CHAP 4 FINITE ELEMENT ANALYSIS OF BEAMS AND FRAMES

INTRODUCTION

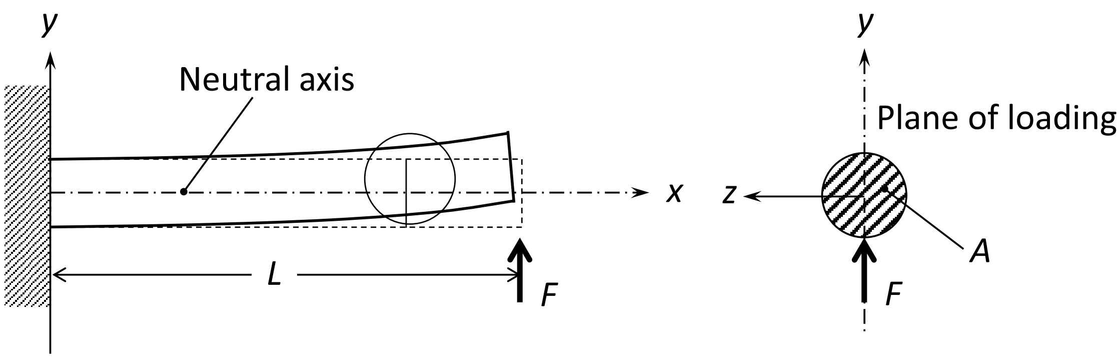
- We learned **Direct Stiffness Method** in Chapter 2
  - Limited to simple elements such as 1D bars
- we will learn **Energy Method** to build beam finite element
  - Structure is in equilibrium when the potential energy is minimum
- **Potential energy**: Sum of strain energy and potential of applied loads
- Interpolation scheme:

$$\Pi = \underbrace{U}_{\text{Strain energy}} + \underbrace{V}_{\text{Potential of applied loads}}$$

$$\underbrace{v(x)}_{\text{Beam deflection}} = \underbrace{\lfloor \mathbf{N}(x) \rfloor}_{\text{Interpolation function}} \cdot \underbrace{\{\mathbf{q}\}}_{\text{Nodal DOF}}$$

BEAM THEORY

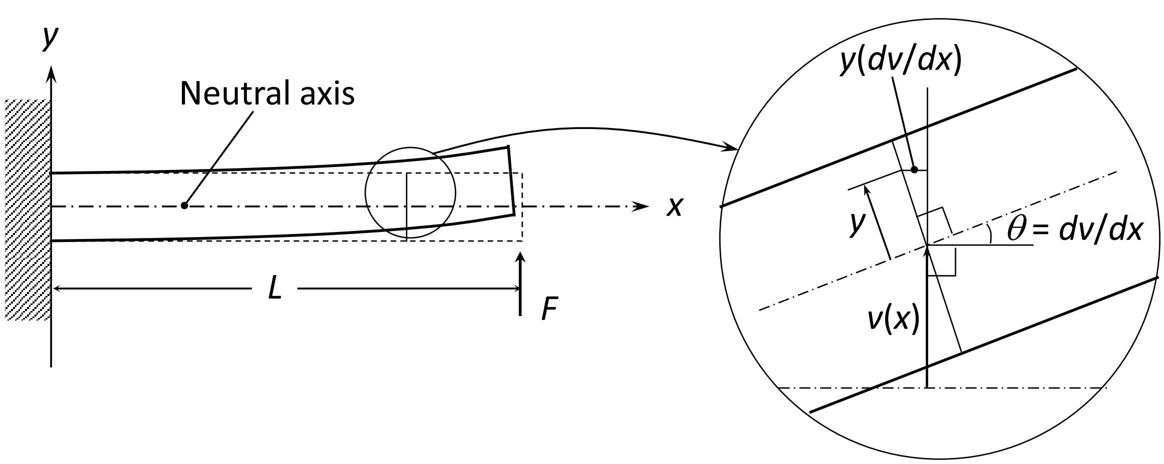
- Euler-Bernoulli Beam Theory
  - can carry the transverse load
  - slope can change along the span (x-axis)
  - Cross-section is symmetric w.r.t. xy-plane
  - The y-axis passes through the centroid
  - Loads are applied in xy-plane (plane of loading)



BEAM THEORY *cont.*

- Euler-Bernoulli Beam Theory cont.
  - Plane sections normal to the beam axis remain plane and normal to the axis after deformation (no shear stress)
  - Transverse deflection (deflection curve) is function of x only: ***v*(*x*)**
  - Displacement in x-dir is function of x and y: ***u*(*x*, *y*)**

$$u(x,y) = u_0(x) - y\frac{dv}{dx} \quad \Rightarrow \quad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} - y\frac{d^2v}{dx^2} \qquad \theta = \frac{dv}{dx}$$



BEAM THEORY *cont.*

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} - y\frac{d^2v}{dx^2}$$

- Euler-Bernoulli Beam Theory cont.
  - Strain along the beam axis: $\varepsilon_0 = du_0 / dx$
  - Strain ε_{xx} varies linearly w.r.t. y; Strain ε_{yy} = 0
  - Curvature: $-d^2v/dx^2$
  - Can assume plane stress in z-dir $\Longrightarrow$ basically uniaxial status

$$\sigma_{xx} = E\varepsilon_{xx} = E\varepsilon_0 - Ey\frac{d^2v}{dx^2}$$

- Axial force resultant and bending moment

$$P = \int_A \sigma_{xx} dA = E\varepsilon_0 \int_A dA - E\frac{d^2v}{dx^2}\cancel{\int_A y dA}$$

$$M = -\int_A y\sigma_{xx} dA = -E\varepsilon_0 \cancel{\int_A y dA} + E\frac{d^2v}{dx^2}\underbrace{\boxed{\int_A y^2 dA}}_{\text{Moment of inertia } I(x)}$$

$\Longrightarrow$

$$P = EA\varepsilon_0$$

$$M = EI\frac{d^2v}{dx^2}$$

EA: axial rigidity

EI: flexural rigidity

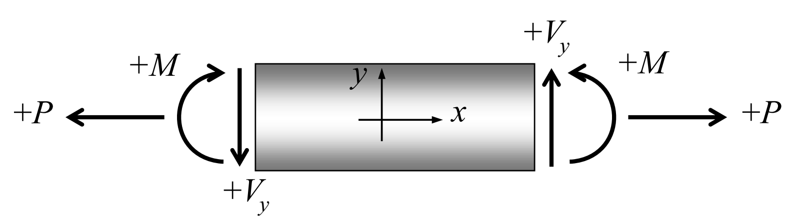
BEAM THEORY *cont.*

- Beam constitutive relation
  - We assume *P* = 0 (We will consider non-zero P in the frame element)
  - Moment-curvature relation:

