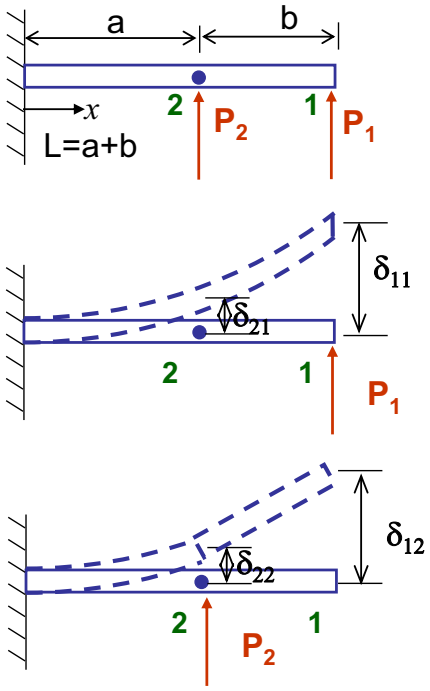
Why symmetry?

- Symmetry is often argued from the requirement that the strain energy must be positive. (e.g. Generalized 3-D Hooke's law)
- One of the derivatives of energy principles is the Betti-Maxwell reciprocity theorem. (Very useful in structural mechanics)
- Betti-Maxwell Reciprocal theorem:
  - If two load sets act on a linearly elastic structure, work done by the first set of loads in acting through the displacements produced by the second set of loads is equal to the work done by the second set of loads in acting through displacements produced by the first set.

Example for Betti-Maxwell Reciprocal Theorem



For this system the Betti_Maxwell Reciprocal theorem provides the condition

$$P_1\delta_{12} = P_2\delta_{21}$$

The displacement v and rotation θ of an end loaded cantilever beam is given by

$$v = \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \qquad \theta = \frac{P}{EI}\left(Lx - \frac{x^2}{2}\right)$$

Using these we can write the displacements as

$$\delta_{11} = \frac{P_1(a+b)^3}{3EI} \qquad \delta_{21} = \frac{P_1 a^2}{6EI}(2a+3b)$$

$$\delta_{22} = \frac{P_2 a^3}{3EI} \qquad \delta_{12} = \frac{P_2 a^3}{3EI} + \frac{P_2 a^2 b}{2EI}$$

Betti-Maxwell theorem applied to finite element equations

The theorem is same as before, except now we use a system of equations

$$\{\mathbf{R}_1\}^T\{\mathbf{D}_2\} = \{\mathbf{R}_2\}^T\{\mathbf{D}_1\}$$

The terms above are scalar quantities (representing work done)

If we can expand the equations by substituting for the force vectors R in terms of the stiffness matrix **K** and displacement vector **D**

$$([\mathbf{K}]\{D_1\})^T\{\mathbf{D}_2\} = ([\mathbf{K}]\{D_2\})^T\{\mathbf{D}_1\}$$

$$\{D_1\}^T[\mathbf{K}]^T\{\mathbf{D}_2\} = \{D_2\}^T[\mathbf{K}]^T\{\mathbf{D}_1\}$$

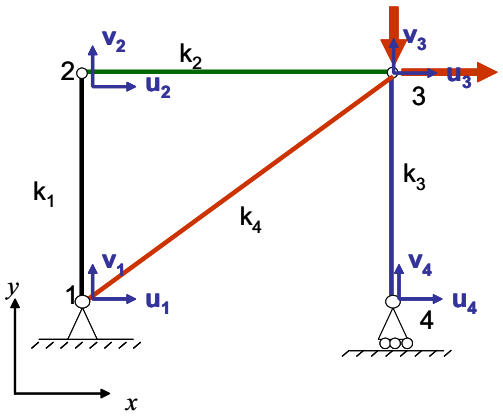
Since they are scalar terms the transpose should be the same

$$\{D_1\}^T[\mathbf{K}]^T\{\mathbf{D}_2\} = \{D_1\}^T[\mathbf{K}]\{\mathbf{D}_2\} \Rightarrow [\mathbf{K}]^T = [\mathbf{K}]$$

