$[\underset{\sim}{k}] = \int_0^L \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor AE\,dx$$

$$[\underset{\sim}{k}] = \frac{AE}{L^4}\int_0^L \begin{bmatrix} 9L-24Lx+16x^2 & -12L^2+40Lx-32x^2 & 3L^2-16Lx+16x^2 \\ & 16\left(L^2-4Lx+4x^2\right) & 4\left(-L^2+6Lx-8x^2\right) \\ \text{symmetric} & & L^2-8Lx+16x^2 \end{bmatrix} dx$$

$$[\underset{\sim}{k}] = \frac{AE}{L^4}\begin{bmatrix} 9L^2x-12Lx^2+\frac{16}{3}x^3 & -12L^2x+20Lx^2-\frac{32}{3}x^3 & 3L^2x-8Lx^2+\frac{16}{3}x^3 \\ & 16\left(L^2x-2Lx^2+\frac{4}{3}x^3\right) & 4\left(-L^2x+3Lx^2-\frac{8}{3}x^3\right) \\ \text{symmetric} & & L^2x-4Lx^2+\frac{16}{3}x^3 \end{bmatrix}_0^L$$

$$[\underset{\sim}{k}] = \frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

3. The boundary value problem for a cantilevered beam can be written as

$$\frac{d^4w}{dx^4} - p(x) = 0, \quad 0 < x < 1$$
$$w(0) = \frac{dw}{dx}(0) = 0, \ \frac{d^2w}{dx^2}(1) = 1, \ \frac{d^3w}{dx^3}(1) = -1 \text{: boundary condtions}$$

Assume $p(x) = x$. Assuming the approximate deflection in the form $\tilde{w}(x) = c_1x^2 + c_2x^3$. Solve for the boundary value problem using the Galerkin method. Compare the approximate solution to the exact solution by plotting the solutions on a graph.

Solution:

(a) Exact solution: By integrating the governing differential equation four times and applying four boundary conditions, we can obtain the exact solution, as

$$w(x) = \frac{1}{120}x^5 - \frac{1}{4}x^3 + \frac{7}{6}x^2$$

(b) Galerkin method

Using the second derivatives of the two trial functions, $\phi_1'' = 2, \phi_2'' = 6x$, we can calculate the coefficient matrix and the vector on the RHS, as

$$K_{11} = \int_0^1 (\phi_1'')^2\,dx = 4$$
$$K_{22} = \int_0^1 (\phi_2'')^2\,dx = 12$$
$$K_{12} = K_{21} = \int_0^1 \phi_1''\phi_2''dx = 6$$
$$F_1 = \int_0^1 x\phi_1 dx - w'''(1)\phi_1(1) + w'''(0)\phi_1(0) + w''(1)\phi_1'(1) - w''(0)\phi_1'(0) = \tfrac{13}{4}$$
$$F_2 = \int_0^1 x\phi_2 dx - w'''(1)\phi_2(1) + w'''(0)\phi_2(0) + w''(1)\phi_2'(1) - w''(0)\phi_2'(0) = \tfrac{21}{5}$$

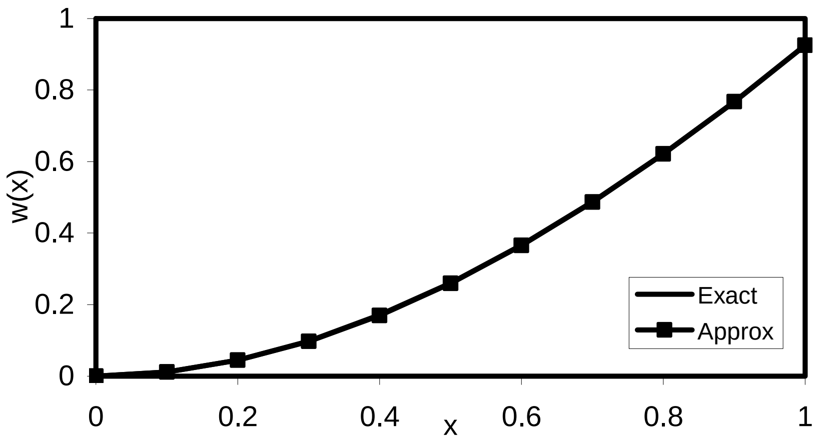
Thus, the matrix equation become

$$\begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} \frac{13}{4} \\ \frac{21}{5} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} \frac{31}{20} \\ -\frac{9}{40} \end{Bmatrix}$$

Thus, the approximate solution becomes

$$\tilde{w}(x) = \frac{23}{20}x^2 - \frac{9}{40}x^3$$

The exact and approximation solutions are shown in the figure below.