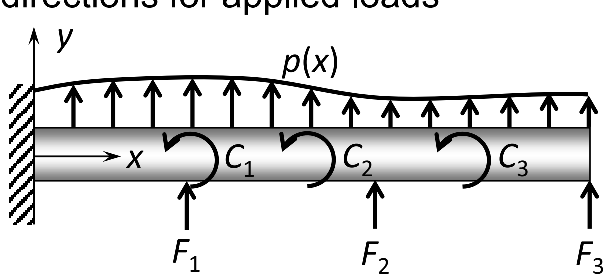
$$M = EI\frac{d^2v}{dx^2}$$

Moment and curvature is linearly dependent

- Sign convention



  - Positive directions for applied loads



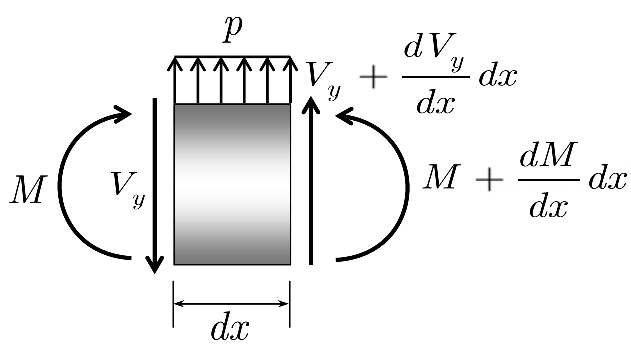
GOVERNING EQUATIONS

- Beam equilibrium equations

$$\sum f_y = 0 \Rightarrow p(x)dx + \left(V_y + \frac{dV_y}{dx}dx \right) - V_y = 0 \quad \Longrightarrow \quad \boxed{\frac{dV_y}{dx} = -p(x)}$$

$$-M + \left(M + \frac{dM}{dx}dx \right) - (pdx)\frac{dx}{2} + V_y dx = 0 \quad \Longrightarrow \quad \boxed{V_y = -\frac{dM}{dx}}$$

  - Combining three equations together: $\boxed{EI\frac{d^4v}{dx^4} = p(x)}$
  - Fourth-order differential equation



STRESS AND STRAIN

- Bending stress

$$\sigma_{xx} = -Ey\frac{d^2v}{dx^2} \quad \Longleftrightarrow \quad M = EI\frac{d^2v}{dx^2}$$

$$\boxed{\sigma_{xx}(x,y) = -\frac{M(x)y}{I}} \quad \text{Bending stress}$$

  - This is only non-zero stress component for Euler-Bernoulli beam

- Transverse shear strain

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} = 0 \quad \Longleftarrow \quad u(x,y) = u_0(x) - y\frac{dv}{dx}$$

  - Euler beam predicts zero shear strain (approximation)
  - Traditional beam theory says the transverse shear stress is $\tau_{xy} = \frac{VQ}{Ib}$
  - However, this shear stress is in general small compared to the bending stress

POTENTIAL ENERGY

- Potential energy $\Pi = U + V$
- Strain energy
  - Strain energy density

$$U_0 = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} = \frac{1}{2}E(\varepsilon_{xx})^2 = \frac{1}{2}E\left(-y\frac{d^2v}{dx^2}\right)^2 = \frac{1}{2}Ey^2\left(\frac{d^2v}{dx^2}\right)^2$$

  - Strain energy per unit length

$$U_L(x) = \int_A U_0(x,y,z)\,dA = \int_A \frac{1}{2}Ey^2\left(\frac{d^2v}{dx^2}\right)^2 dA = \frac{1}{2}E\left(\frac{d^2v}{dx^2}\right)^2 \boxed{\int_A y^2\,dA}$$

Moment of inertia

$$\boxed{U_L(x) = \frac{1}{2}EI\left(\frac{d^2v}{dx^2}\right)^2}$$

  - Strain energy

$$\boxed{U = \int_0^L U_L(x)\,dx = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx}$$

POTENTIAL ENERGY *cont.*

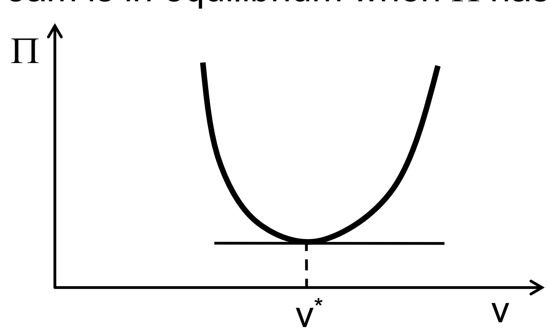
- Potential energy of applied loads

$$V = -\int_0^L p(x)v(x)\,dx - \sum_{i=1}^{N_F} F_i v(x_i) - \sum_{i=1}^{N_C} C_i \frac{dv(x_i)}{dx}$$

- Potential energy

$$\boxed{\Pi = U + V = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L p(x)v(x)\,dx - \sum_{i=1}^{N_F} F_i v(x_i) - \sum_{i=1}^{N_C} C_i \frac{dv(x_i)}{dx}}$$

  - Potential energy is a function of $v(x)$ and slope
  - The beam is in equilibrium when Π has its minimum value



$$\frac{\partial \Pi}{\partial v} = 0$$

RAYLEIGH-RITZ METHOD

1. Assume a deflection shape

$$v(x) = c_1 f_1(x) + c_2 f_2(x) \ldots\ldots + c_n f_n(x)$$

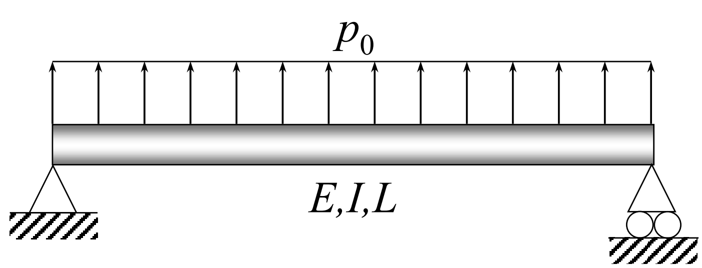
   – Unknown coefficients c_i and known function $f_i(x)$
   – Deflection curve $v(x)$ must satisfy displacement boundary conditions

2. Obtain potential energy as function of coefficients

$$\Pi(c_1, c_2, \ldots c_n) = U + V$$

3. Apply the principle of minimum potential energy to determine the coefficients

EXAMPLE – SIMPLY SUPPORTED BEAM



- Assumed deflection curve

$$v(x) = C \sin \frac{\pi x}{L}$$

- Strain energy

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2 v}{dx^2}\right)^2 dx = \frac{C^2 EI \pi^4}{4L^3}$$

- Potential energy of applied loads (no reaction forces)

$$V = -\int_0^L p(x)v(x)dx = -\int_0^L p_0 C \sin \frac{\pi x}{L} dx = -\frac{2p_0 L}{\pi} C$$

- Potential energy $\Pi = U + V = \dfrac{EI\pi^4}{4L^3} C^2 - \dfrac{2p_0 L}{\pi} C$

- PMPE: $\dfrac{d\Pi}{dC} = \dfrac{EI\pi^4}{2L^3} C - \dfrac{2p_0 L}{\pi} = 0 \quad \Rightarrow \quad C = \dfrac{4p_0 L^4}{EI\pi^5}$

EXAMPLE – SIMPLY SUPPORTED BEAM *cont.*

- Exact vs. approximate solutions

$$C_{\text{approx}} = \frac{p_0 L^4}{76.5EI} \qquad C_{\text{exact}} = \frac{p_0 L^4}{76.8EI}$$

- Approximate bending moment and shear force

$$M(x) = EI\frac{d^2v}{dx^2} = -EIC\frac{\pi^2}{L^2}\sin\frac{\pi x}{L} = -\frac{4p_0L^2}{\pi^3}\sin\frac{\pi x}{L}$$

$$V_y(x) = -EI\frac{d^3v}{dx^3} = -EIC\frac{\pi^3}{L^3}\cos\frac{\pi x}{L} = -\frac{4p_0L}{\pi^2}\cos\frac{\pi x}{L}$$