Sparsity

- Sparsity is a term used to quantify the number of zeros in a stiffness matrix.
- In very large models only a few nodes are connected to each other. This creates a lot of zeros.
- To economize storage (computer memory) commercial programs often use different ways to avoid storing the zero locations.
- Node numbering in a FE model affects the topology of the stiffness matrix
  - For a linear assembly of bar or beam elements you obtain a banded matrix.
  - However for most 3-D complex structures the bandwidth increases and can often result in a “dense” matrix.
- We need to find a topology that favors storage and efficient solving of the equation

Skyline of a Matrix



$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & K_{15} & K_{16} & 0 & 0 \\ K_{21} & K_{22} & 0 & K_{24} & K_{25} & K_{26} & 0 & 0 \\ 0 & 0 & K_{33} & 0 & K_{35} & 0 & 0 & 0 \\ 0 & K_{42} & 0 & K_{44} & 0 & 0 & 0 & 0 \\ K_{51} & K_{52} & K_{53} & 0 & K_{55} & K_{56} & 0 & 0 \\ K_{61} & K_{62} & 0 & 0 & K_{65} & K_{66} & 0 & K_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{68} & 0 & K_{88} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{Bmatrix}$$

- Due to symmetry only terms on the diagonal and above need to be stored.
- Skyline of a matrix encloses the uppermost nonzero coefficients in each column
- The coefficients are then stored in a one-dimensional array.
- The array that describes the profile of the matrix is needed to develop a 1-D storage for the stiffness matrix
- Zeros under the skyline need to be stored as they will become filled in the solution process

Profile of a matrix

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & K_{15} & K_{16} & 0 & 0 \\ K_{21} & K_{22} & 0 & K_{24} & K_{25} & K_{26} & 0 & 0 \\ 0 & 0 & K_{33} & 0 & K_{35} & 0 & 0 & 0 \\ 0 & K_{42} & 0 & K_{44} & 0 & 0 & 0 & 0 \\ K_{51} & K_{52} & K_{53} & 0 & K_{55} & K_{56} & 0 & 0 \\ K_{61} & K_{62} & 0 & 0 & K_{65} & K_{66} & 0 & K_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{68} & 0 & K_{88} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{Bmatrix}$$

The auxiliary array that describes the skyline is

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 5 & 6 & 0 & 3 \end{bmatrix}$$

The sum of the entries is the profile description is the number of coefficients that must be stored and is referred to as the profile of the matrix
The profile for the above case is 21

Note: See textbook example in Figure 2.8-1 on how you can change the profile

Bandwidth of the matrix

$$\begin{matrix}6\\5\\3\\1\\2\\3\\1\\1\end{matrix}\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & K_{15} & K_{16} & 0 & 0 \\ K_{21} & K_{22} & 0 & K_{24} & K_{25} & K_{26} & 0 & 0 \\ 0 & 0 & K_{33} & 0 & K_{35} & 0 & 0 & 0 \\ 0 & K_{42} & 0 & K_{44} & 0 & 0 & 0 & 0 \\ K_{51} & K_{52} & K_{53} & 0 & K_{55} & K_{56} & 0 & 0 \\ K_{61} & K_{62} & 0 & 0 & K_{65} & K_{66} & 0 & K_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{68} & 0 & K_{88} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{Bmatrix}$$

- Semi-bandwith b_i of any row i is equal to the number of columns from the diagonal to the rightmost non-zero term
- A matrix of large profile also has a large bandwidth
- The root mean square of the semi-bandwidths of the rows is used as a measure of the bandwidth of the matrix
- For our example the bandwidth is 3.28
- The solution procedure has to create fewer fills in the matrix when the coefficients are tightly clustered around the diagonal

Solution of Equations

- We have developed FE structural equations of the form

$$[\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{R}\}$$

- and indicated that this can be solved (when the system is non-singular) as

$$\{\mathbf{D}\} = [\mathbf{K}]^{-1}\{\mathbf{R}\}$$

- This must be simply interpreted as solution of the equation set.
- Seldom do we need to obtain the inverse matrix and multiply them as shown above.
- Solution procedure are of two types:
  - Direct solvers (e.g. Gauss elimination)
  - Iterative solvers (e.g. Gauss-Siedel iteration)