4. Solve Problem 3.2-3

$\boxed{3,2\text{-}3}$

(a) $\phi = \sum N_i \phi_i$; use Eqs. 3.2-7

$$\phi = \frac{(3-x)(5-x)(8-x)}{(2)(4)(7)}2 + \frac{(1-x)(5-x)(8-x)}{(-2)(2)(5)}2$$

$$+ \frac{(1-x)(3-x)(8-x)}{(-4)(-2)(3)}2 + \frac{(1-x)(3-x)(5-x)}{(-7)(-5)(-3)}5$$

(b) $$\phi_0 = \frac{3\cdot5\cdot8}{56}2 + \frac{1\cdot5\cdot8}{-20}2 + \frac{1\cdot3\cdot8}{24}2 + \frac{1\cdot3\cdot5}{-105}5$$

$$= \frac{120}{28} - \frac{40}{10} + \frac{24}{12} - \frac{15}{21} = 1.57$$

Similarly

$$\phi_2 = \frac{1\cdot3\cdot6}{28} - \frac{(-1)\cdot3\cdot6}{10} + \frac{(-1)\cdot1\cdot6}{12} - \frac{(-1)\cdot1\cdot3}{21} = 2.086$$

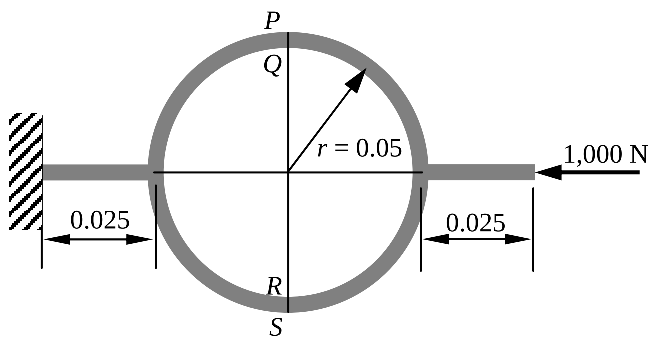
$$\phi_4 = \frac{(-1)\cdot1\cdot4}{28} - \frac{(-3)\cdot1\cdot4}{10} + \frac{(-3)\cdot(-1)\cdot4}{12} - \frac{(-3)\cdot(-1)\cdot1}{21} = 1.914$$

$$\phi_7 = \frac{(-4)\cdot(-2)\cdot1}{28} - \frac{(-6)\cdot(-2)\cdot1}{10} + \frac{(-6)\cdot(-4)\cdot1}{12} - \frac{(-6)\cdot(-4)\cdot(-2)}{21} = 3.371$$

5. The figure shown below depicts a load cell made of aluminum. The ring and the stem both have square cross-section: 0.1×0.1 m^2. Assume the Young's modulus is 72 GPa. The mean radius of the ring is 0.05 m. In a load cell, the axial load is measured from the average of strains at points *P*, *Q*, *R* and *S* as shown in the figure. Points *P* and *S* are on the outside surface, and *Q* and *R* are on the inside surface of the ring. Model the load cell using plane frame elements. The step portions may be modeled using one element each. Use about 20 elements to model the entire ring. Compute the axial strain ε_{xx} at locations *P*, *Q*, *R* and *S* for a load of 1,000 N. Draw the axial force, shear force and bending moment diagrams for one quarter of the ring. The strain can be computed using the beam formula:

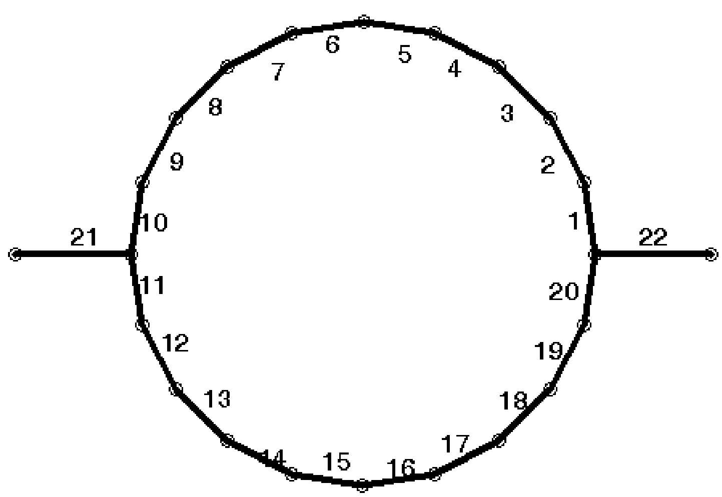
$$\sigma_{xx} = E\varepsilon_{xx} = \frac{P}{A} \pm \frac{Mc}{I}$$

where *P* is the axial force, *M* the bending moment, *A* the cross-sectional area, *I* the moment of inertia, and *c* the distance from the midplane.