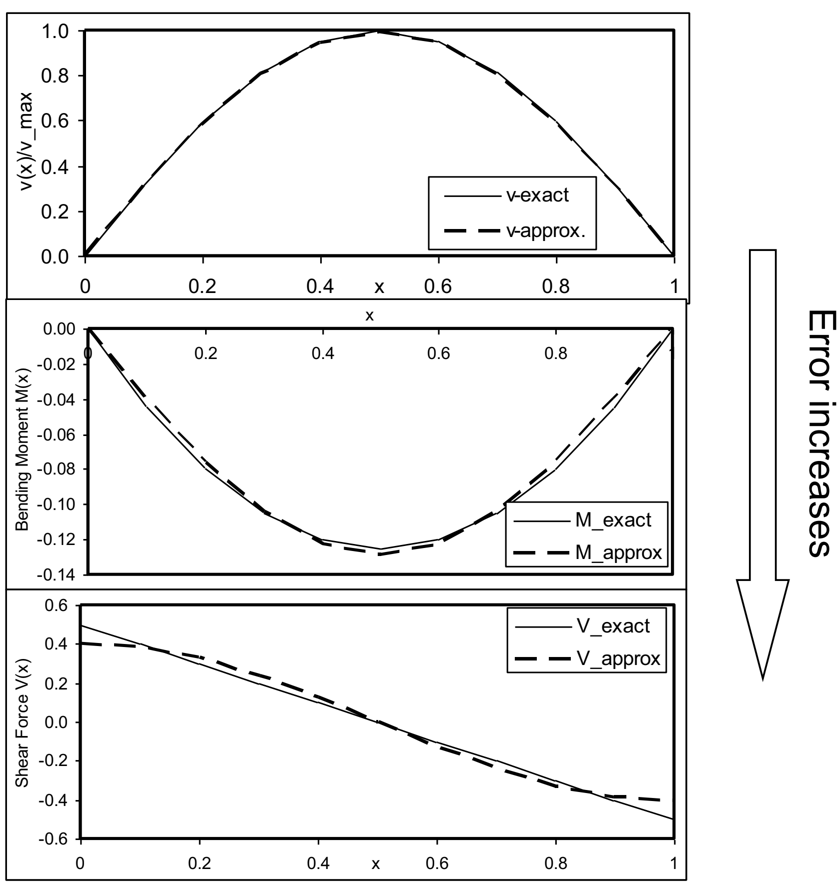
- Exact solutions $v(x) = \dfrac{1}{EI}\left(\dfrac{p_0L^3}{24}x - \dfrac{p_0L}{12}x^3 + \dfrac{p_0}{24}x^4\right)$

$$M(x) = -\frac{p_0L}{2}x + \frac{p_0}{2}x^2$$

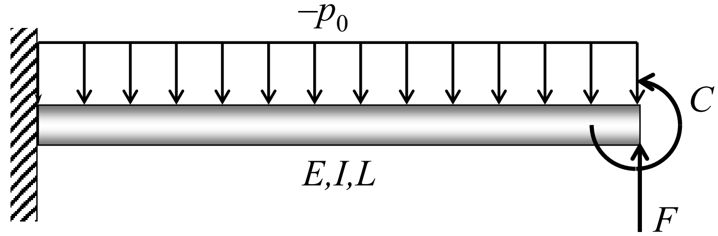
$$V_y(x) = \frac{p_0L}{2} - p_0x$$

EXAMPLE – SIMPLY SUPPORTED BEAM *cont.*

- Deflection
- Bending moment
- Shear force



EXAMPLE – CANTILEVERED BEAM



- Assumed deflection

  $v(x) = a + bx + c_1x^2 + c_2x^3$

- Need to satisfy BC

  $v(0) = 0,\ dv(0)/dx = 0 \implies v(x) = c_1x^2 + c_2x^3$

- Strain energy $U = \dfrac{EI}{2}\displaystyle\int_0^L (2c_1 + 6c_2x)^2\,dx$

- Potential of loads

$$V(c_1, c_2) = -\int_0^L (-p_0)v(x)dx - Fv(L) - C\frac{dv}{dx}(L)$$

$$= c_1\left(\frac{p_0L^3}{3} - FL^2 - 2CL\right) + c_2\left(\frac{p_0L^4}{4} - FL^3 - 3CL^2\right)$$

EXAMPLE – CANTILEVERED BEAM *cont.*

- Derivatives of *U*: $\dfrac{\partial U}{\partial c_1} = 2EI\displaystyle\int_0^L (2c_1 + 6c_2x)dx = EI\left(4Lc_1 + 6L^2c_2\right)$

$$\frac{\partial U}{\partial c_2} = 6EI\int_0^L (2c_1 + 6c_2x)x\,dx = EI\left(6L^2c_1 + 12L^3c_2\right)$$

- PMPE: $\dfrac{\partial \Pi}{\partial c_1} = 0$, $\dfrac{\partial \Pi}{\partial c_2} = 0$ $\implies$

$$EI\left(4Lc_1 + 6L^2c_2\right) = -\frac{p_0L^3}{3} + FL^2 + 2CL$$

$$EI\left(6L^2c_1 + 12L^3c_2\right) = -\frac{p_0L^4}{4} + FL^3 + 3CL^2$$

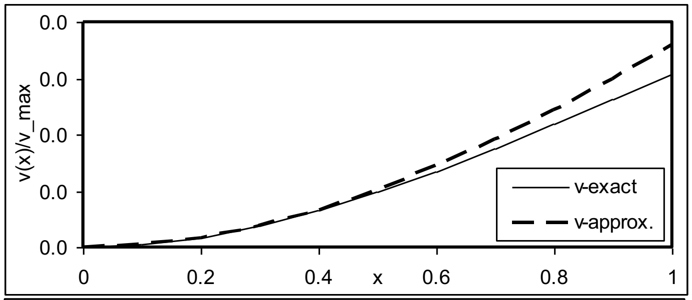
- Solve for c_1 and c_2: $c_1 = 23.75\times10^{-3},\ c_2 = -8.417\times10^{-3}$

- Deflection curve: $v(x) = 10^{-3}\left(23.75x^2 - 8.417x^3\right)$

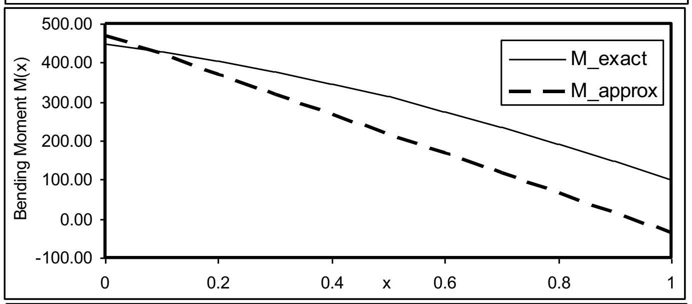
- Exact solution: $v(x) = \dfrac{1}{24EI}\left(5400x^2 - 800x^3 - 300x^4\right)$