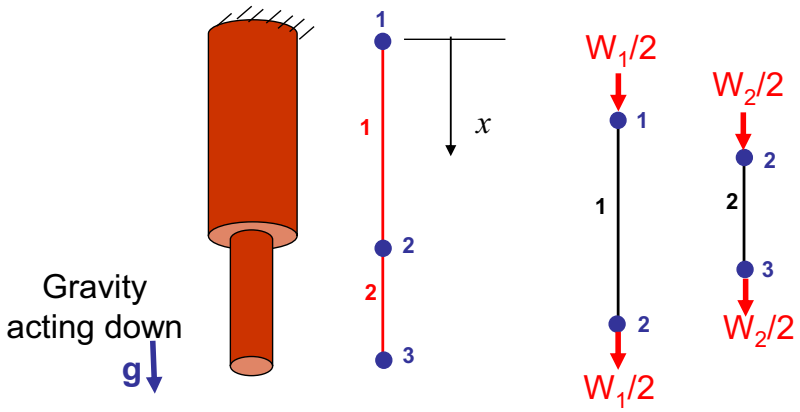
Direct Solvers

- Direct solvers use methods to transform the equations into an upper or lower triangle matrix that facilitates the solution (Gauss elimination), or decomposes the matrix into a product of upper and lower triangle (LU decomposition)
- The effort of solving a system of equations using the direct solvers is a function of its sparsity and profile (or bandwidth)
- Typically the number of operation required to solve a nxn matrix system of equations is nb^2 when the matrix bandwidth is b. (For dense matrix this is $n^3/3$)
- Review basics of Gauss elimination
- For very large structures, the matrix equations can be solved even before the entire equation can be assembled. These methods are called frontal solvers.

Iterative solvers

- These methods start with a guess and iterate till it converges to a solution
  - (e.g Newton-Raphson method for solving algebraic equations)
  - For matrices review Gauss-Siedel Iteration
- The effort required often is difficult to predict in iterative solvers
- The convergence depends on the “condition number” of the stiffness matrix
- Condition number is the ratio of the largest and smallest eigenvalues of the stiffness matrix
- In structures the eigenvalues relate to the natural frequencies of the structure
- Pre-conditioned iterative solvers transform the system of equations to improve its conditioning before solving it.

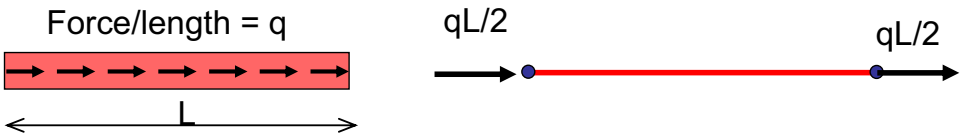
What if the applied load is not a concentrated load?



- The nodal loads can be calculated using equivalent physical deformations or energy
- Note:
  - The weight of each element is assigned to the nodes
  - In the actual structure the stress/strain is zero at the free end where as there is some strain at the element having the free end
  - This error is reduced if the elements are made smaller

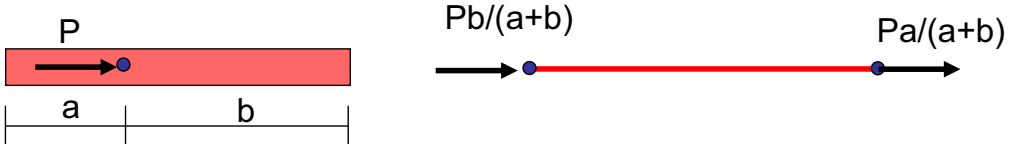
Loads on Bar Elements

Uniformly loaded bar



The strain variation in a uniformly loaded bar is linear.
The bar element only allows a constant variation.
For accuracy more elements are needed.