Solution:

We solved the problem using MATLAB toolbox with 22 frame elements. The finite element model and element numbers are shown in the figure below.



The axial force and bending moment at point P, Q, R, and S are the same and given as

$$P = -493.844 \text{ N}, \quad M = 9.2156 \text{ N} \cdot \text{m}$$

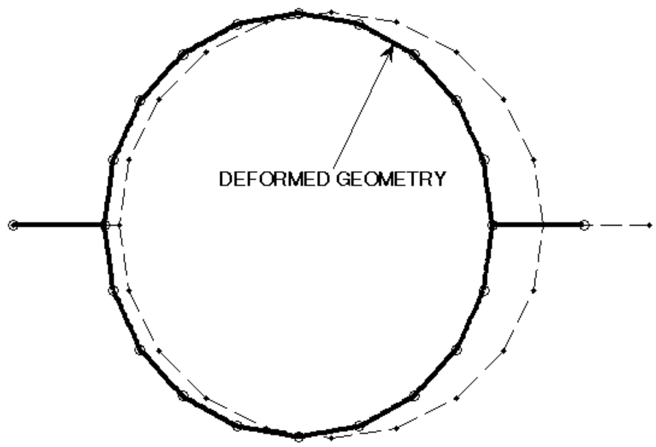
The Points *P* and *S* have the same strain value, and they can be calculated using the above equation as

$$(\varepsilon_{xx})_P = (\varepsilon_{xx})_S = \frac{-493.844}{72 \times 10^9 \times 0.01} + \frac{9.2156 \times .05}{72 \times 10^9 \times 8.33 \times 10^{-6}} = 8.24 \times 10^{-8}$$

On the other hand, the strains in Points Q and R are

$$(\varepsilon_{xx})_Q = (\varepsilon_{xx})_R = \frac{-493.844}{72\times10^9\times0.01} - \frac{9.2156\times.05}{72\times10^9\times8.33\times10^{-6}} = -1.45\times10^{-6}$$

There is a large difference in strains between inside and outside of the cell.
Deformed geometry with magnification is shown in the figure below:



6. In Problem 5, assume that load is applied eccentrically; i.e., the distance between the line of action of the applied force and the center line of load cell, e, is not equal to zero. Calculate the strain at *P*, *Q*, *R* and S for *e* = 0.002 m. What is the average of these strains? Comment on the results. Note: The eccentric load can be replaced by a central load and a couple of 1000×*e* N·m.

Solution:

The same program in Problem 5 can be used for this problem. The only difference is adding additional couple caused by eccentric load. Thus, add the following command: F(66) = 2.0. Due to eccentricity, the axial force and moment are different the two locations. They are given as

$$\begin{aligned} P_{PQ} &= -512.78 \text{ N}, \quad M_{PQ} = 9.1744 \text{ N}\cdot\text{m} \\ P_{RS} &= -474.90 \text{ N}, \quad M_{RS} = 9.2568 \text{ N}\cdot\text{m} \end{aligned}$$

Now, all strain components are different.

$$\begin{aligned} (\varepsilon_{xx})_P &= \frac{-512.78}{72\times10^9\times0.01} + \frac{9.1744\times.05}{72\times10^9\times8.33\times10^{-6}} = 5.24\times10^{-8} \\ (\varepsilon_{xx})_Q &= \frac{-512.78}{72\times10^9\times0.01} - \frac{9.1744\times.05}{72\times10^9\times8.33\times10^{-6}} = -1.48\times10^{-6} \end{aligned}$$

$$
\begin{aligned}
(\varepsilon_{xx})_S &= \frac{-474.90}{72 \times 10^9 \times 0.01} + \frac{9.2568 \times .05}{72 \times 10^9 \times 8.33 \times 10^{-6}} = 11.19 \times 10^{-8} \\
(\varepsilon_{xx})_R &= \frac{-474.90}{72 \times 10^9 \times 0.01} - \frac{9.2568 \times .05}{72 \times 10^9 \times 8.33 \times 10^{-6}} = -1.43 \times 10^{-6}
\end{aligned}
$$

Although the inside strains change significantly, their average value is very close to the inside strain in Problem 5. The outside strains vary less than 3%. In addition, the average value of outside strains is very close to that of Problem 31.