EXAMPLE – CANTILEVERED BEAM *cont.*

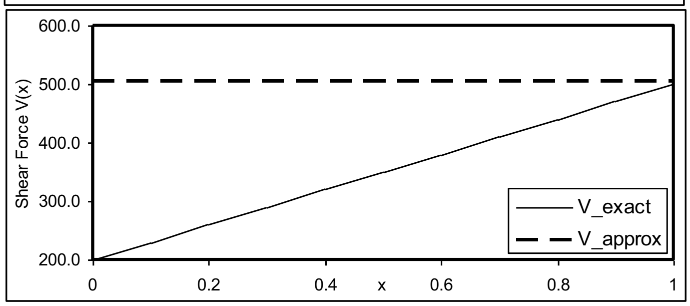
- Deflection

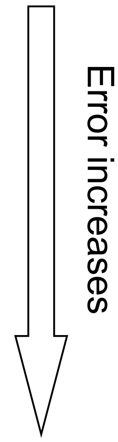


- Bending moment



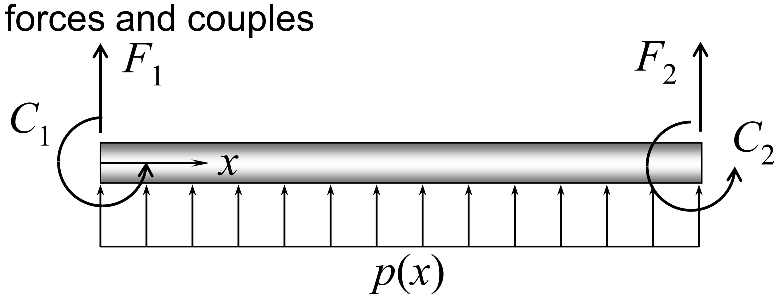
- Shear force





FINITE ELEMENT INTERPOLATION

- Rayleigh-Ritz method approximate solution in the entire beam
  - Difficult to find approx solution that satisfies displacement BC
- Finite element approximates solution in an element
  - Make it easy to satisfy displacement BC using interpolation technique
- Beam element
  - Divide the beam using a set of elements
  - Elements are connected to other elements at nodes
  - Concentrated forces and couples can only be applied at nodes
  - Consider two-node bean element
  - Positive directions for forces and couples
  - Constant or linearly distributed load

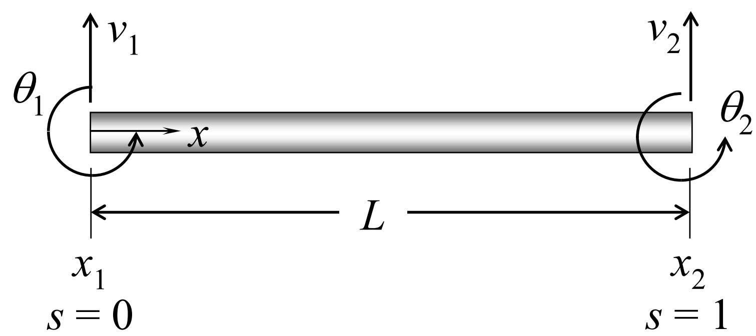


FINITE ELEMENT INTERPOLATION *cont.*

- Nodal DOF of beam element
  - Each node has deflection *v* and slope θ
  - Positive directions of DOFs
  - Vector of nodal DOFs $\{\mathbf{q}\} = \{v_1 \quad \theta_1 \quad v_2 \quad \theta_2\}^T$
- Scaling parameter s
  - Length *L* of the beam is scaled to 1 using scaling parameter *s*

$$s = \frac{x - x_1}{L}, \quad ds = \frac{1}{L}dx,$$

$$dx = L ds, \quad \frac{ds}{dx} = \frac{1}{L}$$



- Will write deflection curve *v*(*s*) in terms of *s*

FINITE ELEMENT INTERPOLATION *cont.*

- Deflection interpolation
  - Interpolate the deflection *v*(*s*) in terms of four nodal DOFs
  - Use cubic function: $v(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3$
  - Relation to the slope: $\theta = \frac{dv}{dx} = \frac{dv}{ds}\frac{ds}{dx} = \frac{1}{L}(a_1 + 2a_2 s + 3a_3 s^2)$
  - Apply four conditions:

$$v(0) = v_1 \qquad \frac{dv(0)}{dx} = \theta_1 \qquad v(1) = v_2 \qquad \frac{dv(1)}{dx} = \theta_2$$

  - Express four coefficients in terms of nodal DOFs

$$v_1 = v(0) = a_0$$

$$\theta_1 = \frac{dv}{dx}(0) = \frac{1}{L}a_1$$

$$v_2 = v(1) = a_0 + a_1 + a_2 + a_3$$

$$\theta_2 = \frac{dv}{dx}(1) = \frac{1}{L}(a_1 + 2a_2 + 3a_3)$$

$$\Rightarrow$$

$$a_0 = v_1$$

$$a_1 = L\theta_1$$

$$a_2 = -3v_1 - 2L\theta_1 + 3v_2 - L\theta_2$$

$$a_3 = 2v_1 + L\theta_1 - 2v_2 + L\theta_2$$

FINITE ELEMENT INTERPOLATION *cont.*

- Deflection interpolation cont.

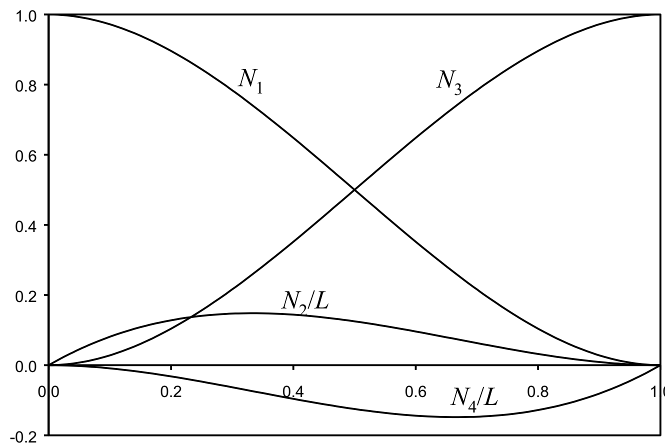
$$v(s) = (1-3s^2+2s^3)v_1 + L(s-2s^2+s^3)\theta_1 + (3s^2-2s^3)v_2 + L(-s^2+s^3)\theta_2$$

$$v(s) = [N_1(s) \quad N_2(s) \quad N_3(s) \quad N_4(s)]\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\boxed{v(s) = \lfloor \mathbf{N} \rfloor \{\mathbf{q}\}}$$

- Shape functions

$$\boxed{\begin{aligned} N_1(s) &= 1-3s^2+2s^3 \\ N_2(s) &= L(s-2s^2+s^3) \\ N_3(s) &= 3s^2-2s^3 \\ N_4(s) &= L(-s^2+s^3) \end{aligned}}$$



  - Hermite polynomials
  - Interpolation property

FINITE ELEMENT INTERPOLATION *cont.*

- Properties of interpolation
  - Deflection is a cubic polynomial (discuss accuracy and limitation)
  - Interpolation is valid within an element, not outside of the element
  - Adjacent elements have continuous deflection and slope
- Approximation of curvature
  - Curvature is second derivative and related to strain and stress

$$\frac{d^2v}{dx^2} = \frac{1}{L^2}\frac{d^2v}{ds^2} = \frac{1}{L^2}[-6+12s,\ L(-4+6s),\ 6-12s,\ L(-2+6s)]\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\boxed{\frac{d^2v}{dx^2} = \frac{1}{L^2}\underset{1\times 4}{\lfloor \mathbf{B} \rfloor}\underset{4\times 1}{\{\mathbf{q}\}}}$$

B: strain-displacement vector

  - **B** is linear function of *s* and, thus, the strain and stress
  - Alternative expression: $\frac{d^2v}{dx^2} = \frac{1}{L^2}\underset{1\times 4}{\lfloor \mathbf{q}^T \rfloor}\underset{4\times 1}{\{\mathbf{B}^T\}}$
  - If the given problem is linearly varying curvature, the approximation is accurate; if higher-order variation of curvature, then it is approximate