Bar loaded between nodes



Often required because mesh does not include a node at the point of force application

Boundary Conditions

Lets us consider a model with 6 DOF. The assembled stiffness matrix must appear as follows, where $K_{ij}=K_{ji}$.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{Bmatrix}$$

Let us assume that the DOFs u_1, u_3 and v_3 are known

The equations can be re-organized as follows

$$\left[\begin{array}{ccc|ccc} K_{22} & K_{23} & K_{24} & K_{21} & K_{25} & K_{26} \\ K_{32} & K_{33} & K_{34} & K_{31} & K_{35} & K_{36} \\ K_{42} & K_{43} & K_{44} & K_{41} & K_{45} & K_{46} \\ \hline K_{1} & K_{13} & K_{14} & K_{11} & K_{15} & K_{16} \\ K_{52} & K_{53} & K_{54} & K_{51} & K_{55} & K_{56} \\ K_{62} & K_{63} & K_{64} & K_{61} & K_{65} & K_{66} \end{array}\right] \left\{\begin{array}{c} v_1 \\ u_2 \\ v_2 \\ \hline u_1 \\ u_3 \\ v_3 \end{array}\right\} = \left\{\begin{array}{c} f_{y1} \\ f_{x2} \\ f_{y2} \\ \hline f_{x1} \\ f_{x3} \\ f_{y3} \end{array}\right\}$$

Partitioning of FE equations

The partitioned set of equations and be expanded and denoted by the following matrix form

$$\begin{bmatrix} K_{22} & K_{22} & K_{22} \\ K_{32} & K_{33} & K_{34} \\ K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ u_2 \\ v_2 \end{Bmatrix} + \begin{bmatrix} K_{21} & K_{25} & K_{26} \\ K_{31} & K_{35} & K_{36} \\ K_{41} & K_{45} & K_{46} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix}$$

$$\begin{bmatrix} K_{12} & K_{13} & K_{14} \\ K_{52} & K_{53} & K_{54} \\ K_{62} & K_{63} & K_{64} \end{bmatrix} \begin{Bmatrix} v_1 \\ u_2 \\ v_2 \end{Bmatrix} + \begin{bmatrix} K_{21} & K_{25} & K_{26} \\ K_{51} & K_{55} & K_{56} \\ K_{61} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{x3} \\ f_{y3} \end{Bmatrix}$$

The equations for the unknown displacements are then

$$\begin{bmatrix} K_{22} & K_{22} & K_{22} \\ K_{32} & K_{33} & K_{34} \\ K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix} - \begin{bmatrix} K_{21} & K_{25} & K_{26} \\ K_{31} & K_{35} & K_{36} \\ K_{41} & K_{45} & K_{46} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \\ v_3 \end{Bmatrix}$$

Note: When all the specified DOF take zero values then the above form is the same as obtained by the row-column elimination procedure

Compact representation of the partitioning method

The partitioned stiffness matrix and force/displacement vectors can be denoted as follows.

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{D}_x \\ \mathbf{D}_c \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_c \\ \mathbf{R}_x \end{Bmatrix}, \quad \begin{matrix} \mathbf{D}_x, \mathbf{R}_x \text{ are the unknown displacements and forces.} \\ \mathbf{D}_c, \mathbf{R}_c \text{ are the unknown displacements and forces.} \end{matrix}$$

Note:
In above equations $\mathbf{K}_{ij}$, $\mathbf{D}_x$, $\mathbf{D}_c$, $\mathbf{R}_x$, $\mathbf{R}_c$ denote matrices and vectors and not entries of a matrix or vector as before (hence the bold lettering)
When the displacements are specified, the reactions forces are unknown
When Forces are specified the displacements are unknown

The solution can now be written as

$$\mathbf{D}_x = [\mathbf{K}_{11}]^{-1}(\{\mathbf{R}_c\} - [\mathbf{K}_{12}]\{\mathbf{D}_c\})$$

The first set is used to get displacements

$$\mathbf{R}_x = [\mathbf{K}_{21}]\{\mathbf{D}_x\} + [\mathbf{K}_{22}]\{\mathbf{D}_c\}$$

The second set is used to calculate reaction forces