FINITE ELEMENT INTERPOLATION *cont.*

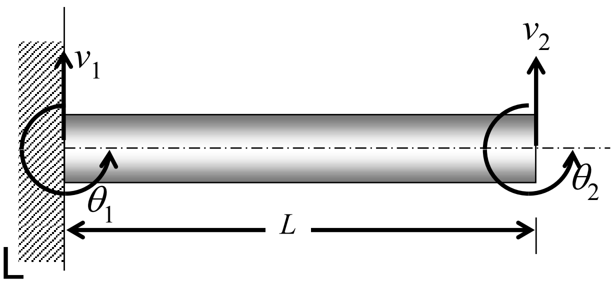
- Approximation of bending moment and shear force

$$M(s) = EI\frac{d^2v}{dx^2} = \frac{EI}{L^2}\lfloor \mathbf{B} \rfloor\{\mathbf{q}\} \qquad \text{Linear}$$

$$V_y = -\frac{dM}{dx} = -EI\frac{d^3v}{dx^3} = \frac{EI}{L^3}[-12 \quad -6L \quad 12 \quad -6L]\{\mathbf{q}\} \qquad \text{Constant}$$

  - Stress is proportional to *M*(*s*); *M*(*s*) is linear; stress is linear, too
  - Maximum stress always occurs at the node
  - Bending moment and shear force are not continuous between adjacent elements

EXAMPLE – INTERPOLATION



- Cantilevered beam
- Given nodal DOFs

$$\{\mathbf{q}\} = \{0, 0, -0.1, -0.2\}^T$$

- Deflection and slope at x = 0.5L
- Parameter s = 0.5 at x = 0.5L
- Shape functions: $N_1(\tfrac{1}{2}) = \dfrac{1}{2}, \quad N_2(\tfrac{1}{2}) = \dfrac{L}{8}, \quad N_3(\tfrac{1}{2}) = \dfrac{1}{2}, \quad N_4(\tfrac{1}{2}) = -\dfrac{L}{8}$
- Deflection at s = 0.5:

$$v(\tfrac{1}{2}) = N_1(\tfrac{1}{2})v_1 + N_2(\tfrac{1}{2})\theta_1 + N_3(\tfrac{1}{2})v_2 + N_4(\tfrac{1}{2})\theta_2$$

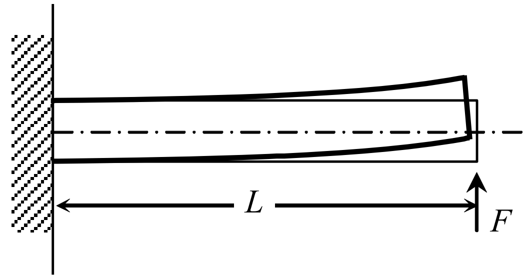
$$= \frac{1}{2}\times 0 + \frac{L}{8}\times 0 + \frac{1}{2}\times v_2 - \frac{L}{8}\times\theta_2 = \frac{v_2}{2} - \frac{L\theta_2}{8} = -0.025$$

- Slope at s = 0.5:

$$\frac{dv}{dx} = \frac{1}{L}\frac{dv}{ds} = \frac{1}{L}\left(v_1\frac{dN_1}{ds} + \theta_1\frac{dN_2}{ds} + v_2\frac{dN_3}{ds} + \theta_2\frac{dN_4}{ds}\right)$$

$$= v_1\frac{1}{L}(-6s + 6s^2) + \theta_1\left(1 - 4s + 3s^2\right) + v_2\frac{1}{L}(6s - 6s^2) + \theta_2\left(-2s + 3s^2\right) = -0.1$$

EXAMPLE



- A beam finite element with length L

$$v_1 = 0, \theta_1 = 0, v_2 = \frac{L^3}{3EI}, \theta_2 = \frac{L^2}{2EI}$$

- Calculate v(s)

$$v(s) = N_1(s)\cancel{v_1} + N_2(s)\cancel{\theta_1} + N_3(s)v_2 + N_4(s)\theta_2$$

$$v(s) = (3s^2 - 2s^3)v_2 + L(-s^2 + s^3)\theta_2$$

- Bending moment

$$M(s) = EI\frac{d^2v}{dx^2} = \frac{EI}{L^2}\frac{d^2v}{ds^2} = \frac{EI}{L^2}\left[(6-12s)v_2 + L(-2+6s)\theta_2\right]$$

$$= \frac{EI}{L^2}\left[(6-12s)\frac{L^3}{3EI} + L(-2+6s)\frac{L^2}{2EI}\right]$$

$= L(1-s) = (L-x)$ Bending moment cause by unit force at the tip

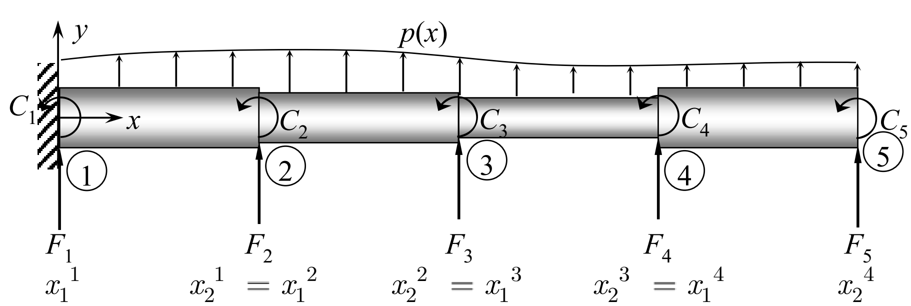
FINITE ELEMENT EQUATION FOR BEAM

- Finite element equation using PMPE
  - A beam is divided by NEL elements with constant sections
- Strain energy
  - Sum of each element's strain energy

$$U = \int_0^{L_T} U_L(x)dx = \sum_{e=1}^{NEL}\int_{x_1^{(e)}}^{x_2^{(e)}} U_L(x)dx = \sum_{e=1}^{NEL} U^{(e)}$$

  - Strain energy of element (e)

$$U^{(e)} = EI\int_{x_1^{(e)}}^{x_2^{(e)}} \frac{1}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{L^3}\int_0^1 \frac{1}{2}\left(\frac{d^2v}{ds^2}\right)^2 ds$$



FE EQUATION FOR BEAM *cont.*

- Strain energy *cont.*
  - Approximate curvature in terms of nodal DOFs

$$\left(\frac{d^2v}{ds^2}\right)^2 = \left(\frac{d^2v}{ds^2}\right)\left(\frac{d^2v}{ds^2}\right) = \underset{1\times 4}{\{\mathbf{q}^{(e)}\}^T} \underset{4\times 1}{\lfloor \mathbf{B} \rfloor^T} \underset{1\times 4}{\lfloor \mathbf{B} \rfloor} \underset{4\times 1}{\{\mathbf{q}^{(e)}\}}$$

  - Approximate element strain energy in terms of nodal DOFs

$$U^{(e)} = \frac{1}{2}\{\mathbf{q}^{(e)}\}^T \left[\frac{EI}{L^3}\int_0^1 \lfloor \mathbf{B} \rfloor^T \lfloor \mathbf{B} \rfloor ds\right]^{(e)} \{\mathbf{q}^{(e)}\} = \frac{1}{2}\{\mathbf{q}^{(e)}\}^T [\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\}$$

- Stiffness matrix of a beam element

$$[\mathbf{k}^{(e)}] = \frac{EI}{L^3}\int_0^1 \begin{bmatrix} -6+12s \\ L(-4+6s) \\ 6-12s \\ L(-2+6s) \end{bmatrix} \begin{bmatrix} -6+12s & L(-4+6s) & 6-12s & L(-2+6s) \end{bmatrix} ds$$

FE EQUATION FOR BEAM *cont.*

- Stiffness matrix of a beam element

$$[\mathbf{k}^{(e)}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Symmetric, positive semi-definite

Proportional to EI

Inversely proportional to L

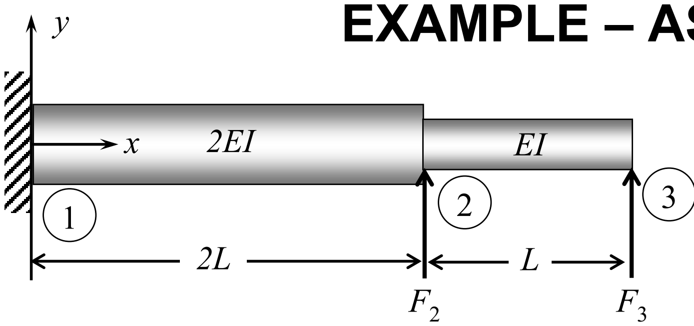
- Strain energy *cont.*

$$U = \sum_{e=1}^{NEL} U^{(e)} = \frac{1}{2}\sum_{e=1}^{NEL}\{\mathbf{q}^{(e)}\}^T [\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\}$$

  - Assembly

$$U = \frac{1}{2}\{\mathbf{Q}_s\}^T [\mathbf{K}_s]\{\mathbf{Q}_s\}$$

EXAMPLE – ASSEMBLY



- Two elements
- Global DOFs

$$\{\mathbf{Q}_s\}^T = \{v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \quad v_3 \quad \theta_3\}$$

$$[\mathbf{k}^{(1)}] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 3 & 3L & -3 & 3L \\ 3L & 4L^2 & -3L & 2L^2 \\ -3 & -3L & 3 & -3L \\ 3L & 2L^2 & -3L & 4L^2 \end{bmatrix} \end{array}\begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \qquad [\mathbf{k}^{(2)}] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}\begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$[\mathbf{K}_s] = \frac{EI}{L^3}\begin{bmatrix} 3 & 3L & -3 & 3L & 0 & 0 \\ 3L & 4L^2 & -3L & 2L^2 & 0 & 0 \\ -3 & -3L & 15 & 3L & -12 & 6L \\ 3L & 2L^2 & 3L & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

FE EQUATION FOR BEAM *cont.*

- Potential energy of applied loads
  - Concentrated forces and couples

$$V = -\sum_{i=1}^{ND}\left(F_i v_i + C_i \theta_i\right) \implies V = -\lfloor v_1\, \theta_1\, v_2 \ldots\ldots \theta_{ND} \rfloor \begin{Bmatrix} F_1 \\ C_1 \\ F_2 \\ \vdots \\ C_{ND} \end{Bmatrix} = \boxed{-\{\mathbf{Q}_s\}^T\{\mathbf{F}_s\}}$$

  - Distributed load (Work-equivalent nodal forces)

$$V = -\sum_{e=1}^{NEL}\int_{x_1^{(e)}}^{x_2^{(e)}} p(x)v(x)\,dx = \sum_{e=1}^{NEL} V^{(e)} \implies V^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} p(x)v(x)\,dx = L^{(e)}\int_0^1 p(s)v(s)\,ds$$

$$\begin{aligned} V^{(e)} &= L^{(e)}\int_0^1 p(s)\left(v_1 N_1 + \theta_1 N_2 + v_2 N_3 + \theta_2 N_4\right)ds \\ &= v_1\left(L^{(e)}\int_0^1 p(s)N_1 ds\right) + \theta_1\left(L^{(e)}\int_0^1 p(s)N_2 ds\right) + v_2\left(L^{(e)}\int_0^1 p(s)N_3 ds\right) + \theta_2\left(L^{(e)}\int_0^1 p(s)N_4 ds\right) \\ &= v_1 F_1^{(e)} + \theta_1 C_1^{(e)} + v_2 F_2^{(e)} + \theta_2 C_2^{(e)} \end{aligned}$$

EXAMPLE – WORK-EQUIVALENT NODAL FORCES

- Uniformly distributed load

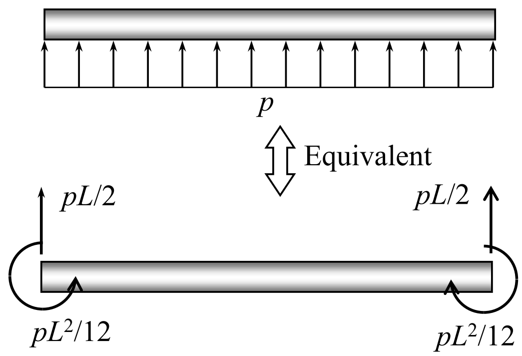
$$F_1 = pL\int_0^1 N_1(s)\,ds = pL\int_0^1 (1-3s^2+2s^3)\,ds = \frac{pL}{2}$$

$$C_1 = pL\int_0^1 N_2(s)\,ds = pL^2\int_0^1 (s-2s^2+s^3)\,ds = \frac{pL^2}{12}$$

$$F_2 = pL\int_0^1 N_3(s)\,ds = pL\int_0^1 (3s^2-2s^3)\,ds = \frac{pL}{2}$$

$$C_2 = pL\int_0^1 N_4(s)\,ds = pL^2\int_0^1 (-s^2+s^3)\,ds = -\frac{pL^2}{12}$$

$$\boxed{\{\mathbf{F}\}^T = \left\{ \frac{pL}{2} \quad \frac{pL^2}{12} \quad \frac{pL}{2} \quad -\frac{pL^2}{12} \right\}}$$



FE EQUATION FOR BEAM *cont.*

- Finite element equation for beam

$$\boxed{\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ pL^2/12 \\ pL/2 \\ -pL^2/12 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ C_1 \\ F_2 \\ C_2 \end{Bmatrix}}$$

- One beam element has four variables
- When there is no distributed load, $p = 0$
- Applying boundary conditions is identical to truss element
- At each DOF, either displacement (v or θ) or force (F or C) must be known, not both
- Use standard procedure for assembly, BC, and solution

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

- Potential energy (quadratic form)

$$\Pi = U + V = \frac{1}{2}\{\mathbf{Q}_s\}^T[\mathbf{K}_s]\{\mathbf{Q}_s\} - \{\mathbf{Q}_s\}^T\{\mathbf{F}_s\}$$

- PMPE
  - Potential energy has its minimum when

$$\boxed{[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}}$$

[**K**$_s$] is symmetric & PSD

- Applying BC
  - The same procedure with truss elements (striking-the-rows and striking-he-columns)

$$[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$$

[**K**] is symmetric & PD

- Solve for unknown nodal DOFs {**Q**}

BENDING MOMENT & SHEAR FORCE

- Bending moment

$$M(s) = EI\frac{d^2v}{dx^2} = \frac{EI}{L^2}\frac{d^2v}{ds^2} = \frac{EI}{L^2}\lfloor \mathbf{B} \rfloor\{\mathbf{q}\}$$

  - Linearly varying along the beam span

- Shear force

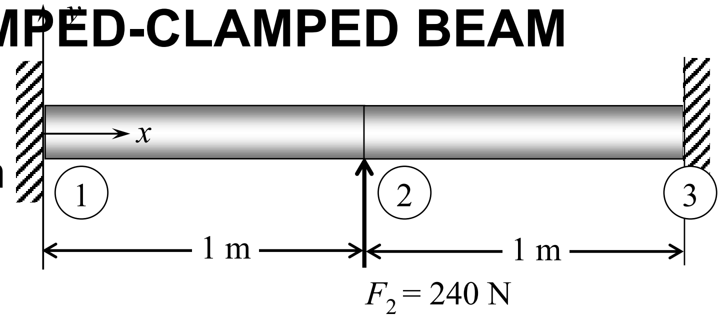
$$V_y(s) = -\frac{dM}{dx} = -EI\frac{d^3v}{dx^3} = -\frac{EI}{L^3}\frac{d^3v}{ds^3} = \frac{EI}{L^3}[-12 \quad -6L \quad 12 \quad -6L]\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

  - Constant
  - When true moment is not linear and true shear is not constant, many elements should be used to approximate it

- Bending stress $\sigma_x = -\frac{My}{I}$
- Shear stress for rectangular section

$$\tau_{xy}(y) = \frac{1.5V_y}{bh}\left(1 - \frac{4y^2}{h^2}\right)$$

EXAMPLE – CLAMPED-CLAMPED BEAM



- Determine deflection & slope at x = 0.5, 1.0, 1.5 m
- Element stiffness matrices

$$[\mathbf{k}^{(1)}] = 1000 \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$[\mathbf{k}^{(2)}] = 1000 \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$1000 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ C_1 \\ 240 \\ 0 \\ F_3 \\ C_3 \end{Bmatrix}$$

EXAMPLE – CLAMPED-CLAMPED BEAM *cont.*

- Applying BC

$$1000 \begin{bmatrix} 24 & 0 \\ 0 & 8 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 240 \\ 0 \end{Bmatrix} \implies \begin{array}{l} v_2 = 0.01 \\ \theta_2 = 0.0 \end{array}$$

- At x = 0.5 $\implies$ s = 0.5 and use element 1

$$v(\tfrac{1}{2}) = v_1 N_1(\tfrac{1}{2}) + \theta_1 N_2(\tfrac{1}{2}) + v_2 N_3(\tfrac{1}{2}) + \theta_2 N_4(\tfrac{1}{2}) = 0.01 \times N_3(\tfrac{1}{2}) = 0.005\,\text{m}$$

$$\theta(\tfrac{1}{2}) = \frac{1}{L^{(1)}} v_2 \left.\frac{dN_3}{ds}\right|_{s=\frac{1}{2}} = 0.015\,\text{rad}$$

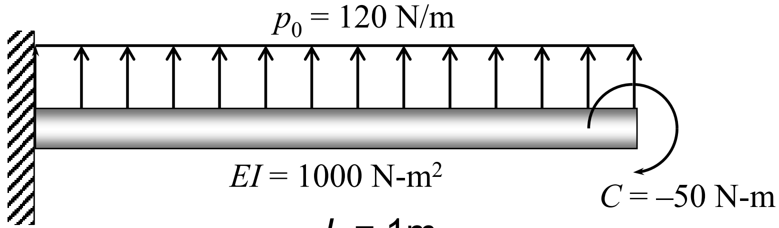
- At x = 1.0 $\implies$ either s = 1 (element 1) or s = 0 (element 2)

$$v(1) = v_2 N_3(1) = 0.01 \times N_3(1) = 0.01\,\text{m} \qquad v(0) = v_2 N_1(0) = 0.01 \times N_1(0) = 0.01\,\text{m}$$

$$\theta(1) = \frac{1}{L^{(1)}} v_2 \left.\frac{dN_3}{ds}\right|_{s=1} = 0.0\,\text{rad} \qquad \theta(0) = \frac{1}{L^{(2)}} v_2 \left.\frac{dN_1}{ds}\right|_{s=0} = 0.0\,\text{rad}$$

Will this solution be accurate or approximate?

EXAMPLE – CANTILEVERED BEAM



L = 1m

- One beam element
- No assembly required
- Element stiffness

$$[\mathbf{K}_s] = 1000\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{array}{l} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

- Work-equivalent nodal forces

$$\begin{Bmatrix} F_{1e} \\ C_{1e} \\ F_{2e} \\ C_{2e} \end{Bmatrix} = p_0 L \int_0^1 \begin{Bmatrix} 1-3s^2+2s^3 \\ (s-2s^2+s^3)L \\ 3s^2-2s^3 \\ (-s^2+s^3)L \end{Bmatrix} ds = p_0 L \begin{Bmatrix} 1/2 \\ L/12 \\ 1/2 \\ -L/12 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 10 \\ 60 \\ -10 \end{Bmatrix}$$

EXAMPLE – CANTILEVERED BEAM *cont.*

- FE matrix equation

$$1000\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1+60 \\ C_1+10 \\ 60 \\ -10-50 \end{Bmatrix}$$

- Applying BC

$$1000\begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -60 \end{Bmatrix} \Longrightarrow \begin{array}{l} v_2 = -0.01\,\text{m} \\ \theta_2 = -0.03\,\text{rad} \end{array}$$

- Deflection curve: $v(s) = -0.01N_3(s) - 0.03N_4(s) = -0.01s^3$
- Exact solution: $v(x) = 0.005(x^4 - 4x^3 + x^2)$

EXAMPLE – CANTILEVERED BEAM *cont.*

- Support reaction (From assembled matrix equation)

$$1000(-12v_2+6\theta_2)=F_1+60 \qquad\qquad F_1=-120\,\mathrm{N}$$
$$1000(-6v_2+2\theta_2)=C_1+10 \qquad\Rightarrow\qquad C_1=-10\,\mathrm{N\cdot m}$$

- Bending moment

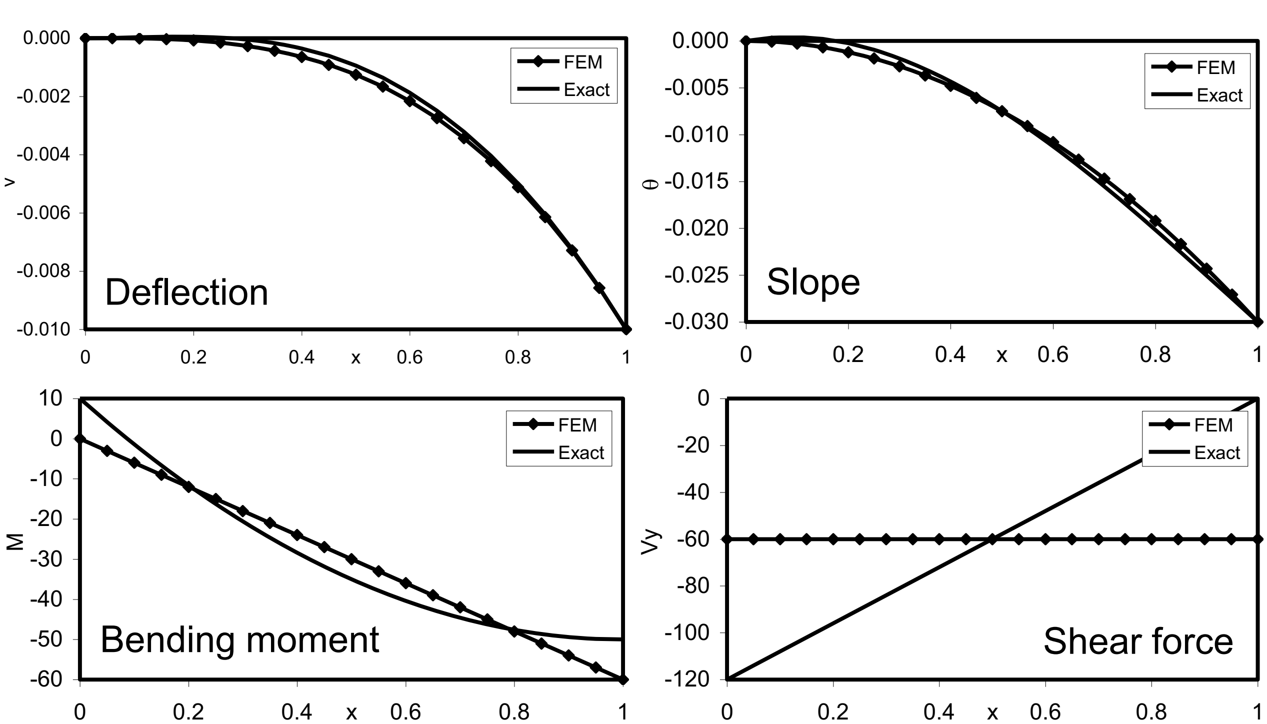
$$\begin{aligned}M(s)&=\frac{EI}{L^2}\lfloor\mathbf{B}\rfloor\{\mathbf{q}\}\\&=\frac{EI}{L^2}\left[(-6+12s)v_1+L(-4+6s)\theta_1+(6-12s)v_2+L(-2+6s)\theta_2\right]\\&=1000[-0.01(6-12s)-0.03(-2+6s)]\\&=-60s\ \mathrm{N\cdot m}\end{aligned}$$

- Shear force

$$\begin{aligned}V_y&=-\frac{EI}{L^3}\left[12v_1+6L\theta_1-12v_2+6L\theta_2\right]\\&=-1000[-12\times(-0.01)+6(-0.03)]\\&=60\,\mathrm{N}\end{aligned}$$

EXAMPLE – CANTILEVERED BEAM *cont.*

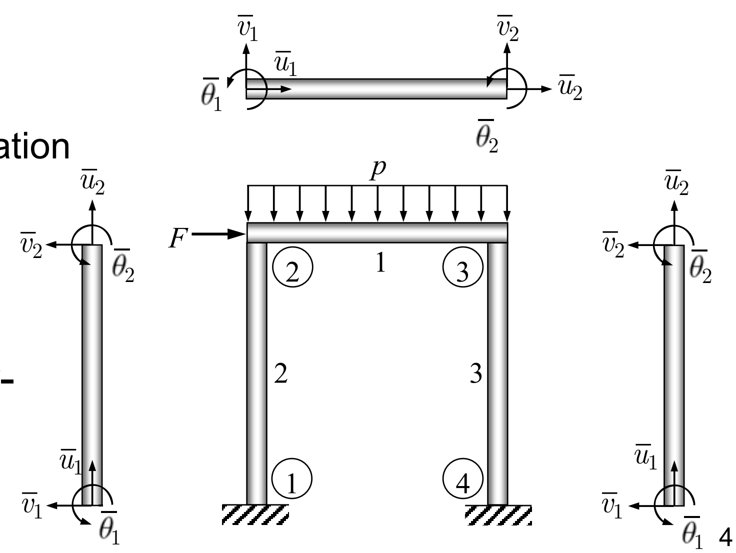
- Comparisons



PLANE FRAME ELEMENT

- Beam
  - Vertical deflection and slope. No axial deformation
- Frame structure
  - Can carry axial force, transverse shear force, and bending moment (Beam + Truss)
- Assumption
  - Axial and bending effects are uncoupled
  - Reasonable when deformation is small
- 3 DOFs per node

  $\{u_i, v_i, \theta_i\}$
- Need coordinate transformation like plane truss



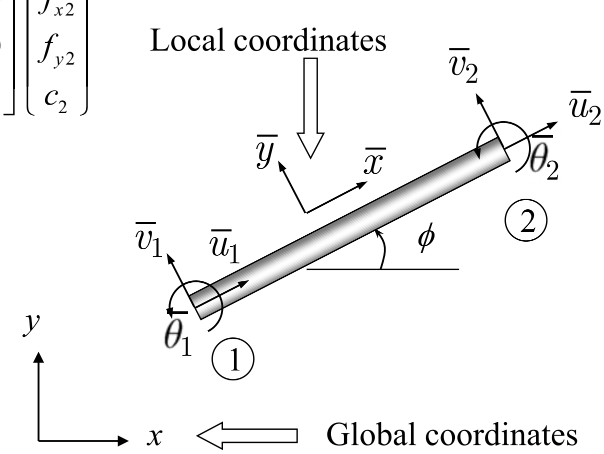
PLANE FRAME ELEMENT *cont.*

- Element-fixed local coordinates $\bar{x}-\bar{y}$
- Local DOFs $\{\bar{u}, \bar{v}, \bar{\theta}\}$ Local forces $\{f_{\bar{x}}, f_{\bar{y}}, \bar{c}\}$
- Transformation between local and global coord.

$$\begin{Bmatrix} f_{\bar{x}1} \\ f_{\bar{y}1} \\ \bar{c}_1 \\ f_{\bar{x}2} \\ f_{\bar{y}2} \\ \bar{c}_2 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{y1} \\ c_1 \\ f_{x2} \\ f_{y2} \\ c_2 \end{Bmatrix}$$

$$\boxed{\{\bar{\mathbf{f}}\} = [\mathbf{T}]\{\mathbf{f}\}}$$

$$\boxed{\{\bar{\mathbf{q}}\} = [\mathbf{T}]\{\mathbf{q}\}}$$



PLANE FRAME ELEMENT *cont.*

- Axial deformation (in local coord.)

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{Bmatrix} f_{\bar{x}1} \\ f_{\bar{x}2} \end{Bmatrix}$$

- Beam bending

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} f_{\bar{y}1} \\ \bar{c}_1 \\ f_{\bar{y}2} \\ \bar{c}_2 \end{Bmatrix}$$

- Basically, it is equivalent to overlapping a beam with a bar
- A frame element has 6 DOFs

PLANE FRAME ELEMENT *cont.*

- Element matrix equation (local coord.)

$$\begin{bmatrix} a_1 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 12a_2 & 6La_2 & 0 & -12a_2 & 6La_2 \\ 0 & 6La_2 & 4L^2a_2 & 0 & -6La_2 & 2L^2a_2 \\ -a_1 & 0 & 0 & a_1 & 0 & 0 \\ 0 & -12a_2 & -6La_2 & 0 & 12a_2 & -6La_2 \\ 0 & 6La_2 & 2L^2a_2 & 0 & -6La_2 & 4L^2a_2 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{c}_1 \\ \bar{f}_{x2} \\ \bar{f}_{y2} \\ \bar{c}_2 \end{Bmatrix} \qquad a_1 = \frac{EA}{L} \quad a_2 = \frac{EI}{L^3}$$

$$\boxed{[\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\} = \{\bar{\mathbf{f}}\}}$$

- Element matrix equation (global coord.)

$$[\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} = [\mathbf{T}]\{\mathbf{f}\} \implies [\mathbf{T}]^T[\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} = \{\mathbf{f}\} \implies [\mathbf{k}]\{\mathbf{q}\} = \{\mathbf{f}\}$$

$$\boxed{[\mathbf{k}] = [\mathbf{T}]^T[\bar{\mathbf{k}}][\mathbf{T}]}$$

- Same procedure for assembly and applying BC

PLANE FRAME ELEMENT *cont.*

- Calculation of element forces
  - **Element forces can only be calculated in the local coordinate**
  - Extract element DOFs {**q**} from the global DOFs {$\mathbf{Q}_s$}
  - Transform the element DOFs to the local coordinate $\{\bar{\mathbf{q}}\} = [\mathbf{T}]\{\mathbf{q}\}$
  - Then, use 1D bar and beam formulas for element forces
  - Axial force $P = \dfrac{AE}{L}\left(\bar{u}_2 - \bar{u}_1\right)$
  - Bending moment $M(s) = \dfrac{EI}{L^2}\lfloor \mathbf{B} \rfloor \{\bar{\mathbf{q}}\}$
  - Shear force $V_y(s) = \dfrac{EI}{L^3}[-12 \quad -6L \quad 12 \quad -6L]\{\bar{\mathbf{q}}\}$
- Other method:

$$\begin{Bmatrix} -V_{\bar{y}1} \\ -\bar{M}_1 \\ +V_{\bar{y}2} \\ \bar{M}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix}$$